

AMALGAMATED PRODUCTS, CRITICAL EXPONENTS AND UNIFORM GROWTH OF GROUPS: A UNIFIED APPROACH

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ABSTRACT. The aim of this note is to advertise a method which turns out to be powerful enough to be used successfully in problems which are apparently unrelated. It is based on a modification of a construction that we first introduced in [2].

1. Introduction

The aim of this note is to advertise a method which turns out to be powerful enough to be used successfully in problems which are apparently unrelated. It is based on a modification of a construction that we first introduced in [2]. The main tool is a (family of) map(s) between two manifolds (or more general metric spaces, see below) that have contraction properties. We shall present here a quick survey of the constructions and two applications. This text is not intended to be exhaustive as far as references are concerned, but it aims at being a description of an approach that we believe could prove more results; the interested readers may read the details in [3] and [4].

1.1. Amalgamated products

Let us consider a n -dimensional compact hyperbolic manifold, i.e., whose sectional curvature is constant and negative (say -1 , for example). Its fundamental group Γ is a uniform lattice in $PO(n, 1)$. We assume that Γ is an amalgamated product of its subgroup A and B over C , that is $\Gamma = A *_C B$. Here C is a subgroup of both A and B and Γ is the quotient of the free product $A * B$ in which we identify an element $c \in C \subset A$ with itself in $C \subset B$.

We denote by d the distance in the n -dimensional simply-connected hyperbolic space \mathbf{H}^n and we define the critical exponent of a discrete group C acting

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on \mathbf{H}^n , by

$$\delta(C) = \inf\{s > 0; \sum_{\gamma \in C} e^{-sd(\gamma x, x)} < +\infty\}.$$

It does not depend on the point $x \in \mathbf{H}^n$ and in [16], Y. Shalom proves the following theorem,

Theorem 1.1. [16] *Let Γ be a lattice in $PO(n, 1)$. Assume that Γ is an amalgamated product of its subgroups A and B over C . Then, the critical exponent $\delta(C)$ of C satisfies $\delta(C) \geq n - 2$.*

This result proves that if this situation occurs C has to be “big”, and the size is measured by the critical exponent. An example is given by any n -dimensional hyperbolic manifold X which contains a compact separating connected totally geodesic hypersurface Y . The Van Kampen theorem then says that the fundamental group Γ of X is isomorphic to the free product of the fundamental groups of the two halves of $X \setminus Y$ amalgamated over the fundamental group C of the incompressible hypersurface Y . Such examples do exist thanks to A. Lubotsky who showed that any standard arithmetic lattice in $PO(n, 1)$ has a finite index subgroup which is an amalgamated product (see [12]). In these cases there is equality in theorem 1.1, i.e., $\delta(C) = n - 2$ where C is the fundamental group of Y , and Y. Shalom suggested in [16] that the equality case in the theorem 1.1 happens only in that case.

In [3], we extend Shalom’s result to variable curvature and prove the equality case. Indeed, we prove the following theorem,

Theorem 1.2. [3] *Let X be a n -dimensional compact riemannian manifold of sectional curvature $K \leq -1$. We assume that the fundamental group Γ of X is an amalgamated product of its subgroups A and B over C and that neither A nor B equals Γ . Then, the critical exponent $\delta(C)$ of C satisfies $\delta(C) \geq n - 2$. The equality $\delta(C) = n - 2$ happens if and only if C cocompactly preserves a totally geodesic isometrically embedded copy \mathbf{H}^{n-1} of the hyperbolic space of dimension $n - 1$. Moreover, in the equality case, the hypersurface $Y^{n-1} := \mathbf{H}^{n-1}/C$ is embedded in X and separates X in two connected components whose fundamental groups are respectively A and B .*

While the proof given by Shalom is representation-theoretic, ours is purely geometric and relies on the construction of a (family of) map(s) with special properties. It will be briefly described in the sequel.

