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Bayesian Model Selection in Weibull Populations[1]

Sang Gil Kang[2]

Abstract

This article addresses the problem of testing whether the shape parameters in k independent Weibull populations are equal. We propose a Bayesian model selection procedure for equality of the shape parameters. The noninformative prior is usually improper which yields a calibration problem that makes the Bayes factor to be defined up to a multiplicative constant. So we propose the objective Bayesian model selection procedure based on the fractional Bayes factor and the intrinsic Bayes factor under the reference prior. Simulation study and a real example are provided.

Keywords : Equality Of The Shape Parameters, Fractional Bayes Factor, Intrinsic Bayes Factor, Reference Prior, Weibull Populations.

1. Introduction

In Bayesian model selection or testing problem, the Bayes factor under proper priors or informative priors have been very successful. However, limited information and time constraints often require the use of noninformative priors. Since noninformative priors such as Jeffreys' prior or reference prior (Berger and Bernardo, 1989, 1992) are typically improper so that such priors are only defined up to arbitrary constants which affects the values of Bayes factors. Spiegelhalter and Smith (1982), O'Hagan (1995) and Berger and Pericchi (1996) have made efforts to compensate for that arbitrariness.

Spiegelhalter and Smith (1982) used the device of imaginary training samples in the context of linear model comparisons to choose the arbitrary constants. But the choice of imaginary training sample depends on the models under comparison, and so, there is no guarantee that the Bayes factor of Spiegelhalter and Smith (1982)

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1) This research was supported by Sangji University Research Fund, 2006.
2) Associate Professor, Department of Computer & Data Information, Sangji University, Wonju, 220-712, Korea.
E-mail : sangkg@mail.sangji.ac.kr

is coherent for multiple model comparisons. Berger and Pericchi (1996) introduced the intrinsic Bayes factor using a data-splitting idea, which would eliminate the arbitrariness of improper priors. O'Hagan (1995) proposed the fractional Bayes factor. For removing the arbitrariness he used to a portion of the likelihood with a so-called the fraction b. These approaches have shown to be quite useful in many statistical areas (Kang, Kim and Lee, 2005, 2006).

The Weibull distribution is widely used in lifetime data analysis in medical and biological sciences, engineering, etc. Many statistical methods has been developed for this distribution. A general review of the Weibull distribution including several references to applications in diverse fields is given by Johnson, Kotz and Balakrishnan (1994) and Lawless (2003) discussed the specific application of this distribution to failure data analysis.

For comparison of the shape parameters of Weibull distributions, Thoman and Bain (1969) proposed a test based on the maximum likelihood estimators and gave some tables of percentage points by Monte Carlo methods. The approximate Bartlett's procedure and the likelihood ratio procedure for testing the common shape parameter are given in Lawless and Mann (1976) and Lawless (2003). Using the data of Nelson (1970), Lawless (2003) showed that the results of the two tests are in broad agreement, and it appears that there is no real evidence of a difference in shape parameters. However there is a little work in this problem from the viewpoint of objective Bayesian framework.

This paper focuses on the objective Bayesian method for testing equality of the shape parameters in the Weibull distributions. For dealing this problem, we use the Bayesian model selection based on the fractional Bayes factor (O'Hagan, 1995) and the intrinsic Bayes factor (Berger and Pericchi, 1996, 1998). An excellent exposition of the objective Bayesian method to model selection is Berger and Pericchi (2001).

The outline of the remaining sections is as follows. In Section 2, we introduce the Bayesian model selection based on the Bayes factor. In Section 3, Using the reference prior, we provide the Bayesian model selection procedure based on the fractional Bayes factor and intrinsic Bayes factor. In Section 4, simulation study and a real example are given.

2. Intrinsic and Fractional Bayes Factors

Models $M_1, M_2, \cdots, M_q$ are under consideration, with the data $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ having probability density function $f_i(\boldsymbol{x} \mid \boldsymbol{\theta}_i)$ under model M_i. The parameter vectors $\boldsymbol{\theta}_i$ are unknown. Let $\pi_i(\boldsymbol{\theta}_i)$ be the prior distributions of model M_i, and let p_i be the prior probabilities of model M_i, $i = 1, 2, \cdots, q$. Then the posterior probability that the model M_i is true is

$$P(M_i \mid \boldsymbol{x}) = \left(\sum_{j=1}^{q} \frac{p_j}{p_i} \cdot B_{ji} \right)^{-1}, \tag{1}$$

where B_{ji} is the Bayes factor of model M_j to model M_i defined by

$$B_{ji} = \frac{\int f_j(\boldsymbol{x} \mid \boldsymbol{\theta}_j)\pi_j(\boldsymbol{\theta}_j)d\boldsymbol{\theta}_j}{\int f_i(\boldsymbol{x} \mid \boldsymbol{\theta}_i)\pi_i(\boldsymbol{\theta}_i)d\boldsymbol{\theta}_i} = \frac{m_j(\boldsymbol{x})}{m_i(\boldsymbol{x})}. \tag{2}$$

