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Bayesian Hypothesis Testing for the Ratio of Two Quantiles in Exponential Distributions

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Abstract

When X and Y have independent exponential distributions, we develop a Bayesian testing procedure for the ratio of two quantiles under reference prior. The noninformative prior such as reference prior is usually improper which yields a calibration problem that makes the Bayes factor to be defined up to a multiplicative constant. So we develop a Bayesian testing procedure based on fractional Bayes factor and intrinsic Bayes factor. We show that the posterior density under the reference prior is proper and propose the Bayesian testing procedure for the ratio of two quantiles using fractional Bayes factor and intrinsic Bayes factor. Simulation study and a real data example are provided.

Keywords : Fractional Bayes Factor, Intrinsic Bayes Factor, Ratio of Quantiles, Reference Prior.

1. Introduction

In Bayesian testing problem, the Bayes factor under proper priors or informative priors have been very successful. However, limited information and time constraints often require the use of noninformative priors. Since noninformative priors such as Jeffreys' prior or reference prior (Berger and Bernardo, 1989, 1992) are typically improper so that such priors are only defined up to arbitrary

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constants which affects the values of Bayes factors. Spiegalhalter and Smith (1982), O'Hagan (1995) and Berger and Pericchi (1996) have made efforts to compensate for that arbitrariness.

Spiegalhalter and Smith (1982) used the device of imaginary training samples in the context of linear model comparisons to choose the arbitrary constants. But the choice of imaginary training sample depends on the models under comparison, and so, there is no guarantee that the Bayes factor of Spiegalhalter and Smith (1982) is coherent for multiple model comparisons. Berger and Pericchi (1996) introduced the intrinsic Bayes factor using a data-splitting idea, which would eliminate the arbitrariness of improper priors. O'Hagan (1995) proposed the fractional Bayes factor. For removing the arbitrariness he used to a portion of the likelihood with a so-called the fraction b. These approaches have shown to be quite useful in many statistical areas (Kang, Kim and Lee, 2005, 2006).

Comparison between two populations is an important problem in statistics and is commonly used in practice. The populations are usually compared with respect to their means to establish superiority of one population over the other or to check if the two populations are equivalent. For example, two drug may be compared with respect to their mean effects to determine the better one. Even though, comparing two populations with respect to means is a common problem, there are situations where one needs to compare the quantiles instead of their means (see Albers and Löhnberg, 1984; Huang and Johnson, 2006).

For comparison of two quantiles, Albers and Löhnberg (1984) presented a bio-medical problem where comparison between the p quantiles of two populations arises. They provided an approximate distribution free confidence interval for the difference of two quantiles. Bristol (1990) suggested a modification to Albers and Löhnberg's method. Guo and Krishnamoorthy (2005) proposed methods for interval estimating and testing the difference between the quantiles of two normal populations and two exponential popuations. Their methods are based on the concepts of generalized p-value and generalized limit. On the other hand, Huang and Johnson (2006) derived confidence regions for the ratio of quantiles from two normal populations. They developed an exact confidence procedure when the ratio of variances is known. And when the ratio of variances is unknown, they obtained confidence intervals for the ratio of quantiles based on large sample methods.

However there is a little work in this problem from the viewpoint of objective Bayesian framework. The present paper focuses on Bayesian testing for the ratio of two quantiles in the exponential distributions. For dealing this problem, we use the fractional Bayes factor (O'Hagan, 1995) and the intrinsic Bayes factor (Berger and Pericchi, 1996).

The exponential distribution plays an important role in the field of reliability. The reasons for using the exponential distribution assumption in reliability applications can be found in the early work of Davis (1952), Epstein and Sobel (1953), and others. Further justification, in the form of theoretical arguments to support the use of the exponential distribution as the failure law of complex equipment, is presented in the book by Barlow and Proschan (1975) and Lawless (2003).

