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Notes on the Ratio and the Right-Tail Probability in a Log-Laplace Distribution

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Abstract

We consider estimation of the right-tail probability in a log-Laplace random variable, As we derive the density of ratio of two independent log-Laplace random variables, the k-th moment of the ratio is represented by a special mathematical function. and hence variance of the ratio can be represented by a psi-function.

Keywords; Hazard Rate, Hypergeometric Function, Log-Laplace Distribution, Psi-Function, Ratio Of Random Variables, Right-Tail Probability.

1. Introduction

For two independent random variables X and Y and a real number c, the probability $P(X<cY)$ induces the following facts, (i) the probability $P(X<cY)$ is the right-tail probability of Y when X is degenerated at 1 and $c=1/t$, (ii) the probability $P(X<cY)$ is the distribution of the ratio $X/(X+Y)$ when $c=t/(1-t)$ for $0<t<1$. Woo(2006) introduced three cases of probability $P(X<cY)$ for any real number c.

For given random variables X and Y, the distribution of the ratio $R=X/(X+Y)$ is of interest in biological and physical sciences, econometrics, engineering and selection. For example, ratios of normal variables appears as sampling distributions in single equation models in simultaneous equations models. Other area of applications include mass to energy ratios in nuclear physics. another important area is the stress-strength model in the right-tail probability.

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Many authors considered inferences on the right-tail probability(or reliability) in several distributions, and in recent Lee & Won(2006) studied inferences on the reliability in an exponentiated uniform distribution. Ali, et al(2006) studied distribution of the ratio of generalized uniform variates. Woo(2006) introduced the reliability, the ratio X/(X+Y), and a skewed-symmetric distribution of two independent random variables, Woo(2007) studied inferences on the reliability in a half-normal distribution.

A log-Laplace random variable is introduced by the following density:

$$f(x)=\frac{1}{2\beta}\cdot x^{-1}\cdot e^{-|\ln x|/\beta}, \quad x>0\ ,\ \beta>0. \tag{1.1}$$

(cf. a log-normal distribution in Rohatgi(1976, p.375))

For example, a log-Laplace distribution has been applied to a distribution of a channel noise in an information system. And we also have known that, if X is a Laplace random variable, then Y=exp(X) is a log-Laplace random variable.

In this paper, we consider estimation of the right-tail probability in a log-Laplace random variable, As we derive the density of the ratio of two independent log-Laplace random variables, the k-th moment of the ratio of two independent log-Laplace random variables is represented by a special mathematical function. and hence variance of the ratio can be represented by a psi-function.

2. The right-tail probability

From the density (1.1)of log-Laplace random variable, we have known the following easily:

(a) When $\beta>1$, the density f(x) in (1.1) is decreasing function of x, $\lim_{x\to 0} f(x)=\infty$ and $\lim_{x\to\infty} f(x)=0$.

(b) When $0<\beta<1$, the density f(x) is increasing on (0,1) and decreasing on $[1,\infty)$, and $\lim_{x\to 0 \text{ or } \infty} f(x)=0$, and hence x=1 is its mode.

From the density (1.1) of log-Laplace random variable X, the cdf of X is given by:

$$F(x)=\begin{cases}\frac{1}{2}x^{1/\beta}\ , & \text{if } 0<x<1\\ 1-\frac{1}{2}x^{-1/\beta}\ , & \text{if } x\geq 1\end{cases}. \tag{2.1}$$

From the density (1.1) of the log-Laplace random variable X and the formula 3.381(4) in Gradsheyn & Ryzhik (1965, p.317), the k-th moment of X is given by:

$$E(X^k) = \frac{1}{1-\beta^2 k^2}, \quad \text{if} \quad \beta \cdot k < 1, \quad k = 1,2,3,\ldots \text{ and } \beta > 0. \tag{2.2}$$

and hence its mean and variance are

$$\frac{1}{1-\beta^2} \quad \text{and} \quad \frac{1}{1-4\beta^2} - \left(\frac{1}{1-\beta^2}\right)^2, \quad \text{if} \quad 0 < \beta < 1/2 \text{ , respectively,}$$

From the k-th moment (2.2) we evaluate mean, variance, skewness and kurtosis to guess the figure of the density (1.1) as the parameter β varies:

Table 1. Mean, variance, skewness, and kurtosis of the density (1.1)

β	mean	variance	skewness	kurtosis
1/20	1.00251	0.00507	0.58660	5.73691
1/15	1.00446	0.00916	0.71558	7.48102
1/10	1.01010	0.01237	1.13382	9.51362
1/5	1.04167	0.10540	3.00469	13.72300
1/4	1.06667	0.19555	5.16153	-
1/3	1.12500	0.53438	-	-
1/2	1.33333	-	-	-

From Table 1, we observe the following: the density (1.1) is skewed to the right, and the density has kurtosis(>3) when $\beta \leqq 1/5$.