The B_{ji} interpreted as the comparative support of the data for the model j to i. The computation of B_{ji} needs specification of the prior distribution $\pi_i(\boldsymbol{\theta}_i)$ and $\pi_j(\boldsymbol{\theta}_j)$. Often in Bayesian analysis, one can use noninformative priors π_i^N. Common choices are the uniform prior, the Jeffreys' prior and the reference prior. The noninformative prior π_i^N is typically improper. Hence the use of noninformative prior $\pi_i^N(\cdot)$ in (2) causes the B_{ji} to contain unspecified constants. To solve this problem, Berger and Pericchi (1996) proposed the intrinsic Bayes factor, and O'Hagan (1995) proposed the fractional Bayes factor.

One solution to this indeterminacy problem is to use part of the data as a training sample. Let $\boldsymbol{x}(l)$ denote the part of the data to be so used and let $\boldsymbol{x}(-l)$ be the remainder of the data, such that

$$0 < m_i^N(\boldsymbol{x}(l)) < \infty, i = 1, \cdots, q. \tag{3}$$

In view (3), the posteriors $\pi_i^N(\boldsymbol{\theta}_i \mid \boldsymbol{x}(l))$ are well defined. Now, consider the Bayes factor, $B_{ji}(l)$, for the rest of the data $\boldsymbol{x}(-l)$, using $\pi_i^N(\boldsymbol{\theta}_i \mid \boldsymbol{x}(l))$ as the priors:

$$B_{ji}(l) = \frac{\int f(\boldsymbol{x}(-l) \mid \boldsymbol{\theta}_j, \boldsymbol{x}(l))\pi_j^N(\boldsymbol{\theta}_j \mid \boldsymbol{x}(l))d\boldsymbol{\theta}_j}{\int f(\boldsymbol{x}(-l) \mid \boldsymbol{\theta}_i, \boldsymbol{x}(l))\pi_i^N(\boldsymbol{\theta}_i \mid \boldsymbol{x}(l))d\boldsymbol{\theta}_i} = B_{ji}^N \cdot B_{ij}^N(\boldsymbol{x}(l)) \tag{4}$$

where

$$B_{ji}^N = B_{ji}^N(\boldsymbol{x}) = \frac{m_j^N(\boldsymbol{x})}{m_i^N(\boldsymbol{x})} \quad \text{and} \quad B_{ij}^N(\boldsymbol{x}(l)) = \frac{m_i^N(\boldsymbol{x}(l))}{m_j^N(\boldsymbol{x}(l))}$$

are the Bayes factors that would be obtained for the full data $\boldsymbol{x}$ and training samples $\boldsymbol{x}(l)$, respectively.

Berger and Pericchi (1996) proposed the use of a minimal training sample to compute $B_{ij}^N(\boldsymbol{x}(l))$. Then, an average over all the possible minimal training samples contained in the sample is computed. Thus the arithmetic intrinsic Bayes factor (AIBF) of M_j to M_i is

$$B_{ji}^{AI} = B_{ji}^{N} \cdot \frac{1}{L} \sum_{l=1}^{L} B_{ij}^{N}(\boldsymbol{x}(l)). \tag{5}$$

where L is the number of all possible minimal training samples. Also the median intrinsic Bayes factor (MIBF) by Berger and Pericchi (1998) of M_j to M_i is

$$B_{ji}^{MI} = B_{ji}^{N} \cdot ME[B_{ij}^{N}(\boldsymbol{x}(l))], \tag{6}$$

where ME indicates the median, here to be taken over all the training sample Bayes factors. Therefore we can also calculate the posterior probability of M_i using (1), where B_{ji} is replaced by B_{ji}^{AI} and B_{ji}^{MI} from (5) and (6).

The fractional Bayes factor (O'Hagan, 1995) is based on a similar intuition to that behind the intrinsic Bayes factor but, instead of using part of the data to turn noninformative priors into proper priors, it uses a fraction, b, of each likelihood function, $L(\theta_i) = f_i(\boldsymbol{x} \mid \theta_i)$, with the remaining $1-b$ fraction of the likelihood used for model discrimination. Then the fractional Bayes factor (FBF) of model M_j versus model M_i is

$$B_{ji}^{F} = B_{ji}^{N} \cdot \frac{\int L^{b}(\boldsymbol{x} \mid \theta_i)\pi_i^{N}(\theta_i)d\theta_i}{\int L^{b}(\boldsymbol{x} \mid \theta_j)\pi_j^{N}(\theta_j)d\theta_j} = B_{ji}^{N} \cdot \frac{m_i^{b}(\boldsymbol{x})}{m_j^{b}(\boldsymbol{x})}. \tag{7}$$

O'Hagan (1995) proposed three ways for the choice of the fraction b. One common choice of b is $b = m/n$, where m is the size of the minimal training sample, assuming that this number is uniquely defined. (see O'Hagan (1995, 1997) and the discussion by Berger and Mortera in O'Hagan (1995)).