The outline of the remaining sections is as follows. In Section 2, we introduce the Bayesian hypothesis testing based on the Bayes factor. In Section 3, Using the reference prior, we provide the Bayesian testing procedure based on the fractional Bayes factor and intrinsic Bayes factor for testing the ratio of two quantiles. In Section 4, simulation study and a real example are given.

2. Intrinsic and Fractional Bayes Factors

Hypotheses H_1, $H_2,\cdots$, H_q are under consideration with the data $\boldsymbol{x}=(x_1,x_2,\cdots,x_n)$ having probability density function $f_i(\boldsymbol{x} \mid \boldsymbol{\theta}_i)$ under model $H_i, i=1,2,\cdots,q$, where the parameter vectors θ_i are unknown. Let $\pi_i(\theta_i)$ be the prior distribution of model H_i, and let p_i be the prior probabilities of model $H_i, i=1,2,\cdots,q$. Then the posterior probability that the model H_i is true is

$$P(H_i \mid \boldsymbol{x}) = \left(\sum_{j=1}^{q} \frac{p_j}{p_i} \cdot B_{ji} \right)^{-1}, \tag{1}$$

where B_{ji} is the Bayes factor of model H_j to model H_i defined by

$$B_{ji} = \frac{\int f_j(\boldsymbol{x} \mid \boldsymbol{\theta}_j)\pi_j(\boldsymbol{\theta}_j)d\boldsymbol{\theta}_j}{\int f_i(\boldsymbol{x} \mid \boldsymbol{\theta}_i)\pi_i(\boldsymbol{\theta}_i)d\boldsymbol{\theta}_i} = \frac{m_j(\boldsymbol{x})}{m_i(\boldsymbol{x})}. \tag{2}$$

The B_{ji} interpreted as the comparative support of the data for the model j to i. The computation of B_{ji} needs specification of the prior distribution $\pi_i(\theta_i)$ and $\pi_j(\theta_j)$. Usually, one can use the noninformative prior such as uniform prior, Jeffreys prior or reference prior in Bayesian analysis. Denote it as π_i^N. The use of noninformative priors $\pi_i^N(\cdot)$ in (2) causes the B_{ji} to contain unspecified constants. To solve this problem, Berger and Pericchi (1996) proposed the intrinsic Bayes factor and O'Hagan (1995) proposed the fractional Bayes factor.

One solution to this indeterminacy problem is to use part of the data as a training sample. Let $\boldsymbol{x}(l)$ denote the part of the data to be so used and let $\boldsymbol{x}(-l)$ be the remainder of the data, such that

$$0 < m_i^N(\boldsymbol{x}(l)) < \infty, i=1,\cdots,q. \tag{3}$$

In view (3), the posteriors $\pi_i^N(\theta_i \mid x(l))$ are well defined. Now, consider the Bayes factor, $B_{ji}(l)$, for the rest of the data $x(-l)$, using $\pi_i^N(\theta_i \mid x(l))$ as the priors:

$$B_{ji}(l) = \frac{\int_{\Theta_j} f(x(-l) \mid \theta_j, x(l)) \pi_j^N(\theta_j \mid x(l)) d\theta_j}{\int_{\Theta_i} f(x(-l) \mid \theta_i, x(l)) \pi_i^N(\theta_i \mid x(l)) d\theta_i} = B_{ji}^N \cdot B_{ij}^N(x(l)) \tag{4}$$

where

$$B_{ji} = B_{ji}^N(x) = \frac{m_j^N(x)}{m_i^N(x)} \text{ and } B_{ij}^N(x(l)) = \frac{m_i^N(x(l))}{m_j^N(x(l))}$$

are the Bayes factors that would be obtained for the full data x and training samples $x(l)$, respectively.