1.2. Uniform growth of groups

Let us recall that a group is said to be virtually nilpotent if it has a finite index subgroup which is nilpotent. We shall consider discrete subgroups of the isometry group of a Cartan-Hadamard manifold. Let Γ be a discrete group and Σ a finite generating set which we suppose to be symmetric, i.e., if $\sigma \in \Sigma$ then $\sigma^{-1} \in \Sigma$. Any element $\gamma \in \Gamma$ is written as a word in the elements of Σ and we

define the length of γ with respect to Σ to be

$$l_\Sigma(\gamma) = \min\left\{\sum_1^p |q_i| / \gamma = \sigma_{i_1}^{q_1} \dots \sigma_{i_p}^{q_p}, \sigma_{i_j} \in \Sigma\right\},$$

and the rate of growth (or entropy) of Γ with respect to Σ by

$$\text{Ent}_\Sigma(\Gamma) = \lim \frac{1}{N} \log \#\{\gamma \in \Gamma / l_\Sigma(\gamma) \leq N\}.$$

It is not difficult to check that the positiveness of Ent_Σ is independent of the finite generating set Σ , we then set the

Definition 1.3. We say that Γ has *exponential growth* if Ent_Σ is positive for some, hence for all, finite generating set Σ .

Definition 1.4. The *algebraic entropy* (or simply entropy) of Γ is defined by,

$$\text{Ent}(\Gamma) = \inf\{\text{Ent}_\Sigma / \Sigma \text{ finite generating set}\}.$$

The group Γ is said to have *uniform exponential growth* if $\text{Ent}(\Gamma) > 0$.

In [4] we prove the following theorem,

Theorem 1.5. [4] *Let (X^n, g) be a Cartan-Hadamard manifold whose sectional curvature satisfies $-a^2 \leq K_g \leq -1$. Let Γ be a discrete and finitely generated subgroup of the isometry group of (X, g) , then either Γ is virtually nilpotent or its algebraic entropy is bounded below by an explicit constant $C(n, a)$.*

There is quite a few families of groups for which uniform growth is proved. The reader is referred to [4] for a list of examples. Let us however mention the following result by A. Eskin, S. Mozes, and H. Oh (see [8]),

Theorem 1.6. [8] *Let $\Gamma \subset GL(d, \mathbf{K})$ be a finitely generated discrete group, where \mathbf{K} is a field of zero characteristic, then either Γ is virtually nilpotent or it has uniform growth.*

This has been extended to any characteristic by E. Breuillard and T. Gelander (see [5]). Our result is thus a non-linear version of these theorems, as it was the case in the previous subsection.

2. The basic construction

For the sake of simplicity we shall work in the simply connected n -dimensional hyperbolic space \mathbf{H}^n ; most of the constructions, however, extend to Cartan-Hadamard spaces.

2.1. Critical exponent

Let $C \subset \text{Isom}(\mathbf{H}^n)$ be a discrete subgroups of isometries of \mathbf{H}^n . The orbits of C accumulate at the boundary at infinity of \mathbf{H}^n on a set called the limit set of C and denoted $\Lambda(C)$ (see figure 1 below).

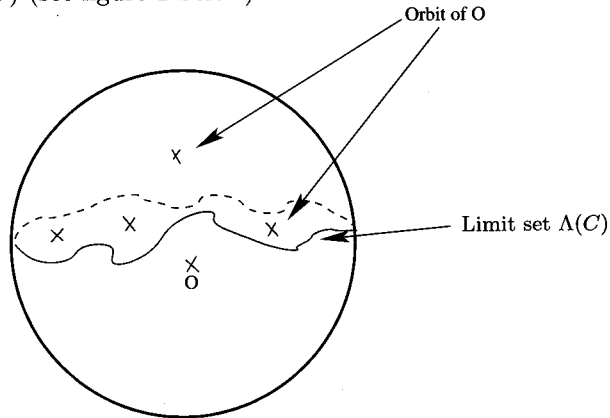


Fig. 1

Let d denote the distance function in \mathbf{H}^n , then if o is a point in \mathbf{H}^n one defines the critical exponent of the group C by,

$$\delta(C) = \inf \left\{ s > 0; \sum_{\gamma \in C} e^{-sd(o, \gamma o)} < +\infty \right\}.$$

The limit set is not rectifiable and in certain cases, for example when C is convex cocompact, $\delta(C)$ is the Hausdorff dimension of $\Lambda(C)$.