3. Bayesian Model Selection Procedures

Let X be a Weibull distribution $Wei(\alpha,\beta)$ with density function

$$f(x \mid \alpha,\beta) = \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1} \exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\},\ x > 0, \tag{8}$$

where $\alpha > 0$ is the scale parameter and $\beta > 0$ is the shape parameter. Suppose that $X_{i1},\cdots,X_{in_i}, i = 1,\cdots,k,$ denote independent random samples from the ith

Weibull populations with the scale parameter α_i and the shape parameter β_i. We are interest to testing the hypotheses $H_1 : \beta_1 = \cdots = \beta_k$ vs. $H_2 : \beta_1 \neq \cdots \neq \beta_k$ based on the fractional Bayes factor and the intrinsic Bayes factor.

The two default models being compared are

$$M_1 : \ f(\boldsymbol{x} \mid \theta_1) = Wei(\boldsymbol{x}_1 \mid \alpha_1, \beta) \cdots Wei(\boldsymbol{x}_k \mid \alpha_k, \beta), \ \pi_1^N(\boldsymbol{\theta}_1)$$

and

$$M_2 : \ f(\boldsymbol{x} \mid \theta_1) = Wei(\boldsymbol{x}_1 \mid \alpha_1, \beta_1) \cdots Wei(\boldsymbol{x}_k \mid \alpha_k, \beta_k), \ \pi_2^N(\boldsymbol{\theta}_2),$$

where $\boldsymbol{x} = (\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k)$, $\boldsymbol{x}_i = (x_{11}, \cdots, x_{in_i})$, $i = 1, \cdots, k$, $\boldsymbol{\theta}_1 = (\alpha_1, \cdots, \alpha_k, \beta)$ and $\boldsymbol{\theta}_2 = (\alpha_1, \cdots, \alpha_k, \beta_1, \cdots, \beta_k)$.

3.1 Bayesian Model Selection based on the Fractional Bayes Factor

Under the model M_1, the reference prior for $\beta (\equiv \beta_1 = \cdots = \beta_k)$ and $(\alpha_1, \cdots, \alpha_k)$ is

$$\pi_1^N(\alpha_1, \cdots, \alpha_k, \beta) \propto \alpha_1^{-1} \cdots \alpha_k^{-1} \beta^{-1}. \tag{9}$$

This reference prior is derived by Kim, Kang and Lee (2006). They showed that the posterior under the above reference prior is proper if $n_1 + \cdots + n_k - k > 0$. And the likelihood function under the model M_1 is

$$L(\alpha_1, \cdots, \alpha_k, \beta \mid \boldsymbol{x}) = \beta^{-n} \left[\prod_{i=1}^{k} \alpha_i^{-n_i \beta} \right] \left[\prod_{i=1}^{n_1} \prod_{i=1}^{n_2} x_{ij}^{\beta - 1} \right] \exp\left\{ - \sum_{i=1}^{k} \sum_{j=1}^{n_i} \left(\frac{x_{ij}}{\alpha_i} \right)^{\beta} \right\}, \tag{10}$$

where $n = n_1 + \cdots + n_k$. Then from the likelihood (10) and the reference prior (9), the element of the FBF under M_1 is given by

$$\begin{aligned} m_1^b(\boldsymbol{x}) &= \int_0^\infty \int_0^\infty \cdots \int_0^\infty L^b(\alpha_1, \cdots, \alpha_k, \beta \mid \boldsymbol{x}) \pi_1^N(\alpha_1, \cdots, \alpha_k, \beta) d\alpha_1 \cdots d\alpha_k d\beta \\ &= \int_0^\infty b^{-bn} \left[\prod_{i=1}^{k} \Gamma(bn_i) \right] \left[\prod_{i=1}^{k} \prod_{j=1}^{n_i} x_{ij}^{-b} \right] \beta^{nb-k-1} \left[\prod_{i=1}^{k} \prod_{j=1}^{n_i} \left(x_{ij}^{\beta} / \sum_{j=1}^{n_i} x_{ij}^{\beta} \right)^{b} \right] d\beta. \end{aligned}$$

