J. Korean Math. Soc. **44** (2007), No. 5, pp. 1065–1078

GLOBAL SOLUTIONS OF SEMIRELATIVISTIC HARTREE TYPE EQUATIONS

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ABSTRACT. We consider initial value problems for the semirelativistic Hartree type equations with cubic convolution nonlinearity $F(u) = (V * |u|^2)u$. Here V is a sum of two Coulomb type potentials. Under a specified decay condition and a symmetric condition for the potential V we show the global existence and scattering of solutions.

1. Introduction

In this paper we consider the following Cauchy problem:

$$\begin{cases} i\partial_t u = \sqrt{m^2 - \Delta}u + F(u) \text{ in } \mathbb{R}^n \times \mathbb{R},\ n \geq 1, \\ u(x,0) = \varphi(x) \text{ in } \mathbb{R}^n. \end{cases} \tag{1}$$

Here $m > 0$ denotes the mass of Bosons in units $\hbar = c = 1$ and $F(u)$ is nonlinear functional of Hartree type such that $F(u) = (V * |u|^2)u$, where $V = V_1 + V_2$ and $*$ denotes the convolution in $\mathbb{R}^n$. We assume that the potentials V_1 and V_2 are real valued functions with the estimate

$$|V_i(x)| \lesssim |x|^{-\gamma_i}, \tag{2}$$

where $0 < \gamma_i < n, i = 1, 2$. The typical examples of V are the Coulomb potential $V(x) = \lambda|x|^{-1}$ corresponding to the case $\gamma_1 = \gamma_2 = 1$ and the Yukawa potential $V(x) = \lambda\frac{e^{-\mu|x|}}{|x|}$ corresponding to the case $\gamma_1 = 1$ and any $\gamma_2 > 0$, where λ is a real number and μ is a nonnegative real number. For the energy conservation we assume that

$$V(x) = V(-x). \tag{3}$$

The equation (1) is called the semirelativistic Hartree equation, which describes the Boson stars with Coulomb potential. See [4, 5, 7] and the references therein.

The main purpose of this paper is to improve the known results in [1, 8] for the local and global existence theory to the equation (1) with a general class of

Received October 28, 2006.

2000 *Mathematics Subject Classification.* 35Q40, 35Q55, 47J35.

Key words and phrases. semirelativistic Hartree type equations, global existence, scattering, Coulomb type potentials.

potentials as above and the scattering theory of the global solutions. For this purpose we study the Cauchy problem (1) in the form of the integral equation:

$$u(t) = U(t)\varphi - i\int_0^t U(t-t')F(u)(t')dt', \tag{4}$$

where

$$(U(t)\varphi)(x) = (e^{-it\sqrt{m^2-\Delta}}\varphi)(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x\cdot\xi - t\sqrt{m^2+|\xi|^2})}\widehat{\varphi}(\xi)\,d\xi.$$

Here we denote the Fourier transform of φ by $\widehat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\varphi(x)\,dx$.

A solution of the equation (1) enjoys two conservation laws to be used for the global existence in the case $0 < \gamma_1, \gamma_2 \leq 2$. If a solution u of (1) has sufficient decay at infinity and smoothness and V satisfies the condition (3), then it satisfies

$$\begin{aligned} \|u(t)\|_{L^2} &= \|\varphi\|_{L^2}, \\ E(u) \equiv K_m(u) + V(u) &= E(\varphi), \end{aligned} \tag{5}$$

where $K_m(u) = \frac{1}{2}\langle\sqrt{m^2-\Delta}\,u, u\rangle$, $V(u) = \frac{1}{4}\langle F(u), u\rangle$ and $\langle,\rangle$ is the complex inner product in L^2. For a rigorous proof of (5) see [8] in the case of $0 < \gamma_1, \gamma_2 \leq 1$ and [11] in the case of $1 < \gamma_1, \gamma_2 \leq 2$.

We show the global existence case by case. In section 2 we consider the potential with $0 < \gamma_1, \gamma_2 \leq 1$. This can be done by adopting exactly the same arguments (conservation laws (5) and contraction mapping theorem) as in [1]. The crucial estimate for local existence is the following Hardy inequality

$$\|V * |u|^2\|_{L^\infty} \lesssim \|u\|^2_{\dot{H}^{\frac{\gamma_1}{2}} \cap \dot{H}^{\frac{\gamma_2}{2}}}. \tag{6}$$

Throughout this paper, $\|\cdot\|_{X\cap Y}$ means $\|\cdot\|_X + \|\cdot\|_Y$ for Banach spaces X and Y. For the global existence we use the time-continuity argument via the energy conservation. On the other hand, from the energy conservation, we get an estimate of solution which is uniform in the mass m on any finite time interval (if m is bounded from above) and then get a strong convergence of solutions of (1) to a solution of the massless equation ($m = 0$). If m is large, then the situation is quite different. The kinetic energy $K_m(u)$ is not bounded globally in time any more. This can be overcome by a phase modulation and a uniform bound of local solutions in H^s with $s \geq \frac{\gamma_1}{2}$. The modulated solution is closely approximated by a solution of a Schrödinger equation of Hartree type if m is sufficiently large. We interpret this phenomenon as a non-relativistic limit and eventually as a semi-classical or vanishing dispersion limit. For the details see Remark 1 below and Propositions 2.4. and 2.5 of [1].

In sections 3, 4 and 5 we consider a case that γ_1 and γ_2 are large. The main tools are the Strichartz estimates and conservations laws (5). If we use the estimate (6) for this case, then on account of the range of γ_1 and γ_2 the right hand side of (6) cannot be bounded by energy (actually the estimate (6) for the case $\gamma_1, \gamma_2 > 1$ is an energy supercritical estimate). Hence we exploit the

well-known Strichartz estimate for the unitary group $U(t)$ which is stated as follows (see [9, 10]):

$$\|U(t)\varphi\|_{L_T^{q_0}H_{r_0}^{s_0-\sigma_0}} \lesssim \|\varphi\|_{H^{s_0}},$$
$$\left\|\int_0^t U(t-t')f(t')\,dt'\right\|_{L_T^{q_1}H_{r_1}^{s_1-\sigma_1}} \lesssim \|f\|_{L_T^1 H^{s_1}}, \tag{7}$$

where $(q_i, r_i), i = 0, 1$, satisfy that for any $\theta \in [0, 1]$

$$\frac{2}{q_i} = (n-1+\theta)\left(\frac{1}{2}-\frac{1}{r_i}\right), \quad 2\sigma_i = (n+1+\theta)\left(\frac{1}{2}-\frac{1}{r_i}\right), \tag{8}$$
$$2 \le q_i, r_i \le \infty, \quad (q_i, r_i) \ne (2, \infty).$$