Remark 2.1. In the example, namely when C is the fundamental group of a compact hyperbolic totally geodesic hypersurface separating in $X = \mathbf{H}^n/\Gamma$, then $\delta(C) = n - 2$.

2.2. Patterson-Sullivan measures

This is a family of measures associated to the action of C . It is defined in a much more general setting (see [13] and [14]). It is supported by the limit set $\Lambda(C)$ and is uniquely defined by the following two properties:

- i) $\forall \gamma \in C, \forall x \in \mathbf{H}^n, \mu_{\gamma x} = \gamma_* \mu_x,$
- ii) $d\mu_x/d\mu_o(\theta) = \exp(-\delta(C)B(x, \theta)), \theta \in \partial\mathbf{H}^n.$

Here $B(x, \theta)$ is the Busemann function centred at $\theta \in \partial\mathbf{H}^n$ computed at $x \in \mathbf{H}^n$ and normalized in order to vanish identically at the origin o (see [1]). The important fact for our discussion is that it yields a C -equivariant map from \mathbf{H}^n to the space of positive measures on the topological space $\Lambda(C)$:

$$\begin{aligned} \mathbf{H}^n &\longrightarrow \mathcal{M}(\Lambda(C)) \\ x &\longmapsto \mu_x \end{aligned}$$

which allows to “transfer” questions to the boundary at infinity.

Remark 2.2. When C acts cocompactly on a totally geodesic hypersurface $\tilde{Y} \subset \mathbf{H}^n$, the measure μ_x is the visual measure of $S^{n-2} \simeq \partial\tilde{Y}$ at x .

2.3. The barycentre map

We construct here a reciprocal map to the previous one. It assigns to each Patterson-Sullivan measure a point in \mathbf{H}^n . Let us define the following function,

$$(1) \quad \mathcal{B}(y, x) = \int_{\Lambda(C)} \exp(B(y, \theta) - B(x, \theta)) d\mu_x(\theta).$$

The negative curvature of the space allows to show that the map $y \mapsto \mathcal{B}(y, x)$ is strictly convex for any fixed x and goes to infinity when y goes to infinity.

Definition 2.3. The unique critical point of $y \mapsto \mathcal{B}(y, x)$ is called the barycentre of the measure μ_x and is denoted $\text{bar}(\mu_x)$.

Let us now set $\tilde{F}(x) = \text{bar}(\mu_x)$. It is easy to check that it is an equivariant map from \mathbf{H}^n to itself. For a general subgroup C of $\text{Isom}(\mathbf{H}^n)$, \tilde{F} is not the identity; it is however the case when the group is cocompact for example. It is important to notice that \tilde{F} is defined by an easy implicit equation, indeed it satisfies

$$(2) \quad \int_{\Lambda(C)} \exp(B(\tilde{F}(x), \theta) - B(x, \theta)) \nabla B_{(\tilde{F}(x), \theta)} d\mu_x(\theta) = 0.$$

Here ∇B stands for the gradient of the Busemann function taken with respect to its first variable (the point in \mathbf{H}^n). This is a vector valued equation obtained by differentiating (1) at its critical point. The map \tilde{F} is the one we aimed at constructing.

Remark 2.4. When C acts cocompactly on a totally geodesic hypersurface in $\tilde{Y} \subset \mathbf{H}^n$, then \tilde{F} is nothing but the orthogonal projection onto this hypersurface (see Fig 2 below).