For the model M_2, the reference prior for $(\alpha_1, \cdots, \alpha_k, \beta_1, \cdots, \beta_k)$ is

$$\pi_2^N(\alpha_1, \cdots, \alpha_k, \beta_1, \cdots, \beta_k) \propto \alpha_1^{-1} \cdots \alpha_k^{-1} \beta_1^{-1} \cdots \beta_k^{-1}. \tag{11}$$

This reference prior is derived by Sun (1997) and the posterior is proper if $n_i > 1, i = 1, \cdots, k$. The likelihood function under the model M_2 is

$$L(\alpha_1,\cdots,\alpha_k,\beta_1,\cdots,\beta_k \mid \boldsymbol{x}) = \prod_{i=1}^{k} \beta_i^{n_i} \alpha_i^{-n_i\beta_i} \left[\prod_{j=1}^{n_i} x_{ij}^{\beta_i - 1} \right] \exp\left\{ -\sum_{j=1}^{n_i} \left(\frac{x_{ij}}{\alpha_i} \right)^{\beta_i} \right\}. \quad (12)$$

Thus from the likelihood (12) and the reference prior (11) the element of FBF under M_2 gives as follows.

$$\begin{aligned} m_2^b(\boldsymbol{x}) &= \int_0^\infty \cdots \int_0^\infty \int_0^\infty \cdots \int_0^\infty L^b(\alpha_1,\cdots,\alpha_k,\beta_1,\cdots,\beta_k \mid \boldsymbol{x}) \\ &\quad \times \pi_2^N(\alpha_1,\cdots,\alpha_k,\beta_1 \cdots,\beta_k) d\alpha_1 \cdots d\alpha_k d\beta_1 \cdots d\beta_k \\ &= b^{-bn} \left[\prod_{i=1}^{k} \Gamma(bn_i) \right] \left[\prod_{i=1}^{k} \prod_{j=1}^{n_i} x_{ij}^{-b} \right] \prod_{i=1}^{k} \left[\int_0^\infty \beta_i^{n_i b - 2} \prod_{j=1}^{n_i} \left(x_{ij}^{\beta_i} / \sum_{j=1}^{n_i} x_{ij}^{\beta_i} \right)^b d\beta_i \right]. \end{aligned}$$

Therefore the element B_{21}^N of the FBF is given by

$$B_{21}^N = \frac{S_2(\boldsymbol{x})}{S_1(\boldsymbol{x})}, \quad (13)$$

where

$$S_1(\boldsymbol{x}) = \int_0^\infty \beta^{n-k-1} \left[\prod_{i=1}^{k} \prod_{j=1}^{n_i} \left(x_{ij}^{\beta} / \sum_{j=1}^{n_i} x_{ij}^{\beta} \right) \right] d\beta$$

and

$$S_2(\boldsymbol{x}) = \prod_{i=1}^{k} \left[\int_0^\infty \beta_i^{n_i - 2} \prod_{j=1}^{n_i} \left(x_{ij}^{\beta_i} / \sum_{j=1}^{n_i} x_{ij}^{\beta_i} \right) d\beta_i \right].$$

And the ratio of marginal densities with fraction b is

$$\frac{m_1^b(\boldsymbol{x})}{m_2^b(\boldsymbol{x})} = \frac{S_1(\boldsymbol{x};b)}{S_2(\boldsymbol{x};b)},$$

where

$$S_1(\boldsymbol{x};b) = \int_0^\infty \beta^{nb-k-1} \left[\prod_{i=1}^{k} \prod_{j=1}^{n_i} \left(x_{ij}^{\beta} / \sum_{j=1}^{n_i} x_{ij}^{\beta} \right)^b \right] d\beta$$

and

$$S_2(\boldsymbol{x};b) = \prod_{i=1}^{k} \left[\int_0^\infty \beta_i^{n_i b - 2} \prod_{j=1}^{n_i} \left(x_{ij}^{\beta_i} / \sum_{j=1}^{n_i} x_{ij}^{\beta_i} \right)^b d\beta_i \right].$$

Thus the FBF of M_2 versus M_1 is given by

$$B_{21}^F = \frac{S_2(\boldsymbol{x})}{S_1(\boldsymbol{x})} \cdot \frac{S_1(\boldsymbol{x};b)}{S_2(\boldsymbol{x};b)}. \quad (14)$$

Note that the calculation of the FBF of M_2 versus M_1 requires an one dimensional numerical integration.

