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The Stable Embeddability on Modules over Complex Simple Lie Algebras

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Abstract

Several partial orders on integral partitions have been studied with many applications such as majorizations, capacities of quantum memory and embeddabilities of matrix algebras. In particular, the embeddability, stable embeddability and strong-stable embeddability problems arise for finite dimensional irreducible modules over a complex simple Lie algebra L. We find a sufficient condition for an L-module strong-stably embeds into another L-module using formal character theory.

Keywords : Embeddability, Irreducible Modules, Stable Embeddability, Strong Stable Embeddability

1. Introduction

One of computational questions in quantum information theory is that when a hybrid quantum memory is as useful as the other. The measurement algebra of a combined memory consisting of an a-state quantum digit and a b-state classical digit is

$$M_a \otimes C^b = \bigoplus_{k=1}^{b} M_a,$$

where M_a is the set of $a \times a$ matrices. In general, the measurement algebra A of a finite memory could be any direct sum of matrix algebras of different dimensions:

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$$A \cong \bigoplus_{k=1}^{n} M_{\lambda_k}.$$

An excellent overview can be found in [Kuperberg, G. (2003)]. The partition $\lambda = \lambda(A) = [\lambda_1, \lambda_2, \cdots, \lambda_n]$ is a list of the dimensions of the matrix algebra of A called the shape of the memory A. The memory capacity comparison problem can be interpolated to the embeddability of an integral partition λ into the other partition μ. A partition λ embeds into μ if there exists a map $\varphi : \{1,2,\cdots,m\} \to \{1,2,\cdots,n\}$ such that

$$\sum_{i \in \varphi^{-1}(j)} \lambda_i \le \mu_j$$

for all j, denoted by $\lambda \hookrightarrow \mu$. This embedding problem is known as a bin-packing problem and the question of whether λ embeds into μ is computable but NP-hard. Furthermore, there is more than one notion by which one memory unit has more capacity than another. For instance, in the presence of an auxiliary memory, one can consider a stable embeddability. A partition λ stably embeds into a partition μ if there exist a partition ν such that $\lambda \times \nu \hookrightarrow \mu \times \nu$, denoted by $\lambda \overset{s}{\hookrightarrow} \mu$ where the multiplication $\lambda \times \mu$ is defined by $[\lambda_i \cdot \mu_j]$. For example, let $\lambda_1 = [2,2,2,2]$, $\lambda_2 = [4,2,2]$, $\mu_1 = [4,1,1,1,1,1,1,1,1]$, $\mu_2 = [5,3]$. Then, λ_1 stably embeds into μ_1 (by setting $\nu = [2,1,1]$) but λ_1 does not supermajorize μ_1 also λ_1 does not embed into μ_1, λ_2 supermajorizes μ_2 but λ_2 does not stably embed into μ_2. These embeddabilities of the matrix algebras have been studied in the relation with the supermajorization order and capacities of quantum memory [Kim, D. and Lee, J. (2007), Kuperberg, G. (2003)].

In the present paper, we consider the embeddability problem for finite dimensional L-modules where L is a complex simple Lie algebra. Due to the Schur's lemma if we know how to decompose the given L-modules into direct sums of irreducible L-modules, it completely determines the embeddability. However, the decomposition problem itself is one of very difficult problems in the representation theory of Lie algebras, for example the honeycomb model by Knutson and Tao [Knutson, A. and Tao, T. (1999), Knutson, A. and Tao, T. (2001), Knutson, A. and Tao, T. and Woodward, C. (2004)]. In particular, the decomposition problem of a tensor product of two irreducible representations is known as the “Clebsch-Gordan Problem”. In fact, the unique decomposition of tensor products into irreducible L-modules was recently proved by Rajan, C. (2004).

If A, B are matrix algebras, then A, B can be represented by integral partitions [Kim, D. and Lee, J. (2007), Kuperberg, G. (2003)]. Therefore, the embeddability problem of can be interpolated as an integral *bin-packing problem* by replacing the shape of A by the size of blocks and the shape of B by the size of bins. This is also closely related with the capacity of quantum memory by assuming all algebras are $*$-algebras. Several partial orders on integral partitions were introduced [Bhatia, R. (1996), Kim, D. and Lee, J. (2007), Kuperberg, G. (2003)]., and they are related with many applications such as majorizations, capacities of quantum memory and embeddabilities of matrix algebras [Csirik, J. and Johnson, D. S. (1991), Nielson, M. and Vidal, G. (2001), Stanley, R. P. (1999)]. Some of them naturally arose in the embeddability problems of L-modules and we will discuss these embeddabilities of matrix algebras. In particular, we find a sufficient condition for the strong-stably embeddability of L-modules in Theorem 1.