We call a pair (q, r, σ) satisfying (8) an *admissible pair*. If $\theta = 0$, it is called wave admissible and if $\theta = 1$, then Schrödinger admissible. Here $H_r^s = (1-\Delta)^{-s/2}L^r$ is the usual Sobolev space and $H^s = H_2^s$. Hereafter, we denote the space $L^q(0, T; B)$ by $L_T^q(B)$ and its norm by $\|\cdot\|_{L_T^q B}$ for a Banach space B, and also $L^q(B)$ with norm $\|\cdot\|_{L^q B}$ by $L^q(0, \infty; B)$, $1 \le q \le \infty$. For the related weighted Strichartz estimates, see [2, 3] in which some global existence and scattering of radial solutions are considered.

On the right hand side of the second inequality of (7), only the space $L_T^1 H^{s_1}$ for f is related because contrary to the case of Klein-Gordon equation, inhomogeneous estimate for $U(t)$ preserves a regularity. One of course can consider the general space like $L^q H_r^s$ with additional regularity. In section 3, we use the wave and Schrödinger admissible pairs to prove the global existence. In [1] the global existence is proved for $0 < \gamma_1 = \gamma_2 < \frac{2n}{n+1}, n \ge 2$. When the potentials are competing each other, the Sobolev embedding argument on the single potential as in [1] is not enough, especially if the difference of γ_1 and γ_2 is big and $\gamma_1, \gamma_2 \ge \frac{n}{n+1}$. To overcome this difficulty, we proceed with an interpolation together with Sobolev embedding for the proof. In case that one of γ_1 and γ_2 is smaller than $\frac{n}{n+1}$ and the other is larger than $\frac{n}{n+1}$, we use the Hardy inequality as in section 2 together with an interpolation argument.

In section 4, we use the end point Schrödinger admissible ones for the small data scattering in the case $2 < \gamma_1, \gamma_2 < n$. This case can be treated in a similar way to the single potential case.

The most difficult case is when $\gamma_1 < 2 < \gamma_2$, or $\gamma_2 < 2 < \gamma_1$. To control potentials of these types we can try to use non-endpoint admissible pair and endpoint one simultaneously, but this does not seem to work. To avoid this difficulty, in section 5 we assume stronger condition on V_1 and V_2 such that $|V_1(x)| \lesssim \chi_{\{|x|\le R\}}|x|^{-\gamma_1}$ and $|V_2(x)| \lesssim \chi_{\{|x|>R\}}|x|^{-\gamma_2}$ for $\gamma_1 < 2$ and $\gamma_2 > 2$. For these potentials V becomes an $L^{\frac{n}{2}}$ function. Hence this enables us to use a wider range of γ_1 and the end point Schrödinger admissible pair to obtain the global existence. For potentials of these types, the case $\gamma_1 = 2$ can be treated on account of the endpoint wave admissible pair.

Unfortunately, we do not have any idea of the global existence for the potentials as stated above in case that $\gamma_2 \leq 2 < \gamma_1$. The decay of V at space infinity becomes slower and the singularity at the origin stronger than those of V satisfying above conditions, which make the competition between the potentials stronger. We believe the global existence at least for the case that one of γ_1 and γ_2 is between $\frac{2n}{n+1}$ and 2. For this we need to establish a new method. As a future work, this problem seems to be worth being pursued on account of not only the mathematical concern but applications to the another equations like Dirac and Klein-Gordon, etc.

If not specified, throughout this paper, the notation $A \lesssim B$ and $A \gtrsim B$ denote $A \leq CB$ and $A \geq C^{-1}B$, respectively. Different positive constants possibly depending on n, m, λ and γ might be denoted by the same letter C. $A \sim B$ means that both $A \lesssim B$ and $A \gtrsim B$ hold. For γ with $0 < \gamma < n$, we use the integral operators $I_{n-\gamma}$ by convolution with the homogeneous potential $|x|^{-\gamma}$ as $I_{n-\gamma}(f)(x) = |\cdot|^{-\gamma} * f(x)$.

2. Case $0 < \gamma_1, \gamma_2 \leq 1$

We can handle this case by using only conservation laws as in [1]. Let us first introduce the following local existence result.

Proposition 1. *Let $0 < \gamma_1, \gamma_2 < n$ and $n \geq 1$. Suppose that V_1 and V_2 satisfy the condition (2) and $\varphi \in H^s(\mathbb{R}^n)$ with $s \geq \max(\frac{\gamma_1}{2}, \frac{\gamma_2}{2})$. Then there exists a positive time T independent of m such that (4) has a unique solution $u \in C([0,T]; H^s)$ with $\|u\|_{L^\infty_T H^s} \leq C\|\varphi\|_{H^s}$, where C does not depend on m.*

Proof. The method of proof is almost the same as of [1]. From the decays of V_1, V_2 and Hardy inequality it follows that

$$\|V * |u|^2\|_{L^\infty} \lesssim \|u\|^2_{H^{\frac{\gamma_1}{2}}} + \|u\|^2_{H^{\frac{\gamma_2}{2}}} \lesssim \|u\|^2_{H^s}. \tag{9}$$

This inequality enables us to get the uniform boundedness of the existence time T and the constant C on the mass m. For the details, see the proof of Proposition 1 in [1] or [8]. □

Lemma 1. *Suppose that $0 < \gamma_1, \gamma_2 < 2$, and $V < 0$ and satisfies (2) and (3). Let u be a local solution of (1) in $C([0,T]; H^{\frac{1}{2}})$ satisfying (5) for all $t \in [0,T]$. Then if $\gamma_1, \gamma_2 < 1$, then*

$$\sup_{t\in[0,T]} \|u(t)\|_{\dot{H}^{\frac{1}{2}}} \leq C(1 + \|\varphi\|^2_{H^{\frac{1}{2}}})^{\frac{2}{1-\gamma}}, \tag{10}$$

where $\gamma = \max(\gamma_1, \gamma_2)$.