Berger and Pericchi (1996) proposed the use of a minimal training sample to compute $B_{ij}^N(x(l))$. Then, an average over all the possible minimal training samples contained in the sample is computed. Thus the Arithmetic Intrinsic Bayes factor (AIBF) of H_j to H_i is

$$B_{ji}^{AI} = B_{ji}^N \cdot \frac{1}{L} \sum_{l=1}^{L} B_{ij}^N(x(l)). \tag{5}$$

where L is the number of all possible minimal training samples. Also the Median Intrinsic Bayes factor (MIBF) by Berger and Pericchi (1998) of H_j to H_i is

$$B_{ji}^{MI} = B_{ji}^N \cdot ME[B_{ij}^N(x(l))], \tag{6}$$

where ME indicates the median, here to be taken over all the training sample Bayes factors. So we can also calculate the posterior probability of H_i using (1), where B_{ji} is replaced by B_{ji}^{AI} and B_{ji}^{MI} from (5) and (6).

The fractional Bayes factor (FBF; O'Hagan, 1995) is based on a similar intuition to that behind the intrinsic Bayes factor but, instead of using part of the data to turn noninformative priors into proper priors, it uses a fraction, b, of each likelihood function, $L(\theta_i) = f_i(x \mid \theta_i)$, with the remaining $1-b$ fraction of the likelihood used for model discrimination. Then the fractional Bayes factor of model H_j versus H_i is

$$B_{ji}^F = B_{ji}^N \cdot \frac{\int L^b(\theta_i)\pi_i^N(\theta_i)d\theta_i}{\int L^b(\theta_j)\pi_j^N(\theta_j)d\theta_j} = B_{ji}^N \cdot \frac{m_i^b(\boldsymbol{x})}{m_j^b(\boldsymbol{x})},$$

and $f_i(\boldsymbol{x} \mid \theta_i)$ is the likelihood function and b specifies a fraction of the likelihood which is to be used as a prior density. He proposed three ways for the choice of the fraction b. One common choice of b is $b = m/n$, where m is the size of the minimal training sample, assuming that this number is uniquely defined. (see O'Hagan (1995, 1997), and the discussion by Berger and Mortera in O'Hagan (1995)).

3. Bayesian Testing Procedures

Let X be an exponential distribution with density function

$$f(x \mid \mu) = \frac{1}{\mu}\exp\left\{-\frac{x}{\mu}\right\},\ x > 0,$$

where $\mu > 0$ is the mean parameter. For any given $0 < p < 1$, the pth quantile of X is given by $-\mu\log(1-p)$. Suppose that $X_1,\cdots,X_{n_1}$ denote independent random samples from exponential distribution with mean μ_1, and $Y_1,\cdots,Y_{n_2}$ denote independent random samples from exponential distribution with mean μ_2. Then the joint probability density function is

$$f(\boldsymbol{x},\boldsymbol{y} \mid \mu_1,\mu_2) = \mu_1^{-n_1}\mu_2^{-n_2}\exp\left\{-\sum_{i=1}^{n_1}\frac{x_i}{\mu_1} - \sum_{i=1}^{n_2}\frac{y_i}{\mu_2}\right\}, \tag{7}$$

where $\mu_1 > 0$ and $\mu_2 > 0$. Also the p_1th quantile of X_i can be expressed as $\eta_1 = -\mu_1\log(1-p_1)$ and the p_2th quantile of Y_i can be expressed as $\eta_2 = -\mu_2\log(1-p_2)$. We want to test the hypotheses $H_1 : \eta_1 \le \eta_2$ vs. $H_2 : \eta_1 > \eta_2$. Since η_1 and η_2 are positive, the hypotheses $H_1 : \eta_1 \le \eta_2$ vs. $H_2 : \eta_1 > \eta_2$ is equivalent to

$$H_1 : \frac{\mu_2}{\mu_1} \le c \quad \text{vs.} \quad \mathrm{H}_2 : \frac{\mu_2}{\mu_1} > \mathrm{c} \tag{8}$$

where $c = \log(1-p_1)/\log(1-p_2)$. Our interest is to develop a Bayesian testing procedure based on the FBF, the AIBF and the MIBF for the hypotheses (8).