The outline of this paper is as follows. We first discuss the definitions of several embeddabilities and study their simple properties in section 2. In section 3, we first review some algebraic background about the weight spaces and tensor product. Then, we discuss stable and strong stable embeddabilities of L-modules.

2. Embeddability, stable embeddability and strong stable embeddability

Let A, B be finite dimensional L-modules. To answer the embeddability problem, we need to decompose A, B into direct sums of irreducible modules,

$$A \cong \bigoplus_{\lambda} a_\lambda V_\lambda, \qquad B \cong \bigoplus_{\mu} b_\lambda V_\lambda,$$

where a_λ, b_λ are the numbers of copies of V_λ, the irreducible module of highest weight λ in the decomposition of A, B respectively. The Schur's lemma states that

$$\dim(Hom_L(V_\lambda, V_\mu)) = \begin{cases} 1 & \text{if} \quad V_\lambda \cong V_\mu \\ 0 & \text{otherwise.} \end{cases}$$

Thus, one can easily see that $A \hookrightarrow B$ if and only if $a_\lambda \le b_\lambda$ for all λ. We says A *stably embeds* into B if there exists an L-module H such that $A \otimes H \hookrightarrow B \otimes H$, denoted by $A \overset{s}{\hookrightarrow} B$. If H is an irreducible L-module, we say A *strong stably embeds* into B, denoted by $A \overset{s.s}{\hookrightarrow} B$. Fundamental implications regarding these three embeddabilities are

$$A \hookrightarrow B \quad \Rightarrow \quad A \overset{s.s}{\hookrightarrow} B \quad \Rightarrow \quad A \overset{s}{\hookrightarrow} B \tag{1}$$

The converses of the above implications are not true in general. We provide counterexamples for the converse of (1) as follows.

Example 2. 1. Let L = **sl**(2,C). Let V_i be an irreducible L-module of highest weight i. Let $A=V_1$, $B=V_3, H=V_1\oplus V_1\oplus V_2$ and $I=V_2\oplus V_2\oplus V_3$. Then

1) A does not embed into B but A strong-stably embed into B.
2) H does not strong-stably embed into I but H stably embeds into I.

Proof. By Schur's lemma, one can see that $A \not\hookrightarrow B$. But if we take $Z=V_2$,

$$A\otimes Z=V_3\oplus V_1 \hookrightarrow V_5\oplus V_3\oplus V_1=B\otimes Z.$$

Let $Z=V_k$. If $k<3$, one can check $H\otimes V_k \not\hookrightarrow I\otimes V_k$. If $k\geq 3$, then

$$H\otimes V_k=V_{k+2}\oplus 2V_{k+1}\oplus V_k\oplus 2V_{k-1}\oplus V_{k-2},$$
$$I\otimes V_k=V_{k+3}\oplus 2V_{k+2}\oplus V_{k+1}\oplus 2V_k\oplus V_{k-1}\oplus 2V_{k-2}\oplus V_{k-3}.$$

Therefore, $H\otimes V_k \not\hookrightarrow I\otimes V_k$. To show $H\overset{s}{\hookrightarrow} I$, we set $Z=V_2\oplus V_3$. Then one can see that

$$H\otimes Z=V_5\oplus 3V_4\oplus 3V_3\oplus 3V_2\oplus 2V_1,$$
$$I\otimes Z=V_6\oplus 3V_5\oplus 3V_4\oplus 3V_3\oplus 3V_2\oplus 3V_1\oplus 3V_0.$$

3. Formal characters and embeddabilities

We will review the weight spaces and the tensor product rules of irreducible modules of complex simple Lie algebras, we refer to [Fulton, W. and Harris, J. (1991), Humphreys, J. E. (1972)] for definitions and notations.

Let L be a complex simple Lie algebra, H a fixed Cartan subalgebra of L, Φ the root system, $\Delta=\{\alpha_1, \alpha_2, \cdots, \alpha_l\}$ a base of Φ, W the Weyl group. Let V be a finite dimensional L-module. Then the Cartan subalgebra H acts diagonally on V, so V can be decomposed into the eigenspaces of H, i. e., $V=\coprod_{\lambda\in H^*} V_\lambda=\{v\in V \mid h(v)=\lambda(h)v\}$. Whenever $V_\lambda\neq 0$, we call it a *weight space* and we call λ a *weight* of V.