If $0 < \gamma_1 < \gamma_2 = 1$, then for sufficiently small $\|\varphi\|_{L^2}$ independent of $\|\varphi\|_{\dot{H}^{\frac{1}{2}}}$

$$\sup_{t\in[0,T]} \|u(t)\|_{\dot{H}^{\frac{1}{2}}} \leq C(1 + \|\varphi\|^2_{\dot{H}^{\frac{1}{2}}})^{\frac{2}{1-\gamma_1}}. \tag{11}$$

If $0 < \gamma_1 = \gamma_2 = 1$, *then for sufficiently small* $\|\varphi\|_{L^2}$ *independent of* $\|\varphi\|_{\dot{H}^{\frac{1}{2}}}$

$$\sup_{t\in[0,T]} \|u(t)\|_{\dot{H}^{\frac{1}{2}}} \le C(1 + \|\varphi\|^2_{\dot{H}^{\frac{1}{2}}}). \tag{12}$$

If $1 < \gamma_i < 2$ *and* $\|\varphi\|_{L^2}$ *is sufficiently small dependently on* $\|\varphi\|_{\dot{H}^{\frac{1}{2}}}$, *then*

$$\sup_{t\in[0,T]} \|u(t)\|_{\dot{H}^{\frac{1}{2}}} \le C(1 + \|\varphi\|^2_{\dot{H}^{\frac{1}{2}}})^{\gamma}. \tag{13}$$

Proof of Lemma 1. From the estimate (9) and L^2 conservation, we have by Sobolev embedding and interpolation that

$$\begin{aligned}
\|u(t)\|^2_{\dot{H}^{\frac{1}{2}}} &\le 2|E(u)| + 2|V(u)| \\
&\le 2|E(\varphi)| + C\sum_{i=1,2} \|u\|^2_{L^{\frac{2n}{n-\gamma_i+1}}} \|u\|^2_{\dot{H}^{\frac{1}{2}}} \\
&\le \|\varphi\|^2_{H^{\frac{1}{2}}} + C\sum_{i=1,2} \|\varphi\|^{4-2\gamma_i}_{L^2} \|\varphi\|^{2\gamma_i}_{\dot{H}^{\frac{1}{2}}} + C\sum_{i=1,2} \|\varphi\|^{4-2\gamma_i}_{L^2} \|u\|^{2\gamma_i}_{\dot{H}^{\frac{1}{2}}}.
\end{aligned}$$

If $\gamma_i < 1$, then by Young's inequality we get (10). If $\gamma_1 < \gamma_2 = 1$, then by the above argument for γ_1 and the smallness of $\|\varphi\|_{L^2}$ that $C\|\varphi\|^2_{L^2} \le \frac{1}{2}$ for $\gamma_2 = 1$ we get (11). If $\gamma_i = 1$, then for small $\|\varphi\|_{L^2}$ so that $2C\|\varphi\|^2_{L^2} \le \frac{1}{2}$ we get (11). If $\gamma_i > 2$, then for any φ with

$$\|\varphi\|_{L^2} \le \min\left(1, (8^{\gamma}C^{\gamma}(1 + \|\varphi\|^2_{\dot{H}^{\frac{1}{2}}})^{\gamma(\gamma-1)})^{-\frac{1}{4-2\gamma}}\right)$$

we have (12). □

Now using the conservation laws (5), we establish the global time existence.

Theorem 1. *Assume that* V_1 *and* V_2 *satisfy the conditions* (2) *and* (3). *Let* $0 < \gamma_1, \gamma_2 \le 1$ *for* $n \ge 2$, $0 < \gamma_1, \gamma_2 < 1$ *for* $n = 1$ *and* $s \ge \frac{1}{2}$. *Let* T^* *be the maximal existence time of the solution* u *as in Proposition* 1. *Then if* $V \ge 0$, *or if* $V < 0$ *and* $\gamma_i < 1$, *or if* $V < 0$ *and at least one of* γ_is *is* 1 *and* $\|\varphi\|_{L^2}$ *is small, then* $T^* = \infty$. *Moreover* $\|u(t)\|_{H^s} \le C\|\varphi\|_{H^s} e^{C\mathcal{E}(\varphi)t}$, *where* C *does not depend on* m *and* $\mathcal{E}(\varphi)$ *is a constant depending only on* γ_i, $\|\varphi\|_{H^{\frac{1}{2}}}$ *and* $E(\varphi)$.

Proof. If $V \ge 0$, then

$$\|u(t)\|^2_{\dot{H}^{\frac{\gamma_i}{2}}} \le C(E(u) + \|\varphi\|^2_{L^2}) = C(E(\varphi) + \|\varphi\|^2_{L^2}). \tag{14}$$

From Lemma 1 it follows that if $\gamma_1, \gamma_2 < 1$, and if one of γ_is is 1 and $\|\varphi\|_{L^2}$ is sufficiently small, then

$$\|u(t)\|^2_{\dot{H}^{\frac{\gamma_i}{2}}} \le C(\|u(t)\|^2_{\dot{H}^{\frac{1}{2}}} + \|\varphi\|^2_{L^2}) = C(1 + \|\varphi\|^2_{H^{\frac{1}{2}}})^{\frac{1}{1-\gamma}} \quad \text{for } \gamma < 1, \tag{15}$$

$$\|u(t)\|^2_{\dot{H}^{\frac{\gamma_i}{2}}} \le C(1 + \|\varphi\|^2_{H^{\frac{1}{2}}})^{\frac{1}{1-\gamma_1}} \quad \text{for one of } \gamma_1 < \gamma_2 = 1. \tag{16}$$

Now let us define a functional $\mathcal{E}$ by

$$\begin{aligned}\mathcal{E}(\varphi) &= E(\varphi) + \|\varphi\|_{L^2}^2 \text{ if } V \geq 0,\\ &= (1 + \|\varphi\|_{H^{\frac{1}{2}}}^2)^{\frac{1}{1-\gamma}} \text{ if } V < 0 \text{ and } \gamma < 1,\\ &= (1 + \|\varphi\|_{\dot{H}^{\frac{1}{2}}}^2)^{\frac{2}{1-\gamma_1}} \text{ if } V < 0 \text{ and } \gamma_1 < \gamma_2 = 1.\end{aligned}$$