3.2 Bayesian Model Selection based on the Intrinsic Bayes Factor

The element B_{21}^N,(13), of the intrinsic Bayes factor is computed in the fractional Bayes factor. So using minimal training sample, we only calculate the marginal densities under M_1 and M_2, respectively. The marginal density of $(X_{1j_1}, X_{1j_2}, \cdots, X_{kl_1}, X_{kl_2})$ is finite for all $1 \le j_1 < j_2 \le n_1, \cdots, 1 \le l_1 < l_2 \le n_k$ under each model (Sun, 1997; Kim, Kang and Lee, 2006). Thus we conclude that any training sample of size $2k$ is a minimal training sample.

The marginal density $m_1^N(x_{1j_1}, x_{1j_2}, \cdots, x_{kl_1}, x_{kl_2})$ under M_1 is given by

$$\begin{aligned} m_1^N(x_{1j_1}, x_{1j_2}, \cdots, x_{kl_1}, x_{kl_2}) &= \int_0^\infty \int_0^\infty \cdots \int_0^\infty f(x_{1j_1}, x_{1j_2}, \cdots, x_{kl_1}, x_{kl_2} \mid \alpha_1, \cdots \alpha_k, \beta) \\ &\quad \times \pi_1^N(\alpha_1, \cdots \alpha_k, \beta) d\alpha_1 \cdots d\alpha_k d\beta \\ &= \int_0^\infty \beta^{k-1} \left[\prod_{i=1}^k \prod_{j=1}^2 \left(x_{ij}^\beta / \sum_{j=1}^2 x_{ij}^\beta \right) \right] d\beta \\ &\equiv T_1(x_{1j_1}, x_{1j_2}, \cdots, x_{kl_1}, x_{kl_2}), \end{aligned}$$

where $1 \le j_1 < j_2 \le n_1, \cdots, 1 \le l_1 < l_2 \le n_k$. And the marginal density $m_2^N(x_{1j_1}, x_{1j_2}, \cdots, x_{kl_1}, x_{kl_2})$ under M_2 is given by

$$\begin{aligned} & m_2^N(x_{1j_1}, x_{1j_2}, \cdots, x_{kl_1}, x_{kl_2}) \\ &= \int_0^\infty \cdots \int_0^\infty \int_0^\infty \cdots \int_0^\infty f(x_{1j_1}, x_{1j_2}, \cdots, x_{kl_1}, x_{kl_2} \mid \alpha_1, \beta_1, \cdots, \alpha_k, \beta_k) \\ &\quad \times \pi_2^N(\alpha_1, \cdots, \alpha_k, \beta_1, \cdots, \beta_k) d\alpha_1 \cdots d\alpha_k d\beta_1 \cdots d\beta_k \\ &= \prod_{i=1}^k \left[\int_0^\infty \prod_{j=1}^2 \left(x_{ij}^{\beta_i} / \sum_{j=1}^2 x_{ij}^{\beta_i} \right) d\beta_i \right] \equiv T_2(x_{1j_1}, x_{1j_2}, \cdots, x_{kl_1}, x_{kl_2}). \end{aligned}$$

Therefore the AIBF of M_2 versus M_1 is given by

$$B_{21}^{AI} = \frac{S_2(x)}{S_1(x)} \cdot \left[\frac{1}{L} \sum_{j_1, j_2} \cdots \sum_{l_1, l_2} \frac{T_1(x_{1j_1}, x_{1j_2}, \cdots, x_{kl_1}, x_{kl_2})}{T_2(x_{1j_1}, x_{1j_2}, \cdots, x_{kl_1}, x_{kl_2})} \right], \tag{15}$$

where $L = [\prod_{i=1}^k n_i(n_i - 1)]/2^k$. And the MIBF of M_2 versus M_1 is given by

$$B_{21}^{MI} = \frac{S_2(x)}{S_1(x)} \cdot ME \left[\frac{T_1(x_{1j_1}, x_{1j_2}, \cdots, x_{kl_1}, x_{kl_2})}{T_2(x_{1j_1}, x_{1j_2}, \cdots, x_{kl_1}, x_{kl_2})} \right]. \tag{16}$$

Note that the calculations of the AIBF and the MIBF of M_2 versus M_1 require an one dimensional integration.

4. Numerical Studies

In order to assess the Bayesian model selection procedures, we evaluate the posterior probability for several configurations (α_i, β_i), n_i, $i=1,\cdots,k$ and k. In particular, for fixed (α_i, β_i) we take 200 independent random samples of X_{ij}, $i=1,\cdots,k$, $j=1,\cdots,$ n_i from the model (8). In our simulation, we examine the cases when the population size k equals 3 and 5.