Assume $X_1, X_2, \ldots, X_n$ be a sample from the log-Laplace density (1.1) with a parameter $\beta > 0$. Then the MLE $\hat{\beta}$ of β is given by:

$$\hat{\beta} = \frac{1}{n} \sum_{i=1}^{n} | \ln X_i | .$$

Since $\sum_{i=1}^{n} | \ln X_i |$ has a gamma distribution with a shape parameter n and a scale parameter β (its mean is $n \cdot \beta$), the MLE is an unbiased estimator of β and has variance β^2/n. Since $\sum_{i=1}^{n} | \ln X_i |$ is a complete sufficient statistics, $\sum_{i=1}^{n} | \ln X_i | /n$ is a known UMVUE of β, from Lehmann-Scheffe Theorem in Rohatgi(1976, p.356). While, as we find a constant "c" such that

$$E[(c \cdot \sum_{i=1}^{n} | \ln X_i | - \beta)^2] \text{ is minimized,}$$

we can recommended a bias estimator $\tilde{\beta}$ of β by:

$$\tilde{\beta}=\frac{1}{n+1}\sum_{i=1}^{n} | \ln X_i | ,$$

which has bias $\frac{1}{n+1}\beta$ and variance $\frac{n}{(n+1)^2}\beta^2$, respectively.

Therefore, we obtain the following:

Fact 1. The bias estimator $\tilde{\beta}$ performs better than the MLE $\hat{\beta}$ in a sense of MSE.

Since a pivot quatity $\frac{2}{\beta}\sum_{i=1}^{n} | \ln X_i |$ has a χ^2-distribution with 2n degree of freedom(df), an $(1-\gamma)100\%$ confidence interval of β is given by:

$$\left(\frac{2\sum_{i=1}^{n} | \ln X_i |}{\chi^2_{1-\gamma/2}(2n)}, \frac{2\sum_{i=1}^{n} | \ln X_i |}{\chi^2_{\gamma/2}(2n)} \right), \tag{2.3}$$

where $p=\int_o^{\chi^2_p(2n)} \chi^2_{2n}(t)dt$, $\chi^2_{2n}(t)$ is the density of χ^2-distribution with 2n-df.

From the cdf (2.1), the right-tail probability R(t) of the log-Laplace random variable is given by:

$$R(t\,;\beta)=\begin{cases} 1-\frac{1}{2}t^{1/\beta}, & \text{if } 0<t<1 \\ \frac{1}{2}t^{-1/\beta}, & \text{if } t \geqq 1 \end{cases} . \tag{2.4}$$

Since $\frac{dR(t;\beta)}{d\beta}\begin{cases} <0, & \text{if } 0<t<1 \\ \geqq 0, & \text{if } t\geqq 1 \end{cases}$, $R(t\,;\beta)$ is a monotone function of β over each interval of t, Since inference on the right-tail probability is equivalent to inference on β.(see McCool(1991)), as applying the result for each interval we obtain the following from Fact 1:

Fact 2. The estimator $R(t;\tilde{\beta})$ performs better than the MLE $R(t;\hat{\beta})$ of the right-tail probability in a sense of MSE.

Remark 1 (interval estimation). Based on the interval (2.4) of β and the right-tail probability $R(t\,;\beta)$ in (2.4), for each interval $(0,1)$ or $(1,\infty)$, we can derive two kinds of confidence interval of $R(t\,;\beta)$ according to monotones on $R(t\,;\beta)$ of β for the corresponding $t\in(0,1)$ or $(1,\infty)$.

As the hazard rate is a theoretical application of the right-tail probability, from definition of the hazard rate in Saunders(2007, p.11), the hazard rate of the log-Laplace random variable is given by:

$$h(t)=\begin{cases}\dfrac{t^{1/\beta-1}/\beta}{2-t^{1/\beta}}, & \text{if } 0<t<1\\ t^{-1}/\beta, & \text{if } t\geqq 1\end{cases}.$$

From characterizations of an increasing failure rate(IFR) and decreasing failure rate(DFR) in Saunders(2007, p.12), we have the following:

Fact 3. (a) A truncated log-Laplace density by f(x) in (1.1) over $[1.\infty)$ has DFR.

(b) A truncated log-Laplace density by f(x) in (1.1) over (0,1) has IFR when $0<\beta<1$.

3. Two independent log-Laplace random variables

For example, (X, Y) can be applied to a pair of two different channel noises in a communication system.

When X and Y be two independent log-Laplace random variables with the density (1.1) having two different parameters β_1 and β_2, respectively. $\ln X$ and $\ln Y$ have symmetric distributions about origin, And hence we obtain the following:

Fact 4. Assume X and Y be two independent log-Laplace random variables with the density (1.1) having two different parameters β_1 and β_2, respectively. Then the reliability $p(X^a< Y^b)=1/2$, for non-zero real numbers a and b.

Assume X and Y be two independent log-Laplace random variables with the density (1.1) having two different parameters β_1 and β_2, respectively.