Then from (14), (15) and (16) and the generalized Leibniz rule, we have

$$\begin{aligned}&\|u(t)\|_{H^s}\\ &\lesssim \|\varphi\|_{H^s} + \int_0^t \|F(u)\|_{H^s}\, dt'\\ &\lesssim \|\varphi\|_{H^s} + \sum_{i=1,2} \int_0^t (\|I_{n-\gamma_i}(|u|^2)\|_{L^\infty} \|u\|_{H^s}\\ &\qquad\qquad + \|I_{n-\gamma_i}(|u|^2)\|_{H^s_{\frac{2n}{\gamma_i}}} \|u\|_{L^{\frac{2n}{n-\gamma_i}}})\, dt' \qquad (17)\\ &\lesssim \|\varphi\|_{H^s} + \int_0^t \|u\|_{H^{\frac{\gamma}{2}}}^2 \|u\|_{H^s}\, dt'\\ &\lesssim \|\varphi\|_{H^s} + \mathcal{E}(\varphi) \int_0^t \|u\|_{H^s}\, dt'.\end{aligned}$$

Gronwall's inequality shows that

$$\|u(t)\|_{H^s} \leq C\|\varphi\|_{H^s} \exp(C\mathcal{E}(\varphi)t).$$

This completes the proof. □

Remark 1. From the uniform boundedness on m for the solution u in Proposition 1 and Theorem 1, we obtain the similar results to those in [1] on the limit problem as $m \to 0$ and $m \to \infty$.

The first is the following. If $u_m \in (C \cap L^\infty)(H^s)$ is the global solution of (4) satisfying the same condition as in Theorem 1, then for any finite time T, $u_m \to u_0$ in $L_T^\infty(H^s)$ with $s \geq \frac{1}{2}$ as $m \to 0$, where u_0 is the global solution to the massless ($m = 0$) equation (1) with $u_0(0) = \varphi$.

Now let us consider the phase modulated function $v_m = e^{imt}u_m$. Then one can easily verify that the function v_m satisfies the equation

$$i\partial_t v_m = (\sqrt{m^2 - \Delta} - m)v_m + F(v_m), \quad v_m(0) = \varphi,$$

and equivalently

$$v_m(t) = U_m(t)\varphi - i\int_0^t U_m(t - t')F(v_m)(t')\, dt', \qquad (18)$$

where $U_m(t) = e^{-it(\sqrt{m^2-\Delta}-m)}$. Let w_m be a solution of the nonlinear Schrödinger equation:

$$i\partial_t w_m = -\frac{1}{2m}\Delta w_m + F(w_m), \quad w_m(0) = \varphi. \qquad (19)$$

Let $T^*_{v_m}$ and $T^*_{w_m}$ be the maximal existence time of the solutions u_m and w_m, respectively. Let $T^* \equiv \inf_{m>1} \min(T^*_{v_m}, T^*_{w_m})$ If $s \geq \frac{\gamma_1}{2}$ and $T < T^*$, then $v_m - w_m \to 0$ in $L^\infty_T(H^s)$ as $m \to \infty$. For the details of proof, see Propositions 2.4 and 2.5 of [1].

3. Case $0 < \gamma_1, \gamma_2 < \frac{2n}{n+1}$

In [1] the global existence was shown with homogeneous potential $|x|^{-\gamma}$ for the ranges of $0 < \gamma < \frac{2n}{n+1}$ and $n \geq 2$ by using the wave admissible Strichartz estimate. Adopting and modifying the method of proof of Theorem 1, we have the following.

Theorem 2. *Assume that V_1 and V_2 satisfy the conditions* (2) *and* (3). *Let $\frac{n}{n+1} \leq \gamma_1, \gamma_2 < \frac{2n}{n+1}$ and $n \geq 2$. Then if $\varphi \in H^{\frac{1}{2}}$ and if $V \geq 0$, or $V < 0$ and $\gamma_1, \gamma_2 < 1$, or $V < 0, \gamma_i \leq 1$ and at least one of γ_is is 1 and $\|\varphi\|_{L^2}$ is sufficiently small, or $V < 0$ and $1 < \gamma_i < \frac{2n}{n+1}$ and $\|\varphi\|_{L^2}$ is sufficiently small, then* (4) *has a unique solution $u \in C([0,\infty); H^{\frac{1}{2}}) \cap L^q_{loc}(H_r^{\frac{1}{2}-\sigma})$, where $q = \frac{4n}{(n-1)\alpha}$, $r = \frac{2n}{n-\alpha}$ and $\sigma = \frac{(n+1)\alpha}{4n}$ for some $\alpha < \frac{2n}{n+1}$ but arbitrarily close to $\frac{2n}{n+1}$.*

Proof. Given n, γ_1 and γ_2 choose a number α so close to $\frac{2n}{n+1}$ that $\min(1 + \frac{(n-1)\alpha}{2n} - \gamma_1, 1 + \frac{(n-1)\alpha}{2n} - \gamma_1) > 0$. Then for some positive number T to be chosen later, let us define a complete metric space $(X_{T,\rho}, d_T)$ with metric d_T by

$$X_{T,\rho} = \left\{ v \in C([0,T]; H^{\frac{1}{2}}) \cap L^q_T(H_r^{\frac{1}{2}-\sigma}) : \|v\|_{L^\infty_T H^{\frac{1}{2}}} + \|v\|_{L^q_T H_r^{\frac{1}{2}-\sigma}} \leq \rho \right\},$$

$$d_T(u,v) = \|u - v\|_{L^\infty_T H^{\frac{1}{2}} \cap L^q_T H_r^{\frac{1}{2}-\sigma}},$$

where q, r, σ are the same indices as stated in Theorem 2.

Now we define a mapping $N : u \mapsto N(u)$ on $X_{T,\rho}$ by

$$N(u)(t) = U(t)\varphi - i \int_0^t U(t-t')F(u)(t')\,dt'. \tag{20}$$

Our strategy is to use the standard contraction mapping argument. From now on, we will prove that the nonlinear mapping N is a contraction on $X_{T,\rho}$, provided T is sufficiently small. We will use another version of Hardy inequality which can be easily shown by splitting integral regions, inside the ball with radius R and its outside and by optimizing over R.