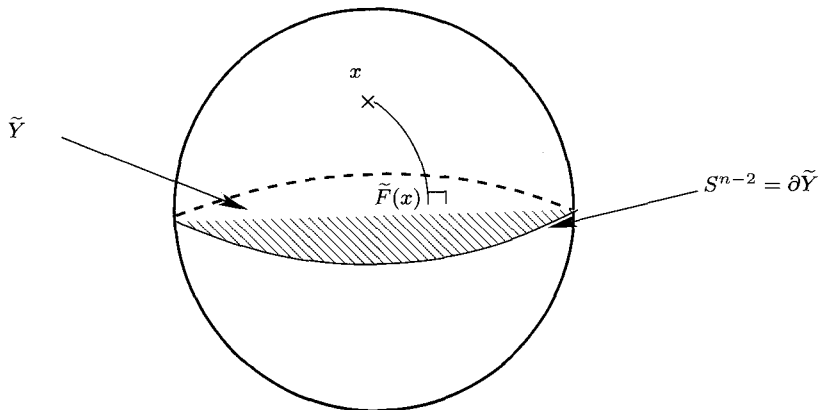


Fig. 2

2.4. Properties of the barycentre map

We define the so-called p -jacobian by,

$$\text{Jac}_p \tilde{F}(x) = \sup \{ |\text{Jac}(\tilde{F}|_E(x))|; E \subset T_x \mathbf{H}^n, \dim E = p \}.$$

Properties

- i) $\text{Jac}_p \tilde{F}(x) \leq \left(\frac{\delta(C)+1}{p} \right)^p, \forall x \in \mathbf{H}^n, 1 \leq p \leq n,$
- ii) If there is equality at some point $x_0 \in \mathbf{H}^n, D_{x_0} \tilde{F}$ is an orthogonal projection onto a p -dimensional subspace $E_0 \subset T_{x_0} \mathbf{H}^n$ composed with an homothety.
- iii) the quotient map $F: \mathbf{H}^n/C \rightarrow \mathbf{H}^n/C$ is homotopic to the identity.

2.5. A quick sketch of proof of theorem 1.2

We sketch the proof of the inequality part of theorem 1.2 in the simple context of a cocompact group Γ acting discretely and without fixed points on \mathbf{H}^n . We assume that Γ is the amalgamation of two subgroups on a third subgroup C . The proof goes along the following steps:

Step 1: Let us first recall that a group is amalgamated if and only if it acts on a simplicial tree with fundamental domain being an edge and the corresponding two vertices (see [15]). We show that the homology group $H_{n-1}(\mathbf{H}^n/C, \mathbf{R})$ is not trivial. This is a simple application of the Mayer-Vietoris exact sequence. It thus exists a non zero $n-1$ -cycle which we can assume to be represented by a compact hypersurface $Z \in \mathbf{H}^n/C$.

Step 2: Now we apply the map F constructed above. the current $F(Z)$ represents the same cycle than Z since F is homotopic to the identity. If F were an immersion then $F(Z)$ would be an immersed submanifold. This is not true in general but by using some standard approximation argument we can assume that it is the case. The properties of F imply that,

$$\text{vol}(F(Z)) \leq \left(\frac{\delta(C)+1}{n-1} \right)^{n-1} \text{vol}(Z).$$

Here the volume is the $n-1$ -volume computed with the induced metric. In particular, by iterating F , if $\delta(C)+1 < n-1$ then $\text{vol}(F^k(Z)) \rightarrow 0$ when k goes to infinity.

Step 3: It is now sufficient to bound from below the volume of a representative of a non zero cycle in $H_{n-1}(\mathbf{H}^n/C, \mathbf{R})$. This is achieved by applying a theorem due to M. Gromov (see [11]). Let i denote the injection $Z \hookrightarrow \mathbf{H}^n/C$, we recall that Z is essential, that is, Z represents a non trivial class in $H_{n-1}(\mathbf{H}^n/C, \mathbf{R})$.