From the hypotheses (8), μ_2/μ_1 is our parameter of interest. Let

$$\theta_1 = \frac{\mu_2}{\mu_1} \text{ and } \theta_2 = \mu_1^{(n_1+n_2)/2n_2}\mu_2^{(n_1+n_2)/2n_1}.$$

With this orthogonal parametrization, the likelihood function of parameters (θ_1, θ_2) from (7) is given by

$$L(\theta_1,\theta_2) = \theta_2^{\frac{-2n_1n_2}{n_1+n_2}} \exp\left[-\theta_2^{\frac{-2n_1n_2}{(n_1+n_2)^2}}\theta_1^{\frac{n_2}{n_1+n_2}}\sum_{i=1}^{n_1}x_i - \theta_2^{\frac{-2n_1n_2}{(n_1+n_2)^2}}\theta_1^{\frac{-n_1}{n_1+n_2}}\sum_{i=1}^{n_2}y_i\right] \quad (9)$$

From the likelihood (9), the reference prior is given by

$$\pi(\theta_1,\theta_2) \propto \theta_1^{-1}\theta_2^{-1} \quad (10)$$

and this reference prior is the unique second order matching prior (Ghosh and Sun, 1997). From the likelihood function (9) and the reference prior (10), the posterior distribution of θ_1 and θ_2 is given by

$$\pi(\theta_1,\theta_2 \mid \boldsymbol{x},\boldsymbol{y}) \propto \theta^{-1}\theta_2^{\frac{-2n_1n_2}{n_1+n_2}-1} \exp\left[-\theta_2^{\frac{-2n_1n_2}{(n_1+n_2)^2}}\theta_1^{\frac{n_2}{n_1+n_2}}\sum_{i=1}^{n_1}x_i - \theta_2^{\frac{-2n_1n_2}{(n_1+n_2)^2}}\theta_1^{\frac{-n_1}{n_1+n_2}}\sum_{i=1}^{n_2}y_i\right].$$

Let $\theta_1 = \mu_2/\mu_1$ and $\theta_2 = \mu_1^{(n_1+n_2)/2n_2}\mu_2^{(n_1+n_2)/2n_1}$. Then the posterior distribution of μ_1 and μ_2 is

$$\pi(\mu_1,\mu_2 \mid \boldsymbol{x},\boldsymbol{y}) \propto \mu_1^{-n_1-1}\mu_2^{-n_2-1} \exp\left\{-\sum_{i=1}^{n_1}\frac{x_i}{\mu_1} - \sum_{i=1}^{n_2}\frac{y_i}{\mu_2}\right\}.$$

So this posterior distribution is proper if $n_1 \geq 1$ and $n_2 \geq 1$.

3.1 Bayesian Testing Procedure based on the Fractional Bayes Factor

From the likelihood (9) and the reference prior (10), the element of the FBF under $H_1 : \theta_1 \leq c$ is given by

$$\begin{aligned} m_1^b(\boldsymbol{x},\boldsymbol{y}) &= \int_0^c \int_0^\infty L^b(\theta_1,\theta_2 \mid \boldsymbol{x},\boldsymbol{y})\pi(\theta_1,\theta_2)d\theta_2 d\theta_1 \\ &= \int_0^c \frac{\Gamma[b(n_1+n_2)](n_1+n_2)^2}{2n_1n_2 b^{b(n_1+n_2)}}\theta_1^{bn_1-1}[\theta_1\sum_{i=1}^{n_1}x_i + \sum_{i=1}^{n_2}y_i]^{-b(n_1+n_2)}d\theta_1. \end{aligned}$$

The element of the FBF under $H_2 : \theta_1 > c$ gives as follows.