Lemma 2. *Let $0 < \gamma < n$. Then for any $0 < \varepsilon < n - \gamma$ we have*

$$\left\| I_{n-\gamma}(|u|^2) \right\|_{L^\infty} \lesssim \|u\|_{L^{\frac{2n}{n-\gamma-\varepsilon}}} \|u\|_{L^{\frac{2n}{n-\gamma+\varepsilon}}}.$$

Taking θ by 0 in the Strichartz estimate (7) and (8), the pair

$$(q,r,\sigma) = \left(\frac{4n}{(n-1)\alpha}, \frac{2n}{n-\alpha}, \frac{(n+1)\alpha}{4n} \right)$$

becomes a wave admissible one. Hence the Strichartz estimate together with the estimate (9), Plancherel theorem, Lemma 2 and generalized Leibniz rules, enables us to deduce that for some small $0 < \varepsilon < n - \gamma_2$,

$$
\begin{aligned}
&\|N(u)\|_{L_T^\infty H^{\frac{1}{2}} \cap L_T^q H_r^{\frac{1}{2}-\sigma}} \\
&\lesssim \|\varphi\|_{H^{\frac{1}{2}}} + \|F(u)\|_{L_T^1 H^{\frac{1}{2}}} \\
&\lesssim \|\varphi\|_{H^{\frac{1}{2}}} + \sum_{i=1,2} \|I_{n-\gamma_i}(|u|^2)\|_{L_T^1 L^\infty} \|u\|_{L_T^\infty H^{\frac{1}{2}}} \\
&\qquad + \sum_{i=1,2} \int_0^T \|I_{n-\gamma_i}(|u|^2)\|_{H^{\frac{1}{2}}_{\frac{2n}{\gamma_i+\varepsilon}}} \|u\|_{L^{\frac{2n}{n-(\gamma_i+\varepsilon)}}} dt \\
&\lesssim \|\varphi\|_{H^{\frac{1}{2}}} + \sum_{i=1,2} \|u\|_{L_T^2 L^{\frac{2n}{n-(\gamma_i+\varepsilon)}}} \|u\|_{L_T^2 L^{\frac{2n}{n-(\gamma_i-\varepsilon)}}} \|u\|_{L_T^\infty H^{\frac{1}{2}}} \\
&\qquad + \sum_{i=1,2} \int_0^T \||u|^2\|_{H^{\frac{1}{2}}_{\frac{2n}{2n-(\gamma_i-\varepsilon)}}} \|u\|_{L^{\frac{2n}{n-(\gamma_i+\varepsilon)}}} dt \\
&\lesssim \|\varphi\|_{H^{\frac{1}{2}}} + \sum_{i=1,2} \|u\|_{L_T^2 L^{\frac{2n}{n-(\gamma_i+\varepsilon)}}} \|u\|_{L_T^2 L^{\frac{2n}{n-(\gamma_i-\varepsilon)}}} \|u\|_{L_T^\infty H^{\frac{1}{2}}} \\
&\qquad + \sum_{i=1,2} \int_0^T \|u\|_{L^{\frac{2n}{n-(\gamma_i-\varepsilon)}}} \|u\|_{H^{\frac{1}{2}}} \|u\|_{L^{\frac{2n}{n-(\gamma_i+\varepsilon)}}} dt.
\end{aligned} \tag{21}
$$

Using Hölder's inequality for time integral, we have

$$
\begin{aligned}
&\|N(u)\|_{L_T^\infty H^{\frac{1}{2}} \cap L_T^q H_r^{\frac{1}{2}-\sigma}} \\
&\lesssim \|\varphi\|_{H^{\frac{1}{2}}} + \sum_{i=1,2} \|u\|_{L_T^2 L^{\frac{2n}{n-(\gamma_i+\varepsilon)}}} \|u\|_{L_T^2 L^{\frac{2n}{n-(\gamma_i-\varepsilon)}}} \|u\|_{L_T^\infty H^{\frac{1}{2}}}.
\end{aligned} \tag{22}
$$

Now if we choose $\varepsilon > 0$ so that $\varepsilon < \min_{i=1,2} \left(n - \gamma_i, 1 + \frac{(n-1)\alpha}{2n} - \gamma_i\right)$, then

$$
2 < \frac{2n}{n-(\gamma_i-\varepsilon)} < \frac{2n}{n-(\gamma_i+\varepsilon)} \le \frac{2n}{n-\alpha-(1-2\sigma)}. \tag{23}
$$

From the interpolation between L^2 and $L^{\frac{2n}{n-\alpha-(1-2\sigma)}}$, it follows that

$$
\|u\|_{L^{\frac{2n}{n-(\gamma_i+\varepsilon)}}} \lesssim \|u\|_{L^2}^{1-\theta_i} \|u\|^{\theta_i}_{L^{\frac{2n}{n-\alpha-(1-2\sigma)}}},
$$

where $\theta_i = \frac{2\gamma_i}{\alpha+(1-2\sigma)}$. Since $1\gamma_i < \frac{2n}{n+1}$, if we choose α sufficiently close to $\frac{2n}{n+1}$, then we can make θ_i be the value in the closed interval $[1, 2]$.

Now using (22) and Sobolev embedding $H_r^{\frac{1}{2}-\sigma} \hookrightarrow L^r \cap L^{\frac{2n}{n-\alpha-(1-2\sigma)}}$, we deduce that

$$\begin{aligned}\|N(u)\|_{L_T^\infty H^{\frac{1}{2}} \cap L_T^q H_r^{\frac{1}{2}-\sigma}} &\le C(\|\varphi\|_{H^{\frac{1}{2}}} + \sum_{i=1,2} T^{1-\theta_i} \|u\|^{3-\theta_i}_{L_T^\infty H^{\frac{1}{2}}} \|u\|^{\theta_i}_{L_T^q H_r^{\frac{1}{2}-\sigma}}) \\ &\le C(\|\varphi\|_{H^{\frac{1}{2}}} + (T^{1-\theta_1} + T^{1-\theta_2})\rho^3)\end{aligned}$$

for some constant C. Here we have used the conventional embedding that if $2(\frac{1}{2}-\sigma) \ge n-\alpha$ then $H_r^{\frac{1}{2}-\sigma} \hookrightarrow L^{r_1}$ for any $r_1 \ge r$. Thus if we choose ρ and T so that $C\|\varphi\|_{H^{\frac{1}{2}}} \le \frac{\rho}{2}$ and $C(T^{1-\theta_1} + T^{1-\theta_2})\rho^3 \le \frac{\rho}{2}$, then we conclude that N maps from $X_{T,\rho}$ to itself.