Now we recall the formal character theory. Let $\Lambda\subset H^*$ be the lattice of integral linear functions. If $V=V(\lambda)$ where $\lambda\in\Lambda^+$, we like to express a formal sum of the weight $\mu\in\prod(\lambda)$, each μ occurring in the sum $m(\mu)$ times. To present it, we use the *group ring* of Λ over **Z**, denoted by $\mathbf{Z}[\Lambda]$, a free **Z** group with basis element $e(\lambda)$ in one-to-one correspondence with elements λ of Λ with the addition denoted by $e(\lambda)+e(\mu)$. It also has a multiplication by $e(\lambda)e(\mu)=e(\lambda+\mu)$. The Weyl group W acts on $\mathbf{Z}[\Lambda]$ by $\sigma e(\lambda)=e(\sigma\lambda)$. Then we define the *formal character* ch_λ as an element $\sum_{\mu\in\prod(\lambda)} m_\lambda(\mu)e(\mu)$ of $\mathbf{Z}[\Lambda]$. There are wonderful theorems by Weyl, Kostant and Steinberg regarding $m(\lambda)$ and tensor products [Humphreys, J. E. (1972), Section 24]. We will use the following formula due to Brauer, R. (1937). First we define

$$t(\lambda)=\begin{cases}0 & \text{if} \quad \lambda \text{ is fixed by } \sigma\neq 1\in W\\ sign(\sigma) & \text{if} \quad \text{nothing but 1 fixes } \lambda \text{ and } \sigma(\lambda)\in\Lambda^+.\end{cases}$$

Then we find

$$ch_\mu * ch_\nu = \sum_{\lambda\in\prod(\mu)} m_\mu(\lambda)t(\lambda+\nu+\delta)ch_{\{\lambda+\nu+\delta\}-\delta} \tag{2}$$

where the braces denotes the unique dominant weight to which the indicated weight is conjugate. For **sl**(3,C), one can find a beautiful explanation with figures of weight diagrams in [Fulton, W. and Harris, J. (1991), Lecture 13]. For Example 2. 1, we recall the Clebsch-Gordan formula for **sl**(2,C), which can be directly proved by using (2),

$$V_i\otimes V_j \cong V_{i+j}\oplus V_{i+j-2}\oplus\cdots\oplus V_{|i-j|},$$

where V_i is an irreducible representation of **sl**(2,C) of highest weight i.

Now we define a partial order "$<$" on $\mathbf{Z}[\Lambda]$,

$$\alpha=\sum_\lambda m_\alpha(\lambda)e(\lambda) < \beta=\sum_\lambda m_\beta(\lambda)e(\lambda) \quad \text{if} \quad m_\alpha(\lambda)\le m_\beta(\lambda) \quad \text{for all } \lambda.$$

Theorem 1. Let L be a complex simple Lie algebra. Let A, B be finite dimensional L-modules. if $ch_A < ch_B$, then $A \overset{s.s}{\hookrightarrow} B$.

Proof. Suppose $ch_A < ch_B$, i. e., $m_\alpha(\lambda)\le m_\beta(\lambda)$ for all λ. Then we fix a sufficiently large η that $\mu+\eta\in\Lambda^+$ for all $\mu\in\prod(B)$. Since $m_\alpha(\lambda)\le m_\beta(\lambda)$ for all λ, it follows that $\mu+\eta\in\Lambda^+$ for all $\mu\in\prod(A)$. For such an η, it is fairly easy to see that $t(\eta+\nu+\delta)=1$ and $\{\eta+\nu+\delta\}=\eta+\nu+\delta$ for all $\nu\in\prod(B)$ (so does for $\nu\in\prod(A)$). Since

$$ch_A * ch_\eta = \sum_{\lambda\in\prod(A)} m_A(\lambda)t(\lambda+\eta+\delta)ch_{\{\lambda+\eta+\delta\}-\delta} = \sum_{\lambda\in\prod(A)} m_A(\lambda)ch_{\lambda+\eta},$$

and

$$ch_B * ch_\eta = \sum_{\lambda\in\prod(B)} m_B(\lambda)t(\lambda+\eta+\delta)ch_{\{\lambda+\eta+\delta\}-\delta} = \sum_{\lambda\in\prod(B)} m_B(\lambda)ch_{\lambda+\eta},$$

we obtain that $m_{A\otimes V_\eta}(\mu)=m_A(\mu-\eta)\le m_B(\mu-\eta)=m_{B\otimes V_\eta}(\mu)$ for all μ. Therefore, $A\otimes V_\eta \hookrightarrow B\otimes V_\eta$.

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