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Characterizations of the Cores of Integer Total Domination Games[1)]

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Abstract

In this paper, we consider cooperative games arising from integer total domination problem on graphs. We introduce two games, rigid integer total domination game and its relaxed game, and focus on their cores. We give characterizations of the cores and the relationship between them.

Keywords : Cooperation Game. Core, Total Domination Function,

1. Introduction

In this paper, we investigate cooperative cost games that arise form integer total domination problems on graphs. Domination problems are widely studied in graph theory. In Haynes, Hedetniemi and Slater[4], overviews of literature on domination problems were given.

Given a graph $G=(V,E;\omega)$ with vertex weight function $\omega: V \rightarrow R_+$ and a given positive integer k, a function $g: V \rightarrow \{0,1,2,\cdots,k\}$ is a k-total dominating function of G if for every vertex, $v \in V$, $\sum_{u \in N(v)} g(u) \geqq k$ where $N(v)$ is the open neighborhood of v in graph G. The k-domination problem is to find a k-dominating function g which minimizes the total weight $\sum_{v \in V} g(v)\omega(v)$. When $k=1$, this problem is just the weighted minimum total dominating set problem.

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The integer k total domination problem has many practical background. For example, let $G=(V,E)$ be a graph in which vertices represent cities and edges represent pairs of cities they are neighbors. There is a need to build a kind of service stations in some cities among them such that each city can receive help from at least k service stations neighboring cities. There is also a fixed cost for building a service station in a certain city. The problem is to determine the number of service stations built in each city such that the total building cost is minimum. This problem is equivalent to the problem of finding a minimum weight k-dominating function on graph G.

A natural question that arises from above example is how to distribute the total cost of building service stations among all the participating cities. In this paper, we introduce two closely related cooperative cost games, a rigid integer domination game and a relaxed integer domination game, to model the cost distribution problem, and focus on an important game solution, core, for both game models.

The main technique we use in this work is linear program duality characterization of cores. The combinatorial optimization techniques have offered much for cooperative games. Especially, integer linear programming and the duality theory have proven itself a very powerful tool in the study of cores. Shapley and Shubik [7] formulated a two-sided market as the assignment game, and showed that the core is exactly the set of optimal solutions of a linear programming dual to the optimal assignment problem. This approach is further exploited in the study of linear production game [5,1], partition game [3], packing and covering games [2], recently dominating set games [8]. Velzen [8] introduced three kinds of cooperative games that arise from the weighted minimum dominating set problem on a graph.

This paper is organized as follows. In section 2, we give some notions from cooperative game theory and introduce two cooperative games that model the cost allocation problems arising from integer total domination problems on graphs. In Section 3, we give the characterizations of cores of the two games.

2. Definition of Integer Total Domination Games

In this section, we introduce two cooperative cost games that model the cost allocation problem arising from integer domination problems on graphs. A cooperative game (in characteristic function form) $\Gamma=(V,c)$ consists of a player set $V=\{1,2,\cdots,n\}$ and a characteristic function $c:2^V\rightarrow R$ with $c(\varnothing)=0$. For each coalition $S\subseteq V$, $c(S)$ represents the revenue or cost achieved by the players in S together. The main issue is how to fairly distribute the total revenue or cost $c(V)$ among all the players. We present the definition here only for cost games, with the understanding that symmetric statement also holds for revenue games.

A vector $z = \{z_1, z_2, \cdots, z_n\}$ is called an imputation if and only if $\sum_{i \in V} z_i = c(V)$ and $z_i \leq c(i)$ for each i. The core of a game $\Gamma = (V, c)$ is defined by

$$Core(\Gamma) = \{z \in R^n : z(V) = c(V) \text{and} z(S) \leq c(S), \forall S \subseteq V\},$$

where $z(S) = \sum_{i \in S} z_i$ for $S \subseteq V$. The constraints imposed on $Core(\Gamma)$, which is called group rationality, ensure that no coalition would have an incentive to split from the grand coalition V, and do better on its own.

The study of the core is closely associated with another important concept, the balanced set. The collection $\mathbb{B}$ of subsets of N is balanced if there exists a set of positive numbers $\beta_S (S \in \mathbb{B})$, such that for each $i \in V$, we have $\sum_{i \in S \in \mathbb{B}} B_S = 1$. A game (V, c) is called balanced if $\sum_{S \in \mathbb{B}} \beta_S c(S) \leq c(V)$ holds for every balanced collection $\mathbb{B}$ with weights $\{\beta_S : S \in \mathbb{B}\}$. With techniques essentially the same as linear programming duality, Shapley [6] proved that a game has non-empty core if and only if it is balanced.