$$m_2^b(\boldsymbol{x},\boldsymbol{y}) = \int_c^\infty \int_0^\infty L^b(\theta_1,\theta_2 \mid \boldsymbol{x},\boldsymbol{y})\pi(\theta_1,\theta_2)d\theta_2 d\theta_1$$
$$= \int_c^\infty \frac{\Gamma[b(n_1+n_2)](n_1+n_2)^2}{2n_1n_2b^{b(n_1+n_2)}}\theta_1^{bn_1-1}[\theta_1\sum_{i=1}^{n_1}x_i+\sum_{i=1}^{n_2}y_i]^{-b(n_1+n_2)}d\theta_1.$$

Therefore the element B_{21}^N of the FBF is given by

$$B_{21}^N = \frac{S_2(\boldsymbol{x},\boldsymbol{y})}{S_1(\boldsymbol{x},\boldsymbol{y})},$$

where

$$S_1(\boldsymbol{x},\boldsymbol{y}) = \int_0^c \theta_1^{n_1-1}[\theta_1\sum_{i=1}^{n_1}x_i+\sum_{i=1}^{n_2}y_i]^{-(n_1+n_2)}d\theta_1$$

and

$$S_2(\boldsymbol{x},\boldsymbol{y}) = \int_c^\infty \theta_1^{n_1-1}[\theta_1\sum_{i=1}^{n_1}x_i+\sum_{i=1}^{n_2}y_i]^{-(n_1+n_2)}d\theta_1.$$

And the ratio of marginal densities with fraction b is

$$\frac{m_1^b(\boldsymbol{x},\boldsymbol{y})}{m_2^b(\boldsymbol{x},\boldsymbol{y})} = \frac{S_1(\boldsymbol{x},\boldsymbol{y};b)}{S_2(\boldsymbol{x},\boldsymbol{y};b)},$$

where

$$S_1(\boldsymbol{x},\boldsymbol{y};b) = \int_0^c \theta_1^{bn_1-1}[\theta_1\sum_{i=1}^{n_1}x_i+\sum_{i=1}^{n_2}y_i]^{-b(n_1+n_2)}d\theta_1$$

and

$$S_2(\boldsymbol{x},\boldsymbol{y};b) = \int_c^\infty \theta_1^{bn_1-1}[\theta_1\sum_{i=1}^{n_1}x_i+\sum_{i=1}^{n_2}y_i]^{-b(n_1+n_2)}d\theta_1.$$

Thus the FBF of H_2 versus H_1 is given by

$$B_{21}^{F} = \frac{S_2(\boldsymbol{x},\boldsymbol{y})}{S_1(\boldsymbol{x},\boldsymbol{y})} \cdot \frac{S_1(\boldsymbol{x},\boldsymbol{y};b)}{S_2(\boldsymbol{x},\boldsymbol{y};b)} \tag{11}$$

Note that the calculation of the FBF of H_2 versus H_1 requires one dimensional integration.

3.2 Bayesian Testing Procedure based on the Intrinsic Bayes Factor

The element B_{21}^{N} of the AIBF and the MIBF is computed in the derivation of the FBF. So using minimal training sample, we only calculate the marginal densities under H_1 and H_2, respectively. The marginal density of (X_i, Y_j) is finite for all $1 \le i \le n_1, 1 \le j \le n_2$ under each hypothesis. Thus we conclude that any training sample of size two is a minimal training sample.