The posterior probabilities of M_1 being true are computed assuming equal prior probabilities. Table 1 and 2 show the results of the averages and the standard deviations in parentheses of posterior probabilities. From Table 1 and 2, the FBF, the AIBF and the MIBF give fairly reasonable answers for all configurations, (α_i, β_i), $i=1,\cdots,k$. Also the FBF, the AIBF and the MIBF give a similar behavior for all sample sizes. However for unclear situations like (0.5,0.5, 0.5,1.5,1.5) with the unbalanced data the MIBF accepts the model M_1 but the AIBF reject the model M_1, and also for the case (1,1,1,3,3) with unbalanced data the MIBF favours the model M_1. Thus from the results of Table 1 and 2, the AIBF give more reasonable results than the MIBF and the FBF.

Example. Nelson (1972) described the time to breakdown of a electrical insulating fluid subjected to a constant voltage stress in a life test experiment. The data are breakdown times for seven groups of specimens, each group involving a different voltage level. For this data sets, Lawless (2003) concluded that there is no real evidence of difference in shape parameters of Weibull distributions by the likelihood ratio test and the approximate Bartlett's test and showed that the results of the two tests are in broad agreement.

The following data is a portion of the data sets (Nelson, 1972) to compare the likelihood ratio test with the Bayesian testing procedures.

Voltage Level	Breakdown Times
1 (30 kV)	17.05, 22.66, 21.02, 175.88, 139.07, 144.12, 20.46, 43.4, 194.9, 47.3, 7.74
2 (32 kV)	0.4, 82.85, 9.88, 89.29, 215.1, 2.75, 0.79, 15.93, 3.91, 0.27, 0.69, 100.58, 27.8, 13.95, 53.24
3 (34 kV)	0.96, 4.15, 0.19, 0.78, 8.01, 31.75, 7.35, 6.5, 8.27, 33.91, 32.52, 3.16, 4.85, 2.78, 4.67, 1.31, 12.06, 36.71, 72.89
4 (36 kV)	1.97, 0.59, 2.58, 1.69, 2.71, 25.5, 0.35, 0.99, 3.99, 3.67, 2.07, 0.96, 5.35, 2.9, 13.77
5 (38 kV)	0.47, 0.73, 1.4, 0.74, 0.39, 1.13, 0.09, 2.38

Table 1: The Averages and the Standard Deviations in Parentheses of Posterior Probabilities