A game $\Gamma = (V, c)$ is called a monotonic game if it satisfies $c(S) \leqq c(T)$ for every $S \subseteq T \subseteq V$. Let $\Gamma = (V, c)$ be a balanced monotonic game and $z \in Core(\Gamma)$, it holds that $z_i = c(V) - \sum_{j \in V \setminus \{i\}} z_j \geqq c(V) - c(V) \setminus \{i\} \geqq 0$ for every $i \in V$. That is, each core element of a monotonic balanced game is non-negative.

Let $G = (V, E)$ be an undirected graph with vertex set V and edge set E. Two distinct vertices $u, v \in V$ are called adjacent if $(u, v) \in E$. For any non-empty set $V' \subseteq V$, the induced subgraph by V', denoted by $G[V']$, is a subgraph of G whose vertex set is V' and whose edge set is the set of edges having both endpoints in V'. The open neighborhood of vertex $v \in V$ is $N(v) = \{u \in V : (u, v) \in E\}$. For any subset $S \subseteq V$, we define the closed neighboring set of S to be the union of the open neighborhoods of all vertices in S, denoted by $N(S) = \cup_{v \in S} N(v)$.

Given a graph $G=(V,E;\omega)$ with vertex weight function $\omega: V \rightarrow R_{+}$ and a given positive integer k, a function $g: V \rightarrow \{0,1,2,\cdots,k\}$ is a k- total dominating function of G if for every vertex $v \in V$, $\sum_{u \in N(v)} g(u) \geq k$. Thus, if S is a dominating set of graph G and we define the function g where $g(v)=1$ if $v \in S$ and $g(v)=0$ if $v \notin S$, then g is a 1-dominating function of G. The k -total domination problem is to find a k-total dominating function g which minimizes the total weight $\sum_{v \in V} g(v)\omega(v)$.

A function $f: V \rightarrow \{0,1,2,\cdots,k\}$ is said to k- totally dominate a set $S \subseteq V$, if for each vertex $v \in S$, $\sum_{u \in N(v)} f(u) \geq k$. In the rest of this paper, for convenience, we denote $\sum_{u \in S} f(u)$ by $f(S)$ and f_i, respectively.

Given a graph $G=(V,E;\omega)$ with vertex weight function $\omega: V \rightarrow R_{+}$, the rigid integer total domination game(rigid ITD game) $\Gamma=(V,c)$ corresponding to G is defined as follows:

1. The player set is $V=\{1,2,\cdots,n\}$;
2. For each coalition $S \subseteq V$

$$c(S)=min\{\sum_{i \in S} g_i\omega_i \mid g: S \rightarrow \{0,1,2,\cdots,k\} \text{and} \forall j \in S, \sum_{i \in N(j) \cap S} g_i \geq k$$

That is, the cost $c(S)$ is the minimum weight of k- total dominating function in the induced graph $G[S]$. In this game model, each coalition can not use the cities not belonging to itself. However, in some situations it makes sense that coalitions are allowed to build service stations in any cities, not restricted to their own cities, as long as the coalition members can receive help from k service stations in its own or within r-neighboring cities. For this reason, we define another related game. The relaxed integer total domination game(relaxed ITD game) $\tilde{\Gamma}=(N,\tilde{c})$ corresponding to G is defined as follows:

1. The player set is $V=\{1,2,\cdots,n\}$;
2. For each coalition $S \subseteq V$,

$$\tilde{c}(S) = min\{\sum_{i\in V} f_i\omega_i \mid f : V \rightarrow \{0,1,2,\cdots,k\} \text{ and } \forall j \in S, \sum_{i\in N(j)} f_i \geq k\}$$

That is, the value $\tilde{c}(S)$ is the minimum weight of a function which k-totally dominates the set S. Obviously, this game is monotonic, i.e., for any subset S and T with $S \subseteq T$, $\tilde{c}(S) \leq \tilde{c}(T)$.

Since coalitions have more possibilities of using the cities in the relaxed ITD game than in the rigid ITD game, it holds that $c(S) \geq \tilde{c}(S)$ for all $S \subset V$. The grand coalition V has the same possibilities in both games, $c(V) = \tilde{c}(V)$.

3. Characterization of the Cores

In this section, we present characterizations of the cores for the rigid and relaxed ITD games. For convenience, we introduce a kind of vertex subset, called basic T-set, which plays an important role in the description of the core elements for the both ITD games.

Definition Let $G=(V,E)$ be a graph. A subset $B \subset V$ is called a basic T-set of G if it satisfies one of the following conditions:

(1) $G[B] = K_3$ (the complete graph with 3 vertices);

(2) $|B| \geq 2$, there exists a vertex $v \in B$ such that $G[B]$ is a v-star, i.e, $B \subseteq \{v\} \cup N(v)$ and any two vertices in $B \setminus \{v\}$ are not adjacent ($G[B] = K_2$ is included in this case).

The set of all basic T-sets of G is denoted by $\mathfrak{I}$.