Definition 2.5. Let g be any Riemannian metric on Z and $z \in Z$,

- i) we define $\text{sys}_{g,i}(Z, z) =$ The shortest g -length of a loop c at z such that $i(c)$ is not contractible in \mathbf{H}^n/C ,
- ii) and the relative systole $\text{sys}_{g,i}(Z) = \inf\{\text{sys}_{g,i}(Z, z); z \in Z\}$.

These invariants satisfy the following strong property.

Theorem 2.6. [10] *There exists a constant C_{n-1} depending on the dimension only such that if Z is an essential submanifold one has,*

$$\text{vol}_g(Z) \geq C_{n-1}(\text{sys}_{g,i}(Z))^{n-1}.$$

Here the volume is computed with the metric g . Now it is not difficult to check that, if we take the metric induced by the injection i , the relative systole is bounded below by the injectivity radius of the manifold $X = \mathbf{H}^n/\Gamma$ which is independent of the representative Z . This gives a lower bound on the volumes of the iterates $F^k(Z)$ and yields a contradiction if $\delta(C) + 1 < n - 1$. We thus have proved that $\delta(C) \geq n - 2$.

Step 4: The last step is the equality case. The rough idea is that if one would have an area minimizing essential submanifold (or a pseudo-manifold in general, see the definition in [10]) then it would have to be fixed by F . In general this is not the case and the idea is to iterate F in order to produce such a minimizing object. We however have to prevent the image of F from shifting to infinity. This can be done by using the existence of a L^2 -harmonic form such that $\int_Z \omega \neq 0$, a result due to G. Carron and E. Pedon (see [7]). We finally show that there exists a point $x_0 \in \mathbf{H}^n$ such that $\text{Jac}_{n-1} \tilde{F}(x_0) = 1$ and this is sufficient to conclude. In the case of a general negatively curved Cartan-Hadamard manifold, that is for theorem 1.2, the equality case is much more difficult and the reader is referred to [3] for all the details.

3. A variation on the same theme

We now intend to sketch the proof of theorem 1.5. We keep the notation of the subsection 1.2. For the sake of simplicity we suppose that Γ acts without fixed points and that all its elements are hyperbolic (there are translations along axis). Margulis lemma (see [6]) states that there exists a number $\mu = \mu(n, a) > 0$ such that, for all $x \in X$, the subgroup of Γ

$$\Gamma_\mu(x) = \text{subgroup generated by } \{ \gamma \in \Gamma; \rho(x, \gamma x) < \mu \}$$

is almost nilpotent, i.e., it has a finite index subgroup which is nilpotent. Here ρ denotes the distance in the manifold X . We can show the following lemma.

Lemma 3.1. *If Γ is not almost nilpotent, then for all finite generating set Σ , there exist two elements σ_1 and σ_2 in Σ such that,*

$$L = \inf_{x \in X} \max \{ \rho(x, \sigma_1 x), \rho(x, \sigma_2 x) \} \geq \mu.$$

The proof consists in analyzing all the cases; if the lemma is not true then any subgroup of Γ generated by two elements in Σ is almost nilpotent, one then shows that this implies that Γ itself is almost nilpotent. For the sake of simplicity again we shall assume from now on that the infimum in the above

lemma is achieved at some point $x_0 \in X$. Let us call σ_1 and σ_2 the elements in Σ such that,

$$\max \{ \rho(x_0, \sigma_1 x_0), \rho(x_0, \sigma_2 x_0) \} = L,$$

and define Λ to be the subgroup of Γ generated by σ_1 and σ_2 ; we denote $S = \{ \sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1} \}$. Theorem 1.5 is a corollary of the

Proposition 3.2. *With the above notations one has,*

$$\text{Ent}_S(\Lambda) \geq c(\mu) = c(n, a),$$

for some computable constant $c(n, a)$.