For any $u, v \in X_{T,\rho}$, we have

$$\begin{aligned}&d_T(N(u), N(v)) \\ &\lesssim \|F(u) - F(v)\|_{L_T^1 H^{\frac{1}{2}}} \\ &\lesssim \sum_{i=1,2} (\|I_{n-\gamma_i}(|u|^2 - |v|^2)u\|_{L_T^1 H^{\frac{1}{2}}} + \|I_{n-\gamma_i}(|v|^2)(u-v)\|_{L_T^1 H^{\frac{1}{2}}}).\end{aligned} \tag{24}$$

By Lemma 2 and Hölder's inequality, we have for sufficiently small $\varepsilon > 0$

$$\begin{aligned}&\|I_{n-\gamma_i}(|u|^2 - |v|^2)u\|_{L_T^1 H^{\frac{1}{2}}} \\ &\lesssim \|I_{n-\gamma_i}(|u|^2 - |v|^2)\|_{L_T^2 L^\infty} \|u\|_{L_T^\infty H^{\frac{1}{2}}} \\ &\qquad + \|I_{n-\gamma_i}(|u|^2 - |v|^2)\|_{L_T^2 H^s_{\frac{2n}{\gamma_i+\varepsilon}}} \|u\|_{L_T^2 L^{\frac{2n}{n-(\gamma_i+\varepsilon)}}} \\ &\lesssim \rho \||u|^2 - |v|^2\|^{\frac{1}{2}}_{L_T^1 L^{\frac{n}{n-(\gamma_i+\varepsilon)}}} \||u|^2 - |v|^2\|^{\frac{1}{2}}_{L_T^1 L^{\frac{n}{n-(\gamma_i-\varepsilon)}}} \\ &\qquad + \rho\|u-v\|_{L_T^\infty H^{\frac{1}{2}}} (\|u\|_{L_T^2 L^{\frac{2n}{n-(\gamma_i-\varepsilon)}}} + \|v\|_{L_T^2 L^{\frac{2n}{n-(\gamma_i-\varepsilon)}}}) \\ &\qquad + \rho\|u-v\|_{L_T^2 L^{\frac{2n}{n-(\gamma_i-\varepsilon)}}} (\|u\|_{L_T^\infty H^{\frac{1}{2}}} + \|v\|_{L_T^\infty H^{\frac{1}{2}}}).\end{aligned} \tag{25}$$

Since $L^2 \cap H_r^{\frac{1}{2}-\sigma} \hookrightarrow L^{\frac{2n}{n-(\gamma_i \pm \varepsilon)}}$, by another Hölder's inequality with respect to the time variable, we have

$$\|I_{n-\gamma_i}(|u|^2 - |v|^2)u\|_{L_T^1 H^{\frac{1}{2}}} \lesssim (T + T^{1-\frac{2}{q}})\rho^2 d_T(u,v).$$

Similarly,

$$\begin{aligned}&\|I_{n-\gamma_i}(|v|^2)(u-v)\|_{L_T^1 H^{\frac{1}{2}}} \\ &\lesssim \|I_{n-\gamma_i}(|v|^2)\|_{L_T^1 L^\infty} \|u-v\|_{L_T^\infty H^{\frac{1}{2}}} \\ &\qquad + \|I_{n-\gamma_i}(|v|^2)\|_{L_T^2 L^{\frac{2n}{\gamma_i+\varepsilon}}} \|u-v\|_{L_T^2 L^{\frac{2n}{n-(\gamma_i+\varepsilon)}}} \\ &\lesssim \|v\|_{L_T^2 L^{\frac{2n}{n-(\gamma_i-\varepsilon)}}} \|v\|_{L_T^2 L^{\frac{2n}{n-(\gamma_i+\varepsilon)}}} d_T(u,v) \\ &\qquad + \|v\|_{L_T^\infty H^{\frac{1}{2}}} \|v\|_{L_T^2 L^{\frac{2n}{n-(\gamma_i-\varepsilon)}}} \|u-v\|_{L_T^2 L^{\frac{2n}{n-(\gamma_i+\varepsilon)}}}.\end{aligned} \tag{26}$$

Hence we get

$$\|I_{n-\gamma_i}(|v|^2)(u-v)\|_{L^1_T H^{\frac{1}{2}}} \lesssim (T+T^{1-\frac{2}{q}})\rho^2 d_T(u,v).$$

Substituting these two estimates into (24) and then using the fact $CT^{1-\frac{2}{q}}\rho^2 \leq \frac{1}{2}$ for smaller T, we conclude that N is a contraction mapping.

For the global existence we adopt the time-continuity argument. Let T^* be the maximal existence time for the local solution u constructed as above. Then we claim that $T^* = \infty$. In fact, if V and φ satisfy the conditions as stated in the theorem, then from the estimates in Lemma 1 and (22), we have

$$\|u\|_{L^q_T H^{\frac{1}{2}-\sigma}_r} \lesssim \mathcal{E}(\varphi) + \sum_{i=1,2} T^{1-\theta_i}\mathcal{E}(\varphi)^{\frac{3-\theta_i}{2}}\|u\|^{\theta_i}_{L^q_T H^{\frac{1}{2}-\sigma}_r},$$

where $\mathcal{E}(\varphi)$ is defined as in the proof of Theorem 1, if $\gamma_i \leq 1$ and it is defined by $(1+\|\varphi\|^2_{\dot{H}^{\frac{1}{2}}})^\gamma$ for $\gamma_i > 1$. If we assume that $T^* < \infty$, then for sufficiently small T depending on $\mathcal{E}(\varphi)$,

$$\|u\|_{L^q(T_{j-1},T_j;H^{\frac{1}{2}-\sigma}_r)} \leq C\mathcal{E}(\varphi),$$

where $T_j - T_{j-1} = T$ for $j \leq k-1$ and $T_k = T^*$. This means that

$$\|u\|^q_{L^q(0,T^*;H^{\frac{1}{2}-\sigma}_r)} \leq \sum_{1\leq j\leq k}\|u\|^q_{L^q(T_{j-1},T_j;H^{\frac{1}{2}-\sigma}_r)} \leq (kC\mathcal{E}(\varphi))^q < \infty.$$

This is a contradiction to the hypothesis $T^* < \infty$. We have just finished the proof. □

Now using Theorem 1 and Theorem 2, we can also treat the potentials V_1, V_2 with $\gamma_1 < \frac{n}{n+1}, \frac{n}{n+1} \leq \gamma_2 < \frac{2n}{n+1}$ or $\frac{n}{n+1} \leq \gamma_1 < \frac{2n}{n+1}, \gamma_2 < \frac{n}{n+1}$, respectively.