Then, from the formulas 3.381(1, 3, & 4) in Gradsheyn & Ryzhik (1965, p.317) and the difference density of two independent random variables in Rohatgi(1976, p.141), we obtain the density of $U\equiv \ln X-\ln Y$ as the following:

$$f_U(u)=\frac{1}{4(\beta_1+\beta_2)}(e^{-|u|/\beta_1}+e^{-|u|/\beta_2})+\frac{1}{4(\beta_1-\beta_2)}(e^{-|u|/\beta_1}-e^{-|u|/\beta_2}),$$

$$\text{if } -\infty<u<\infty\ , \tag{3.1}$$

which is well-defined over $0<\beta_2<\beta_1<1$.

The ratio R of two random variables X and Y is well-known by:

$$\text{R}=\frac{X}{X+Y}=\frac{1}{1+e^{-U}}.$$

From the density (3.1) of U, the density and distribution of the ratio R can be derived by a transform of the random variable U, $R=\frac{1}{1+e^{-U}}$,(0<R<1) as the following:

$$f_R(r)=\begin{cases}[\frac{\beta_1}{2(\beta_1^2-\beta_2^2)}(\frac{1-r}{r})^{1/\beta_1}-\frac{\beta_2}{2(\beta_1^2-\beta_2^2)}(\frac{1-r}{r})^{1/\beta_2}]\frac{1}{r(1-r)}, & \text{if } 1/2\leqq r<1\\ [\frac{\beta_1}{2(\beta_1^2-\beta_2^2)}(\frac{1-r}{r})^{-1/\beta_1}-\frac{\beta_2}{2(\beta_1^2-\beta_2^2)}(\frac{1-r}{r})^{-1/\beta_2}]\frac{1}{r(1-r)}, & \text{if } 0<r<1/2\end{cases}$$

$$\text{if } 0<\beta_2<\beta_1<1. \tag{3.2}$$

Remark 2. From the density of R, the density of the ratio is symmetric at 1/2, and hence mean of the ratio R is 1/2.

From the density (3.2) of the ratio R, the (incomplete) beta function, and the formula 6.6.1 in Abramowitz & Stegun(1972, p.263), we obtain the k-th moment of the ratio R as the following:

For given positive integer $k\geqq 2$,

$$\begin{aligned}E(R^k)=&\frac{\beta_1}{2(\beta_1^2-\beta_2^2)}[B(k-1/\beta_1,1/\beta_1)-B_{1/2}(k-1/\beta_1,1/\beta_1)]\\ &-\frac{\beta_2}{2(\beta_1^2-\beta_2^2)}[B(k-1/\beta_2,1/\beta_2)-B_{1/2}(k-1/\beta_2,1/\beta_2)]\\ &+\frac{\beta_1}{2(\beta_1^2-\beta_2^2)}B_{1/2}(k+1/\beta_1,-1/\beta_1)-\frac{\beta_2}{2(\beta_1^2-\beta_2^2)}B_{1/2}(k+1/\beta_2,-1/\beta_2),\end{aligned} \tag{3.3}$$

if $k>1/\beta_2$ and $0<\beta_2<\beta_1<1$, where $B(a,b)$ is the beta function for a>0, b>0 and $B_x(a,b)$ is the incomplete beta function for 0<x<1 in Abramowitz & Stegun(1972, p.263).

To transform the 2nd moment of the ratio R, from the formula 6.6.8 in Abramowitz & Stegun(1972, p.263), we obtain the following:

$$B_{1/2}(k+1/\beta,-1/\beta)=\frac{1}{k+1/\beta}\cdot 2^{-(k+1/\beta)}\cdot {}_2F_1(k+1/\beta,1+1/\beta\,;k+1/\beta+1;1/2),$$

$$\text{if } \beta>1/k \text{ for a positive integer } k\geqq 2, \tag{3.4}$$

where ${}_2F_1(a,b;c;x)$ is the generalized hypergeometric function.

Especially if k=2 *in* (3.4), *from the formulas* 15.2.17, 15.1.8, 15.1.28, and 6.3.5 in Abramowitz & Stegun(1972, p.558, p,556, p.557, and p.258, in its order of evaluation), we can represent the incomplete beta function $B_{1/2}(a,b)$ by the psi-function:

Fact 5. $B_{1/2}(2+1/\beta,-1/\beta)=\frac{1}{2}-\frac{1+1/\beta}{2}\cdot[\psi(\frac{1}{2\beta}+\frac{1}{2})-\psi(\frac{1}{2\beta})-\frac{2\beta^2}{1+\beta}]$,
if $\beta>1/2$, where $\psi(x)$ is the psi-function.

Remark 3. (a) For β_i, i=1,2 in (3.3), to evaluate k-th moment of the ratio, $1>\beta_1>\beta_2>1/k$ must be satisfied for each positive integer k$\geqq 2$.

(b) For 0<x<1, the incomplete beta function $B_x(a,b)$ can be evaluated by numerical values.

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