The marginal density $m_1^N(x_i, y_j)$ under $H_1: \theta_1 \le c$ is given by

$$\begin{aligned} m_1^N(x_i,y_j) &= \int_0^c \int_0^\infty f(x_i,y_j \mid \theta_1,\theta_2)\pi(\theta_1,\theta_2)d\theta_2 d\theta_1 \\ &= 2c[y_j(cx_i+y_j)]^{-1}, \end{aligned}$$

where $1 \le i \le n_1, 1 \le j \le n_2$. And the marginal density $m_2^N(x_i,y_j)$ under $H_2: \theta_1 > c$ is given by

$$\begin{aligned} m_2^N(x_i,y_j) &= \int_c^\infty \int_0^\infty f(x_i,y_j \mid \theta_1,\theta_2)\pi(\theta_1,\theta_2)d\theta_2 d\theta_1 \\ &= 2x_i^{-1}(cx_i+y_j)^{-1}, \end{aligned}$$

where $1 \le i \le n_1, 1 \le j \le n_2$. Therefore the AIBF of H_2 versus H_1 is given by

$$B_{21}^{AI} = \frac{S_2(\boldsymbol{x},\boldsymbol{y})}{S_1(\boldsymbol{x},\boldsymbol{y})} \cdot \left[\frac{1}{L} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \frac{cx_i}{y_j} \right], \tag{12}$$

where $L = n_1 n_2$. And the MIBF of H_2 versus H_1 is given by

$$B_{21}^{MI} = \frac{S_2(\boldsymbol{x},\boldsymbol{y})}{S_1(\boldsymbol{x},\boldsymbol{y})} \cdot ME\left[\frac{cx_i}{y_j} \right]. \tag{13}$$

Note that the calculations of the AIBF and the MIBF of H_2 versus H_1 require one dimensional integration.

4. Numerical Studies

In order to assess the Bayesian testing procedures, we evaluate the posterior probability for several configurations (p_1, p_2), (μ_1, μ_2) and (n_1, n_2). In particular, for fixed (μ_1, μ_2) we take 5,000 independent random samples of X and Y from the model (7). In our simulation, we examine the cases when $p_1 = 0.5$, $p_2 = 0.1, 0.3$ $0.5, 0.7, 0.9$, $(\mu_1, \mu_2) = (5,5), (5,10), (10,5)$ and $(n_1, n_2) = (5,5), (5,10), (10,10), (10,15)$.

The posterior probabilities of H_1 being true are computed assuming equal prior probabilities. Table 1 shows the results of the averages and the standard deviations in parentheses of posterior probabilities. From the Table 1, the FBF and the MIBF give fairly reasonable answers for all configurations, (p_1, p_2) and (μ_1, μ_2). Also the FBF and the MIBF give a similar behavior for all sample sizes. However the AIBF is tending to favor the hypothesis H_2. This is because the ratio of the marginal density value is unstable when the value of minimal training sample is extremely small value in denominator and large value in nominator or vice versa.

Also for the unequal sample sizes, the FBF favors the hypothesis H_2 than the MIBF from the case $(p_1, p_2) = (0.5, 0.5)$ and $(\mu_1, \mu_2) = (5,5)$. Thus the MIBF gives fairly reasonable results.

Table 1: The averages and the standard deviations in parentheses of posterior probabilities