Theorem 3. *Assume that V_1 and V_2 satisfy the conditions* (2) *and* (3). *Let $0 < \gamma_1 < \frac{n}{n+1}, \frac{n}{n+1} \leq \gamma_2 < \frac{2n}{n+1}$ and $n \geq 2$. Then if $\varphi \in H^{\frac{1}{2}}$ and if $V \geq 0$, or $V < 0$ and $\gamma_2 < 1$, or $V < 0, 1 \leq \gamma_2 < \frac{2n}{n+1}$ and $\|\varphi\|_{L^2}$ is sufficiently small, then* (4) *has a unique solution $u \in C([0,\infty); H^{\frac{1}{2}}) \cap L^q_{loc}(H^{\frac{1}{2}-\sigma}_r)$, where $q = \frac{4n}{(n-1)\alpha}$, $r = \frac{2n}{n-\alpha}$ and $\sigma = \frac{(n+1)\alpha}{4n}$ for some $\alpha < \frac{2n}{n+1}$ but arbitrarily close to $\frac{2n}{n+1}$.*

Proof. For the proof we have only to consider the boundedness of nonlinear term $F(u)$. It can be easily seen by (17) and (21) that

$$\begin{aligned}\|F(u)\|_{L^1_T H^{\frac{1}{2}}} &\leq \|(V_1 * |u|^2)u\|_{L^1_T H^{\frac{1}{2}}} + \|(V_2 * |u|^2)u\|_{L^1_T H^{\frac{1}{2}}} \\ &\lesssim T\|u\|^3_{L^\infty_T H^{\frac{1}{2}}} + T^{1-\theta_2}\|u\|^{3-\theta_2}_{L^\infty_T H^{\frac{1}{2}}}\|u\|^{\theta_2}_{L^q_T H^{\frac{1}{2}-\sigma}_r},\end{aligned}$$

which enables us to conclude the local existence by contraction argument and the global one by time-continuity argument. □

4. Case $2 < \gamma_1, \gamma_2 < n$

In this case, the small data scattering is considered as in [1].

Theorem 4. *Assume that V_1 and V_2 satisfy the condition (2). Let $2 < \gamma_1, \gamma_2 < n$, $n \geq 3$ and $s > s_0 \equiv \max_{i=1,2} s_i$, where $s_i = \frac{\gamma_i}{2} - \frac{n-2}{2n}$. Then if $\varphi \in H^s$ with $\|\varphi\|_{H^{s_0}}$ sufficiently small, (4) has a unique solution $u \in (C \cap L^\infty)(H^s) \cap L^2(H^{s-\frac{n+2}{2n}}_{\frac{2n}{n-2}})$. Moreover there is $\varphi^+ \in H^s$ such that*

$$\|u(t) - U(t)\varphi^+\|_{H^s} \to 0 \text{ as } t \to \infty.$$

Proof. We will use the Strichartz estimate (7) with $\theta = 1$ and endpoint admissible pair $(q, r, \sigma) = \left(2, \frac{2n}{n-2}, \frac{n+2}{2n}\right)$.

To proceed the same strategy as proofs of above theorems, let us define a complete metric space $(Y^s_{R,\rho}, d)$ with metric d by

$$Y^s_{R,\rho} = \Big\{ v \in L^\infty(H^s) \cap L^2(H^{s-\sigma}_r) : \|v\|_{L^\infty H^s \cap L^2 H^{s-\sigma}_r} \leq R, \\ \|v\|_{L^\infty H^{s_0} \cap L^2 H^{s_0-\sigma}_r} \leq \rho \Big\},$$

$$d(u, v) = \|u - v\|_{L^\infty H^s \cap L^2 H^{s-\sigma}_r}.$$

Then from the estimate (22) with s instead of $\frac{1}{2}$, we have

$$\|N(u)\|_{L^\infty H^s \cap L^2 H^{s-\sigma}_r} \leq C\|\varphi\|_{H^s} + \sum_{i=1,2} \|u\|_{L^2 L^{\frac{2n}{n-(\gamma_i+\varepsilon)}}} \|u\|_{L^2 L^{\frac{2n}{n-(\gamma_i-\varepsilon)}}} \|u\|_{L^\infty H^s}.$$

If we choose $\varepsilon > 0$ so small that

$$\varepsilon < \min_{i=1,2} \left(n - \gamma_i, \gamma_i - 2, 2s + \frac{n-2}{n} - \gamma_i \right),$$

then we have

$$\frac{2n}{n-2} \leq \frac{2n}{n-(\gamma_i-\varepsilon)} < \frac{2n}{n-2-2(s_i-\sigma)} < \frac{2n}{n-(\gamma_i+\varepsilon)} \leq \frac{2n}{n-2-2(s-\sigma)}$$

and hence by Sobolev embedding

$$\|N(u)\|_{L^\infty H^s \cap L^2 H^{s-\sigma}_r} \leq C\|\varphi\|_{H^s} + \|u\|_{L^2 H^{s_0-\sigma}_r} \|u\|_{L^2 H^{s-\sigma}_r} \|u\|_{L^\infty H^s}.$$

Similarly, we also have

$$\begin{aligned} \|N(u)\|_{L^\infty H^{s_0} \cap L^2 H^{s_0-\sigma}_r} &\leq C\|\varphi\|_{H^{s_0}} + C\|u\|_{L^2 H^{s_0-\sigma}_r} \|u\|_{L^2 H^{s-\sigma}_r} \|u\|_{L^\infty H^{s_0}} \\ &\leq C\|\varphi\|_{H^{s_0}} + C\rho^2 R. \end{aligned}$$

Hence if for any given φ we choose ρ satisfying that $C\|\varphi\|_{H^{s_0}} \leq \frac{\rho}{2}$ and $C\rho^2 R \leq \frac{\rho}{2}$, then

$$\|N(u)\|_{L^\infty H^{s_0} \cap L^2 H^{s_0-\sigma}_r} \leq \rho$$

and hence for some R and smaller ρ such that $C\|\varphi\|_{H^s} \leq \frac{R}{2}$ and $C\rho R^2 \leq \frac{R}{2}$, we have

$$\|N(u)\|_{L^\infty H^s \cap L^2 H^{s-\sigma}_r} \leq \frac{R}{2} + C\rho R^2 \leq R.$$

This implies that N maps $Y^s_{R,\rho}$ to itself. Similarly, from (24)–(26), one can show that $d(N(u), N(v)) \le \frac{1}{2} d(u, v)$. This proves the existence part.