Corollary 3.3. *If Γ is not almost nilpotent, then for any finite generating set Σ ,*

$$\text{Ent}_\Sigma(\Gamma) \geq c(n, a) > 0,$$

and hence $\text{Ent}(\Gamma) \geq c(n, a)$.

The proof of the proposition is done in two steps.

Step 1: Let us denote $B_S(4)$ the ball of radius 4 in Λ for the word distance associated to the generating set S . We assume that there exists $\gamma \in B_S(4)$ such that

$$l(\gamma) = \inf \{ \rho(x, \gamma x); x \in X \} \geq \delta > 0,$$

where δ is a number to be chosen later. The element γ is an hyperbolic isometry. Now there exists $\sigma_i \in S$ such that γ and $\sigma_i \gamma \sigma_i^{-1}$ have distinct axis (we shall set $i = 1$). Indeed, if not then $\sigma_i, i = 1, 2$ would preserve the axis of γ and then Λ would be virtually abelian. A quantitative Ping-Pong argument then shows that there exists a number N such that one of the two semigroup $\langle \gamma^N, (\sigma_1 \gamma \sigma_1^{-1})^N \rangle$ or $\langle \gamma^N, (\sigma_1 \gamma \sigma_1^{-1})^{-N} \rangle$ is free. Here $\langle \cdot, \cdot \rangle$ stands for “semigroup generated by” and by semigroup we mean the products of nonnegative powers of the generators. This works as soon as $N > 4/\delta$ (see [4]). This shows that

$$\text{Ent}_S(\Lambda) \geq \frac{\delta \log 2}{24 + 6\delta}.$$

Remark 3.4. The question of finding a number N which free (or “semifree”) any two elements in a given group is interesting in itself. One can look for such a number depending on the group under consideration or, even better, valid for a large family of groups, for example fundamental groups of hyperbolic manifolds. At this stage we do not have any satisfactory general answer.

Step 2: We are now in the case where for all $\gamma \in B_S(4)$, the displacement of γ satisfies $l(\gamma) \leq \delta < L$. We take $\delta = \log(\cosh(L/4))$. We shall construct a map from the Cayley graph of (Λ, S) to X with properties comparable to the barycentre map of the previous section. Let us consider,

$$\begin{aligned} X &\longrightarrow \mathbf{R} \\ y &\longmapsto \sum_{\gamma \in \Lambda} e^{\rho(y, \gamma x_0)} e^{-cd_\Lambda(s, \gamma)} \end{aligned}$$

where d_Λ is the distance on the Cayley graph of (Λ, S) such that each edge has length 1, c is a real number such that the series converges, e is the neutral element of Λ and s is a point on the cayley graph (considered as a 1-dimensional graph). We may call this series the Poincaré series associated to Λ . As in the subsection 2.1 one can show that the above map is convex and goes to infinity with y . It thus has unique critical point x which is a minimum. We set

$$F_c(s) = x.$$

This gives a map from the Cayley graph to X which is equivariant for the action of Λ on its Cayley graph and on X as a subgroup of Γ (see Fig 3 below).

Remark 3.5. We left aside some technical details. For example, the distance functions are not differentiable everywhere. This can be circumvented easily and the reader is referred to [4].