(μ_1,μ_2)	(p_1,p_2)	(n_1,n_2)	$P^F(H_1 \mid \boldsymbol{x},\boldsymbol{y})$	$P^{AI}(H_1 \mid \boldsymbol{x},\boldsymbol{y})$	$P^{MI}(H_1 \mid \boldsymbol{x},\boldsymbol{y})$
5, 5	0.5, 0.1	5, 5	0.948 (0.079)	0.864 (0.186)	0.945 (0.085)
		5, 10	0.975 (0.063)	0.936 (0.142)	0.979 (0.057)
		10, 10	0.995 (0.018)	0.978 (0.073)	0.995 (0.019)
		10, 15	0.998 (0.015)	0.990 (0.057)	0.998 (0.014)
	0.5, 0.3	5, 5	0.713 (0.193)	0.513 (0.267)	0.710 (0.204)
		5, 10	0.723 (0.220)	0.542 (0.291)	0.755 (0.210)
		10, 10	0.811 (0.182)	0.607 (0.285)	0.810 (0.184)
		10, 15	0.823 (0.187)	0.642 (0.288)	0.838 (0.178)
	0.5, 0.5	5, 5	0.504 (0.220)	0.312 (0.225)	0.505 (0.231)
		5, 10	0.458 (0.229)	0.284 (0.223)	0.509 (0.233)
		10, 10	0.504 (0.243)	0.278 (0.231)	0.504 (0.247)
		10, 15	0.476 (0.249)	0.266 (0.231)	0.504 (0.252)
	0.5, 0.7	5, 5	0.322 (0.201)	0.178 (0.165)	0.324 (0.210)
		5, 10	0.249 (0.184)	0.135 (0.138)	0.295 (0.202)
		10, 10	0.232 (0.196)	0.103 (0.129)	0.234 (0.201)
		10, 15	0.187 (0.175)	0.081 (0.111)	0.209 (0.189)
	0.5, 0.9	5, 5	0.155 (0.151)	0.079 (0.104)	0.159 (0.161)
		5, 10	0.091 (0.102)	0.046 (0.064)	0.121 (0.125)
		10, 10	0.053 (0.089)	0.020 (0.045)	0.054 (0.092)
		10, 15	0.032 (0.060)	0.012 (0.028)	0.038 (0.069)
5, 10	0.5, 0.1	5, 5	0.842 (0.146)	0.681 (0.255)	0.839 (0.155)
		5, 10	0.881 (0.154)	0.760 (0.260)	0.896 (0.143)
		10, 10	0.946 (0.091)	0.844 (0.211)	0.945 (0.093)
		10, 15	0.961 (0.081)	0.886 (0.189)	0.965 (0.075)
	0.5, 0.3	5, 5	0.492 (0.217)	0.300 (0.219)	0.492 (0.229)
		5, 10	0.439 (0.234)	0.266 (0.219)	0.488 (0.239)
		10, 10	0.481 (0.243)	0.260 (0.224)	0.482 (0.247)
		10, 15	0.458 (0.247)	0.251 (0.221)	0.487 (0.249)
	0.5, 0.5	5, 5	0.277 (0.190)	0.149 (0.146)	0.281 (0.200)
		5, 10	0.204 (0.167)	0.108 (0.121)	0.248 (0.188)
		10, 10	0.179 (0.178)	0.076 (0.110)	0.181 (0.182)
		10, 15	0.132 (0.145)	0.054 (0.081)	0.149 (0.159)
	0.5, 0.7	5, 5	0.147 (0.143)	0.074 (0.097)	0.151 (0.152)
		5, 10	0.084 (0.094)	0.042 (0.059)	0.112 (0.119)
		10, 10	0.048 (0.085)	0.018 (0.043)	0.048 (0.088)
		10, 15	0.026 (0.051)	0.010 (0.026)	0.031 (0.059)
	0.5, 0.9	5, 5	0.052 (0.082)	0.025 (0.049)	0.055 (0.089)
		5, 10	0.020 (0.036)	0.010 (0.021)	0.030 (0.051)
		10, 10	0.005 (0.018)	0.002 (0.008)	0.005 (0.019)
		10, 15	0.002 (0.006)	0.001 (0.002)	0.002 (0.008)

Example. The following data, given by Proschan (1963), are time intervals of successive failures of the air conditioning equipment in Boeing 720 aircraft. For aircraft 1, the Kolmogorov-Smirnov test statistic is 0.1143 and its p-value is 0.88. For aircraft 2, the Kolmogorov-Smirnov test statistic is 0.1791 and its p-value is 0.62. So we can assume that the time between successive failures for each plane is exponentially distributed. The sample means for aircraft 1 and aircraft 2 are 64.125 and 82.562, respectively.