$\beta_1,\cdots,\beta_k$	$\alpha_1,\cdots,\alpha_k$	$n_1,\cdots,n_k$	$P^F(M_1 \mid x)$	$P^{AI}(M_1 \mid x)$	$P^{MI}(M_1 \mid x)$
0.5,0.5,0.5	1,1,1	3,3,5	0.645 (0.157)	0.687 (0.178)	0.707 (0.168)
		5,5,5	0.678 (0.175)	0.771 (0.184)	0.789 (0.171)
		5,5,7	0.714 (0.158)	0.792 (0.157)	0.809 (0.143)
	1,1,5	3,3,5	0.644 (0.156)	0.684 (0.178)	0.704 (0.165)
		5,5,5	0.682 (0.155)	0.778 (0.156)	0.795 (0.142)
		5,5,7	0.723 (0.151)	0.795 (0.155)	0.809 (0.144)
	0.1,1,10	3,3,5	0.643 (0.158)	0.687 (0.169)	0.706 (0.157)
		5,5,5	0.692 (0.137)	0.789 (0.139)	0.806 (0.126)
		5,5,7	0.727 (0.162)	0.802 (0.164)	0.819 (0.152)
0.5,0.5,1.5	1,1,1	3,3,5	0.500 (0.208)	0.440 (0.259)	0.477 (0.253)
		5,5,5	0.436 (0.228)	0.511 (0.260)	0.545 (0.250)
		5,5,7	0.405 (0.242)	0.419 (0.272)	0.451 (0.273)
	1,1,5	3,3,5	0.479 (0.210)	0.420 (0.254)	0.454 (0.249)
		5,5,5	0.363 (0.225)	0.429 (0.264)	0.464 (0.262)
		5,5,7	0.374 (0.249)	0.383 (0.282)	0.414 (0.283)
	0.1,1,10	3,3,5	0.502 (0.206)	0.449 (0.254)	0.485 (0.250)
		5,5,5	0.398 (0.228)	0.467 (0.264)	0.501 (0.260)
		5,5,7	0.406 (0.249)	0.421 (0.284)	0.453 (0.284)
1,1,1	1,1,1	3,3,5	0.649 (0.155)	0.678 (0.182)	0.697 (0.167)
		5,5,5	0.695 (0.140)	0.780 (0.148)	0.794 (0.140)
		5,5,7	0.721 (0.183)	0.786 (0.185)	0.801 (0.174)
	1,1,5	3,3,5	0.641 (0.152)	0.668 (0.172)	0.685 (0.163)
		5,5,5	0.678 (0.159)	0.760 (0.176)	0.775 (0.164)
		5,5,7	0.718 (0.173)	0.782 (0.180)	0.796 (0.171)
	0.1,1,10	3,3,5	0.650 (0.150)	0.680 (0.168)	0.698 (0.155)
		5,5,5	0.677 (0.160)	0.762 (0.169)	0.779 (0.158)
		5,5,7	0.694 (0.188)	0.757 (0.200)	0.771 (0.191)
1,1,3	1,1,1	3,3,5	0.516 (0.209)	0.440 (0.251)	0.471 (0.243)
		5,5,5	0.360 (0.222)	0.401 (0.256)	0.434 (0.253)
		5,5,7	0.372 (0.250)	0.359 (0.274)	0.383 (0.276)
	1,1,5	3,3,5	0.521 (0.186)	0.443 (0.228)	0.476 (0.221)
		5,5,5	0.376 (0.231)	0.416 (0.265)	0.448 (0.259)
		5,5,7	0.367 (0.251)	0.357 (0.278)	0.384 (0.279)
	0.1,1,10	3,3,5	0.523 (0.210)	0.449 (0.252)	0.476 (0.247)
		5,5,5	0.394 (0.231)	0.437 (0.266)	0.467 (0.262)
		5,5,7	0.391 (0.254)	0.384 (0.281)	0.411 (0.284)
3,3,3	1,1,1	3,3,5	0.662 (0.129)	0.622 (0.168)	0.645 (0.163)
		5,5,5	0.677 (0.162)	0.690 (0.191)	0.707 (0.185)
		5,5,7	0.728 (0.151)	0.730 (0.173)	0.748 (0.162)
	1,1,5	3,3,5	0.666 (0.112)	0.619 (0.146)	0.640 (0.136)
		5,5,5	0.677 (0.171)	0.685 (0.204)	0.704 (0.199)
		5,5,7	0.717 (0.168)	0.719 (0.193)	0.735 (0.188)
	0.1,1,10	3,3,5	0.659 (0.126)	0.611 (0.167)	0.636 (0.152)
		5,5,5	0.684 (0.163)	0.700 (0.192)	0.719 (0.184)
		5,5,7	0.716 (0.177)	0.716 (0.201)	0.733 (0.195)
0.5,1,3	1,1,1	3,3,5	0.352 (0.218)	0.260 (0.230)	0.291 (0.232)
		5,5,5	0.198 (0.202)	0.221 (0.237)	0.250 (0.245)
		5,5,7	0.162 (0.200)	0.150 (0.210)	0.170 (0.222)
	1,1,5	3,3,5	0.372 (0.229)	0.286 (0.251)	0.318 (0.252)
		5,5,5	0.203 (0.196)	0.230 (0.233)	0.261 (0.244)
		5,5,7	0.206 (0.208)	0.191 (0.219)	0.213 (0.229)
	0.1,1,10	3,3,5	0.345 (0.215)	0.253 (0.228)	0.289 (0.233)
		5,5,5	0.191 (0.193)	0.211 (0.226)	0.239 (0.234)
		5,5,7	0.172 (0.194)	0.157 (0.205)	0.179 (0.218)

Table 2: The Averages and the Standard Deviations in Parentheses of Posterior Probabilities