Let T be a k-total dominating set of graph $G=(V,E)$. It is easy to see that T can be partitioned into several basic T-sets $B_1, B_2, \cdots, B_t \in \mathbb{B}$, (i.e. $B_i \cap B_j = \emptyset$ and $\cup_{i=1}^{t} B_i = T$), and correspondingly, the vertex set V can be partitioned into t disjoint subset $V_1, V_2, \cdots, V_t$ such that $B_i \subseteq V_i \subseteq N(B_i)$ ($i = 1,2,\cdots,t$) and $\cup_{i=1}^{t} V_i = V$. Now, we provide a characterization of the core elements of the relaxed ITD game and the rigid ITD game.

Let $\Gamma=(V,c)$ and $\tilde{\Gamma}=(V,\tilde{c})$ be the corresponding rigid ITD game and relaxed ITD game, respectively. Now we provide efficient core descriptions of the cores of both k-domination games in terms of coalitions corresponding to basic T

-set.

Theorem 3.1 Let $G=(V,E;\omega)$ be a graph with vertex weight function $\omega : V \rightarrow R_{+}$, and $\tilde{\Gamma}=(V,\tilde{c})$ be the corresponding relaxed ITD game. It holds that $z \in Core(\tilde{\Gamma})$ if and only if

(1) $z \geq 0$, and $z(V)=\tilde{c}(V)$;
(2) for each basic $T-set$ $B \in \mathfrak{I}$,, $z(N(B)) \leq kw(B)$.

Proof. Suppose that $z \in Core(\tilde{\Gamma})$. Then (1) holds because $\tilde{\Gamma}=(V,\tilde{c})$ is a monotonic game. For each basic T-set $B \in \mathfrak{I}$, the function $f : N(B) \rightarrow \{0,1,2,\cdots,k\}$ such that $f_j = k$ if $j \in B$ and $f_j = 0$ if $j \notin B$. Clearly, this function k-totally dominates the set $N(B)$, which implies that $N(B)$ is a coalition with cost at most $kw(B)$. That is, $z(N(B) \leq \tilde{c}(N(B)) \leq kw(B)$.

To prove its sufficiency, we show that $z(S) \leq \tilde{c}(S)$ holds for all $S \subseteq V$. Let $S \subseteq V$ be an arbitrary subset and $f^* : V \rightarrow \{0,1,2,\cdots,k\}$ be an optimal weighted function which k-totally dominates the subset$S$. That is, $\tilde{c}(S) = \sum_{j \in S} f_j^* w_j$.

Then we have $z(S) \leq \frac{1}{k} \sum_{B \in \mathbb{B}} f_B^* z(N(B)) \leq \sum_{B \in \mathbb{B}} f_B^* w(B) \leq \tilde{c}(S)$,

where the first two inequalities hold because $z \geq 0$ and f^* k-total dominates the subset S, the second inequality holds because of our assumption (2). Hence $z \in Core(\tilde{\Gamma})$.

In the next theorem we provide a description of the core of rigid ITD game which is similar to that of relaxed ITD game given above.

Theorem 3.2 Let $G=(V,E;\omega)$ be a graph with vertex weight function $\omega : V \rightarrow R_{+}$, and $\Gamma=(V,c)$ be the corresponding rigid ITD game. It holds that $z \in Core(\Gamma)$ if and only if

(1) $z(V)=c(V)$;
(2) for each basic T-set $B \in \mathfrak{I}$ and each subset $S : B \subseteq S \subseteq N(B), z(S) \leq kw(B)$.

Proof. Suppose that $z \in Core(\Gamma)$. Then we have $z(V)=c(V)$. For each

$B \in \mathfrak{I}$, and each subset $S: B \subseteq S \subseteq N(B)$, the function $g: S \rightarrow \{0,1,2,\cdots,k\}$ such that $g_j = k$ if $j \in B \cap S$ and $g_j = 0$ if $j \notin B \cap S$ is a k-total dominating function in the induced graph $G[S]$, it implies that S is a coalition with cost at most $kw(S)$. Hence $z(S) \leq kw(B)$.

Now we prove its sufficiency. Let $S \subseteq V$ be an arbitrary coalition and $g^*: S \rightarrow \{0,1,2,\cdots,k\}$ be a minimum weight k-total dominating function in the induced graph $G[S]$, that is, $\sum_{j \in S} g_j^* w_j = c(S)$. Since there exists a multiset Ξ of basic T-sets (may be repeated) such that every basic T-set is contained in exactly k v-stars in the collection Ξ, for each basic T-set $B \in \mathbb{B}$ and each subset S, $z(S) = \frac{1}{k} \sum_{B \in \Xi} g^*_{B \cap S}\, z(S) \leq \sum_{B \in \Xi} g^*_{B \cap S}\, w(B) \leq c(S)$, where the inequality holds followed from our assumption (2).

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