Aircraft 1 : 50 44 102 72 22 39 3 15 197 188 79 88 46 5 5 36 22 139 210 97 30 23 13 14

Aircraft 2 : 102 209 14 57 54 32 67 59 134 152 27 14 230 66 61 34

Table 1 (Continued)

(μ_1, μ_2)	(p_1, p_2)	(n_1, n_2)	$P^F(H_1 \mid \boldsymbol{x}, \boldsymbol{y})$	$P^{AI}(H_1 \mid \boldsymbol{x}, \boldsymbol{y})$	$P^{MI}(H_1 \mid \boldsymbol{x}, \boldsymbol{y})$
10, 5	0.5, 0.1	5, 5	0.987 (0.033)	0.956 (0.107)	0.986 (0.038)
		5, 10	0.997 (0.019)	0.991 (0.053)	0.998 (0.015)
		10, 10	1.000 (0.003)	0.999 (0.012)	1.000 (0.003)
		10, 15	1.000 (0.000)	1.000 (0.003)	1.000 (0.000)
	0.5, 0.3	5, 5	0.875 (0.132)	0.729 (0.250)	0.870 (0.143)
		5, 10	0.914 (0.130)	0.814 (0.237)	0.927 (0.119)
		10, 10	0.968 (0.063)	0.894 (0.176)	0.966 (0.067)
		10, 15	0.979 (0.054)	0.932 (0.147)	0.981 (0.051)
	0.5, 0.5	5, 5	0.719 (0.194)	0.525 (0.265)	0.715 (0.203)
		5, 10	0.731 (0.217)	0.551 (0.289)	0.764 (0.205)
		10, 10	0.823 (0.173)	0.623 (0.281)	0.821 (0.177)
		10, 15	0.837 (0.176)	0.660 (0.284)	0.851 (0.168)
	0.5, 0.7	5, 5	0.547 (0.215)	0.347 (0.235)	0.545 (0.227)
		5, 10	0.514 (0.237)	0.331 (0.242)	0.560 (0.237)
		10, 10	0.580 (0.240)	0.343 (0.254)	0.579 (0.244)
		10, 15	0.551 (0.248)	0.326 (0.251)	0.578 (0.248)
	0.5, 0.9	5, 5	0.339 (0.205)	0.188 (0.173)	0.341 (0.214)
		5, 10	0.263 (0.188)	0.145 (0.145)	0.312 (0.205)
		10, 10	0.245 (0.203)	0.110 (0.135)	0.246 (0.206)
		10, 15	0.204 (0.186)	0.089 (0.118)	0.227 (0.198)

Table 2 : The Bayes Factors and Posterior Probabilities

(p_1, p_2)	(0.5, 0.2)	(0.5, 0.3)	(0.5, 0.416)	(0.5, 0.5)	(0.5, 0.7)
c	3.106	1.943	1.289	1.000	0.576
p-value	0.9950	0.8897	0.4925	0.2103	0.0056
BF_{21}^F	0.0099	0.1576	0.8809	2.4972	66.6921
$P^F(H_1 \mid \boldsymbol{x}, \boldsymbol{y})$	0.9902	0.8639	0.5317	0.2859	0.0148
BF_{21}^{AI}	0.0215	0.3330	1.8348	5.1876	140.1275
$P^{AI}(H_1 \mid \boldsymbol{x}, \boldsymbol{y})$	0.9789	0.7502	0.3528	0.1616	0.0071
BF_{21}^{MI}	0.0107	0.1652	0.9101	2.5731	69.5040
$P^{MI}(H_1 \mid \boldsymbol{x}, \boldsymbol{y})$	0.9894	0.8582	0.5235	0.2799	0.0142

The values of the Bayes factor and the posterior probability of H_2 versus H_1 are given Table 2. We assume that the prior probabilities are equal. Also the p-values based on F-test (see Guo and Krishnamoorthy, 2005) are given Table 2. From the results of Table 2, the FBF and the MIBF give fairly reasonable answers for the values of p_1 and p_2. But the p-value does not give reasonable answers for some cases and favors the hypothesis H_1. Also the AIBF is tending to favor the hypothesis H_2.

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