$\beta_1,\cdots,\beta_k$	$\alpha_1,\cdots,\alpha_k$	$n_1,\cdots,n_k$	$P^F(M_1 \mid \boldsymbol{x})$	$P^{AI}(M_1 \mid \boldsymbol{x})$	$P^{MI}(M_1 \mid \boldsymbol{x})$
0.5,0.5,0.5, 0.5,0.5	1,1,1,1,1	3,3,3,3,3 3,3,3,5,5	0.617 (0.177) 0.751 (0.182)	0.739 (0.204) 0.813 (0.176)	0.793 (0.188) 0.859 (0.143)
	1,1,1,5,5	3,3,3,3,3 3,3,3,5,5	0.597 (0.177) 0.765 (0.187)	0.710 (0.213) 0.827 (0.179)	0.774 (0.189) 0.872 (0.146)
	0.1,0.5, 1,5,10	3,3,3,3,3 3,3,3,5,5	0.623 (0.162) 0.736 (0.184)	0.740 (0.185) 0.808 (0.171)	0.800 (0.154) 0.855 (0.141)
0.5,0.5, 0.5,1.5,1.5	1,1,1,1,1	3,3,3,3,3 3,3,3,5,5	0.430 (0.222) 0.485 (0.270)	0.500 (0.287) 0.420 (0.307)	0.590 (0.262) 0.509 (0.307)
	1,1,1,5,5	3,3,3,3,3 3,3,3,5,5	0.422 (0.197) 0.494 (0.267)	0.491 (0.259) 0.428 (0.306)	0.580 (0.241) 0.509 (0.308)
	0.1,0.5, 1,5,10	3,3,3,3,3 3,3,3,5,5	0.463 (0.186) 0.508 (0.270)	0.543 (0.237) 0.441 (0.313)	0.640 (0.205) 0.526 (0.310)
1,1, 1,1,1	1,1,1,1,1	3,3,3,3,3 3,3,3,5,5	0.625 (0.161) 0.736 (0.184)	0.725 (0.189) 0.787 (0.182)	0.776 (0.168) 0.835 (0.148)
	1,1,1,5,5	3,3,3,3,3 3,3,3,5,5	0.626 (0.157) 0.734 (0.199)	0.725 (0.194) 0.787 (0.208)	0.781 (0.163) 0.833 (0.170)
	0.1,0.5, 1,5,10	3,3,3,3,3 3,3,3,5,5	0.619 (0.149) 0.750 (0.190)	0.716 (0.186) 0.806 (0.193)	0.772 (0.151) 0.845 (0.173)
1,1, 1,3,3	1,1,1,1,1	3,3,3,3,3 3,3,3,5,5	0.438 (0.184) 0.521 (0.257)	0.466 (0.245) 0.416 (0.291)	0.556 (0.227) 0.485 (0.287)
	1,1,1,5,5	3,3,3,3,3 3,3,3,5,5	0.468 (0.194) 0.515 (0.260)	0.506 (0.249) 0.423 (0.299)	0.582 (0.231) 0.497 (0.296)
	0.1,0.5, 1,5,10	3,3,3,3,3 3,3,3,5,5	0.448 (0.193) 0.499 (0.269)	0.480 (0.251) 0.399 (0.297)	0.559 (0.229) 0.472 (0.297)
3,3, 3,3,3	1,1,1,1,1	3,3,3,3,3 3,3,3,5,5	0.668 (0.107) 0.773 (0.152)	0.671 (0.172) 0.721 (0.210)	0.731 (0.132) 0.764 (0.189)
	1,1,1,5,5	3,3,3,3,3 3,3,3,5,5	0.665 (0.125) 0.795 (0.128)	0.667 (0.175) 0.752 (0.182)	0.715 (0.156) 0.792 (0.156)
	0.1,0.5, 1,5,10	3,3,3,3,3 3,3,3,5,5	0.671 (0.122) 0.753 (0.169)	0.670 (0.182) 0.696 (0.234)	0.718 (0.155) 0.749 (0.206)
0.3,0.5, 1,3,5	1,1,1,1,1	3,3,3,3,3 3,3,3,5,5	0.156 (0.158) 0.081 (0.151)	0.155 (0.199) 0.040 (0.109)	0.232 (0.225) 0.058 (0.135)
	1,1,1,5,5	3,3,3,3,3 3,3,3,5,5	0.151 (0.147) 0.088 (0.172)	0.147 (0.177) 0.050 (0.140)	0.221 (0.204) 0.069 (0.166)
	0.1,0.5, 1,5,10	3,3,3,3,3 3,3,3,5,5	0.152 (0.149) 0.071 (0.130)	0.146 (0.180) 0.032 (0.083)	0.225 (0.202) 0.052 (0.119)

Table 3: p-values, Bayes Factor Values and Posterior Probabilities

M_1	p-value	B_{21}^F	$P^F(M_1 \mid \boldsymbol{x})$	B_{21}^{AI}	$P^{AI}(M_1 \mid \boldsymbol{x})$	B_{21}^{MI}	$P^{MI}(M_1 \mid \boldsymbol{x})$
$\beta_1 = \beta_3 = \beta_4$	0.568	0.067	0.937	0.038	0.963	0.035	0.966
$\beta_2 = \beta_3 = \beta_5$	0.059	0.272	0.786	0.303	0.768	0.275	0.784

The p-values of likelihood ratio test (Lawless, 2003) and the values of the Bayes factor and the posterior probability of M_1 are given in Table 3. Note that the maximum likelihood estimators of the shape parameters β_i, $i = 1,\cdots,5$, are 1.059, 0.561, 0.771, 0.889, and 1.363, respectively. We assume that the prior probabilities are equal. From the results of Table 3, the likelihood ratio test, the

FBF, the AIBF and the MIBF give fairly same answers. Also posterior probabilities by the MIBF, the AIBF and the FBF give the similar behavior.

In Weibull populations, we developed the objective Bayesian model selection procedure based on the fractional Bayes factor and intrinsic Bayes factor for testing the equality of the shape parameters under the reference prior. From our numerical results, the developed model selection procedures give fairly reasonable answers for all parameter configurations. From our numerical results, we recommend to use the AIBF than the FBF and the MIBF in practical application.

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