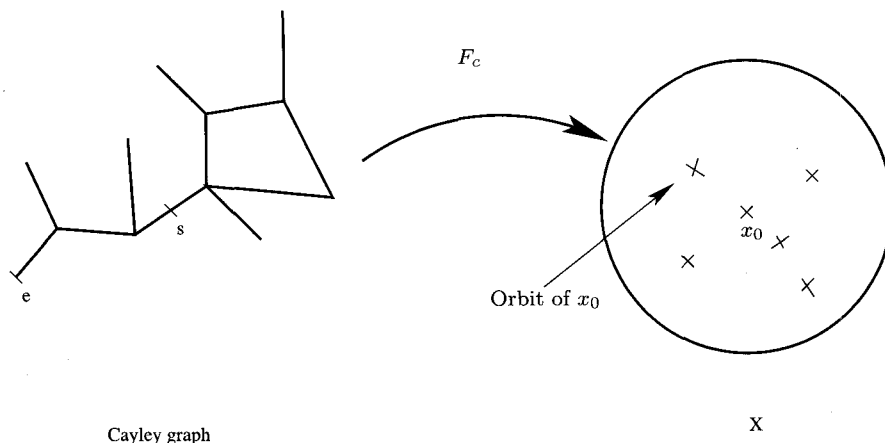


Fig. 3

The maps F_c can be defined by an implicit equation as before, ignoring some technical details. They have nice properties which we summarize now.

Properties

- i) F_c is defined for c big enough (see below),
- ii) It is Λ -equivariant,
- iii) It is Lipschitzian and, $\forall s, |D_s F_c| \leq c$.

We give a heuristic proof of the proposition. We call “asymptotic stretch” or simply “stretch” the number,

$$E = \limsup_{\gamma \rightarrow \partial\Lambda} \frac{\rho(x_0, \gamma x_0)}{d_\Lambda(e, \gamma)}.$$

Then for big γ 's, $\rho(x_0, \gamma x_0) \sim E d_\Lambda(e, \gamma)$ which yields,

$$-\rho(x_0, \gamma x_0) + c d_\Lambda(e, \gamma) \sim (-E + c) d_\Lambda(e, \gamma).$$

The series used to define F_c converges then when the decay of the summand is greater than the growth of the number of elements in Λ of a given word length, that is when $-E + c > \text{Ent}_S(\Lambda)$. The critical value of c (this is a critical exponent in the sense of subsection 2.1) is thus $E + \text{Ent}_S(\Lambda)$. Now, the properties of F show that,

$$L \leq \max_{i=1,2} \{ \rho(F_c(e), F_c(\sigma_i e)) \} = \max_{i=1,2} \{ \rho(F_c(e), \sigma_i F_c(e)) \} \leq c \simeq E + \text{Ent}_S(\Lambda).$$

If we can prove that $L > E$ we get,

$$\text{Ent}_S(\Lambda) \geq L - E > 0,$$

which is the desired inequality (if $L - E$ can be computed explicitly). One can understand L as being the stretch at x_0 . Since E is the asymptotic stretch, $L - E = 0$ means intuitively that the group is elementary (see Fig 4 below).

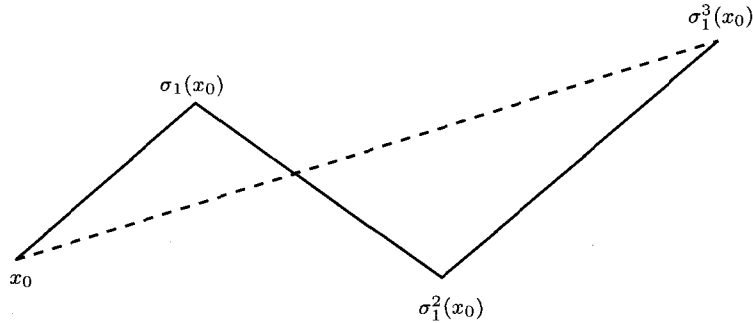


Fig. 4

We emphasise that this is not a proof, it is just a heuristic and intuitive description of the ideas involved in the proof of theorem 1.5. We can be a little bit more precise.

Definition 3.6. Let $\eta > 0$ and $\gamma \in \Lambda$, we say that γ is η -straight (at x_0) if

$$\rho(x_0, \gamma x_0) \geq (L - \eta)l_S(\gamma).$$

We recall that l_S is the word length in Λ computed with the generating set S .