To prove the scattering, let us define a function φ^+ by

$$\varphi^+ = \varphi - i \int_0^\infty U(-t') F(u)(t')\, dt'.$$

Then since $u \in Y^s_\rho$, we have $\varphi^+ \in H^s$, and therefore

$$\begin{aligned} \|u(t) - U(t)\varphi^+\|_{H^s} &\lesssim \int_t^\infty \|F(u)\|_{H^s}\, dt' \\ &\lesssim \|u\|_{L^\infty H^s} \int_t^\infty \|u\|^2_{H^{s-\sigma}_r}\, dt' \to 0 \ \text{ as } \ t \to \infty. \end{aligned}$$

□

Remark 2. In case that $\gamma_i > 2$, more generally, we may assume that the potentials V_1 and V_2 are time dependent. If they are uniformly bounded with respect to the time variable, every estimate concerning potentials works very well. For example, we can take $V_1(x,t) = a(t)\widetilde{V_1}(x)$ and $V_2(x,t) = b(t)\widetilde{V_2}(x)$ for some functions $a, b \in L^\infty(\mathbb{R})$ and $\widetilde{V_i}$ with $|\widetilde{V_i}(x)| \lesssim |x|^{-\gamma_i}$.

5. Case $0 < \gamma_1 \le 2 < \gamma_2 < n$

In this section, we consider the potentials V_1 and V_2 with $0 < \gamma_1 \le 2 < \gamma_2 < n$ such that

$$|V_1(x)| \lesssim \chi_{\{|x|\le 1\}} |x|^{-\gamma_1}, \quad |V_2(x)| \lesssim \chi_{\{|x|>1\}} |x|^{-\gamma_2}.$$

Then $V \in L^{\frac{n}{2}}$ if $0 < \gamma_1 < 2, \gamma_2 > 2$, and also $V \in L^{\frac{n-1}{2}}$ if $\gamma_1 = 2$, $\gamma_2 > \frac{2n}{n-1}$. Thus we have

$$\begin{aligned} &\|V * |u|\|_{L^\infty} \lesssim \|V\|_{L^{\frac{n}{2}}} \|u\|^2_{L^{\frac{2n}{n-2}}} \quad \text{for} \quad 0 < \gamma_1 < 2, \gamma_2 > 2, n \ge 3, \\ &\|V * |u|\|_{L^\infty} \lesssim \|V\|_{L^{\frac{n-1}{2}}} \|u\|^2_{L^{\frac{2(n-1)}{n-3}}} \quad \text{for} \quad \gamma_1 = 2, \gamma_2 > \frac{2n}{n-1}, n \ge 4. \end{aligned} \tag{27}$$

With this potential estimate we show the following theorem.

Theorem 5. (1) *Let $0 < \gamma_1 < 2 < \gamma_2 < n$ and $n \ge 3$. Then if $s \ge \frac{n+2}{2n}$, then there exists $\rho > 0$ such that for any $\varphi \in H^s$ with $\|\varphi\|_{H^s} \le \rho$,* (4) *has a unique solution $u \in (C \cap L^\infty)(H^s) \cap L^2(H^{s-\frac{n+2}{2n}}_{\frac{2n}{n-2}})$.*

(2) *Let $\gamma_1 = 2, \gamma_2 > \frac{2n}{n-1}$ and $n \ge 4$. Then if $s \ge \frac{n+1}{2n}$, then there exists $\rho > 0$ such that for any $\varphi \in H^s$ with $\|\varphi\|_{H^s} \le \rho$,* (4) *has a unique solution $u \in (C \cap L^\infty)(H^s) \cap L^2(H^{s-\frac{n+1}{2n}}_{\frac{2(n-1)}{n-3}})$.*

Moreover there is $\varphi^+ \in H^s$ such that

$$\|u(t) - U(t)\varphi^+\|_{H^s} \to 0 \ \text{ as } \ t \to \infty,$$

where u is the solution constructed as above.

Proof. For the simplicity of proof, we consider only the estimate of nonlinear term $F(u)$. The remaining parts follow readily from the same argument as in the proof of Theorem 4.

As for the first part (1), we use the endpoint Strichartz estimate with Schrödinger admissible pair $(q,r,\sigma) = (2, \frac{2n}{n-2}, \frac{n+2}{2n})$ and have

$$\|F(u)\|_{L^1H^s} \lesssim \|V * |u|^2\|_{L^1L^\infty}\|u\|_{L^\infty H^s} + \|V * |u|^2\|_{L^2H^s_n}\|u\|_{L^2L^{\frac{2n}{n-2}}}.$$

Using (27), Young's convolution inequality $(1 + \frac{1}{n} = \frac{2}{n} + \frac{n-2}{2n} + \frac{1}{2})$ for the term $\|V * |u|^2\|_{L^2H^s_n}$, we have

$$\|F(u)\|_{L^1H^s} \lesssim \|u\|^2_{L^2L^{\frac{2n}{n-2}}}\|u\|_{L^\infty H^s}.$$

The condition $s \geq \frac{n+2}{2n}$ is necessary for the embedding $H^{s-\frac{n+2}{2n}}_{\frac{2n}{n-2}} \hookrightarrow L^{\frac{2n}{n-2}}$.

For the second part (2), we use the endpoint Strichartz estimate with wave admissible pair $(q,r,\sigma) = (2, \frac{2(n-1)}{n-3}, \frac{n+1}{2n})$ and have

$$\|F(u)\|_{L^1H^s} \lesssim \|V * |u|^2\|_{L^1L^\infty}\|u\|_{L^\infty H^s} + \|V * |u|^2\|_{L^2H^s_{n-1}}\|u\|_{L^2L^{\frac{2(n-1)}{n-3}}}.$$

If we apply Young's inequality as above with $(1 + \frac{1}{n-1} = \frac{2}{n-1} + \frac{n-3}{2(n-1)} + \frac{1}{2})$, then we have that

$$\|F(u)\|_{L^1H^s} \lesssim \|u\|^2_{L^2L^{\frac{2(n-1)}{n-3}}}\|u\|_{L^\infty H^s}.$$

For the embedding we need $s \geq \frac{n+1}{2n}$. This completes the proof of the theorem. □

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