The triangular inequality shows that $\rho(x_0, \gamma x_0) \leq l_S(\gamma)L$ by definition of x_0 . The aim is now to show that in our context there are lots of elements in Λ which are not η -straight. Here the negative curvature is crucial and the idea is that if $\gamma \in B_S(4)$ is such that $l(\gamma) \leq \delta < L$ then x_0 is far from the axis of γ . Precisely, we proved

Lemma 3.7. *If $\forall \gamma \in B_S(4)$, $l(\gamma) \leq \delta$, every reduced word of word-length exactly 6 in Λ is non η -straight, for*

$$\eta = \frac{1}{1000} \left(1 - \frac{\cosh(L/4)}{\cosh(L/2)} \right)^4.$$

An immediate consequence is

Lemma 3.8. *If $\forall \gamma \in B_S(4)$, $l(\gamma) \leq \delta$, the Poincaré series converges $\forall c > L - \eta + \text{Ent}_S(\Lambda)$.*

Here $L - \eta$ plays the role of the asymptotic stretch of the heuristic approach so that η is the bound from below of $\text{Ent}_S(\Lambda)$. In order to get a lower bound depending on n and a one replaces L by $\mu(n, a)$. The proof of these lemmas rely on an obvious geometric estimate. Precisely if $\gamma \in \Lambda$ is such that $l(\gamma) \leq \delta$ and $\rho(x_0, \gamma x_0) > L/2$ then,

$$\rho(x_0, \gamma^2 x_0) \leq 2\rho(x_0, \gamma x_0) - \left(1 - \frac{e^\delta}{\cosh(L/4)}\right).$$

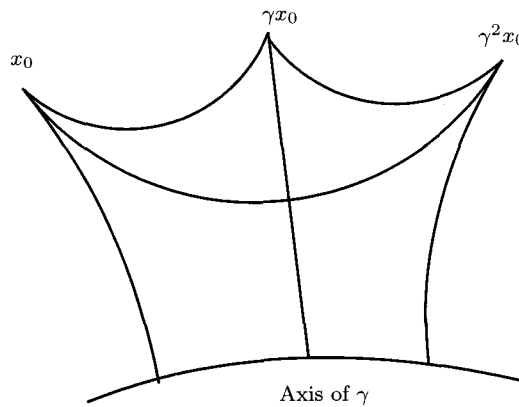


Fig. 5

- Remark 3.9.*
- (1) In [4] a serious difficulty comes from the possibility of having elliptic elements in Γ . However the arguments in step 2 apply when σ_1 and σ_2 are elliptic in which case $l(\sigma_i) = 0$.
 - (2) The negative upper bound on the sectional curvature is crucial for most geometric arguments in this proof whereas the lower bound is used only for the Margulis lemma. It is not yet clear whether the theorem 1.5 is true with the assumption $K_g \leq -1$ only.

4. Conclusion

The construction of special maps between spaces is a powerful tool which is illustrated by the success of harmonic maps in proving superrigidity results. Harmonic maps are related to the energy. The above constructions, which are different from the original one described in [2], give maps whose main property is the decrease of p -volumes and either the critical exponent of a group or its growth, which are two version of the entropy, can be estimated. In the

first construction the map goes from one space to itself and is equivariant with respect to the group action. Intuitively it projects the whole space onto a minimising pseudo-manifold for the $n - 1$ -volume which we think of the “best” representative of the cycle. When we try to adapt the construction to two different spaces a new invariant has to be used, it is what we called the asymptotic stretch which relates the two metrics (again, this is only intuitive). In this second case the map also tries to find the “best” position for putting the Cayley graph into the manifold.

There is room for more variations on this theme and the authors are working on different approaches adapted to other context. It is however important to notice that these maps, as well as harmonic maps, cannot do everything as it is proved in the series of works by T. Farrell and P. Ontaneda (see [9], for example).

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