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PARTS FORMULAS INVOLVING INTEGRAL TRANSFORMS ON FUNCTION SPACE

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Abstract. In this paper we establish several integration by parts formulas involving integral transforms of functionals of the form $F(y) = f(\langle\theta_1, y\rangle, \ldots, \langle\theta_n, y\rangle)$ for s-a.e. $y \in C_0[0,T]$, where $\langle\theta, y\rangle$ denotes the Riemann-Stieltjes integral $\int_0^T \theta(t)\, dy(t)$.

1. Introduction and definitions

In a unifying paper [9], Lee defined an integral transform $\mathcal{F}_{\alpha,\beta}$ of analytic functionals on an abstract Wiener space. For certain values of the parameters α and β and for certain classes of functionals, the Fourier-Wiener transform [2], the Fourier-Feynman transform [3] and the Gauss transform are special cases of his integral transform $\mathcal{F}_{\alpha,\beta}$. In [5], Chang, Kim and Yoo established an interesting relationship between the integral transform and the convolution product for functionals on an abstract Wiener space. In this paper we establish several integration by parts formulas involving integral transforms, convolution products, and the first variations of functionals of the form $F(y) = f(\langle\theta_1, y\rangle, \ldots, \langle\theta_n, y\rangle)$ for s-a.e. $y \in C_0[0,T]$, where $\langle\theta, y\rangle$ denotes the Riemann-Stieltjes integral $\int_0^T \theta(t)\, dy(t)$.

Let $C_0[0,T]$ denote one-parameter Wiener space; that is the space of all real-valued continuous functions $x(t)$ on $[0,T]$ with $x(0) = 0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$ and let m denote Wiener measure. $(C_0[0,T], \mathcal{M}, m)$ is a complete measure space and we denote the Wiener integral of a Wiener integrable functional F by

$$\int_{C_0[0,T]} F(x)\, m(dx). \tag{1.1}$$

Let α and β be nonzero complex numbers. Next we state the definitions of the integral transform $\mathcal{F}_{\alpha,\beta}F$, the convolution product $(F * G)_\alpha$ and the first

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variation δF for functionals defined on $K = K[0,T]$, the space of complex-valued continuous functions defined on $[0,T]$ which vanish at $t = 0$.

Definition 1.1. Let F be a functional defined on K. Then the integral transform $\mathcal{F}_{\alpha,\beta}F$ of F is defined by

$$\mathcal{F}_{\alpha,\beta}(F)(y) \equiv \mathcal{F}_{\alpha,\beta}F(y) \equiv \int_{C_0[0,T]} F(\alpha x + \beta y)\, m(dx), \quad y \in K \tag{1.2}$$

if it exists [5, 7, 8, 9].

Definition 1.2. Let F and G be functionals defined on K. Then the convolution product $(F * G)_\alpha$ of F and G is defined by

$$(F * G)_\alpha(y) \equiv \int_{C_0[0,T]} F\Big(\frac{y + \alpha x}{\sqrt{2}}\Big) G\Big(\frac{y - \alpha x}{\sqrt{2}}\Big)\, m(dx), \quad y \in K \tag{1.3}$$

if it exists [5, 6, 7, 13, 14].

Definition 1.3. Let F be a functional defined on K and let $w \in K$. Then the first variation δF of F is defined by

$$\delta F(y|w) \equiv \frac{\partial}{\partial t} F(y + tw)|_{t=0}, \quad y \in K \tag{1.4}$$

if it exists [1, 4, 7, 11].

Let $\{\theta_1, \theta_2, \ldots\}$ be a complete orthonormal set of real-valued functions in $L_2[0,T]$ and assume that each θ_j is of bounded variation on $[0,T]$. Then for each $y \in K$ and $j \in \{1, 2, \ldots\}$, the Riemann-Stieltjes integral $\langle \theta_j, y \rangle \equiv \int_0^T \theta_j(t)\, dy(t)$ exists. Furthermore

$$|\langle \theta_j, y \rangle| = |\theta_j(T) y(T) - \int_0^T y(t)\, d\theta_j(t)| \le C_j \|y\|_\infty \tag{1.5}$$

with

$$C_j = |\theta_j(T)| + \mathrm{Var}(\theta_j, [0,T]), \tag{1.6}$$

where $\mathrm{Var}(\theta_j, [0,T])$ denote the total variation of θ_j on $[0,T]$.

Next we describe the class of functionals which is related to this paper. For $0 \le \sigma < 1$, let E_σ be the space of all functionals $F : K \to \mathbb{C}$ of the form

$$F(y) = f(\langle \vec{\theta}, y \rangle) = f(\langle \theta_1, y \rangle, \ldots, \langle \theta_n, y \rangle) \tag{1.7}$$

for some positive integer n, where $f(\vec{\lambda}) = f(\lambda_1, \ldots, \lambda_n)$ is an entire function of the n complex variables $\lambda_1, \ldots, \lambda_n$ of exponential type; that is to say,

$$|f(\vec{\lambda})| \le A_F \exp\{B_F |\vec{\lambda}|^{1+\sigma}\} \tag{1.8}$$

for some positive constants A_F and B_F, where $|\vec{\lambda}|^{1+\sigma} = \sum_{j=1}^{n} |\lambda_j|^{1+\sigma}$.

In addition we use the notation

$$F_j(y) = f_j(\langle \vec{\theta}, y \rangle)$$

where $f_j(\vec{\lambda}) = \frac{\partial}{\partial \lambda_j} f(\lambda_1, \ldots, \lambda_n)$ for $j = 1, \ldots, n$.

Recently [7], Kim, Kim and Skoug established the results that if F and G are elements of E_σ then $\mathcal{F}_{\alpha,\beta}F$, $(F*G)_\alpha$, $\delta F(\cdot|w)$ and $\delta F(y|\cdot)$ are also elements of E_σ and examined various relationships holding among $\mathcal{F}_{\alpha,\beta}F$, $\mathcal{F}_{\alpha,\beta}G$, $(F*G)_\alpha$, δF and δG. For related work see [2, 5, 6, 7, 9, 11, 13, 14] and for a detailed survey of previous work see [12].

2. Integration by parts formulas

We begin this section by introducing three existence theorems for the integral transform, the convolution product and the first variation of functionals in E_σ are established in [7]. Although they considered only for functionals in E_0 in [7], as they commented in Remark 5.6 of their paper, their results can be extended for functionals in E_σ.

Theorem 2.1. *Let $F \in E_\sigma$ be given by (1.7). Then the integral transform $\mathcal{F}_{\alpha,\beta}F$ exists, belongs to E_σ and is given by the formula*

$$\mathcal{F}_{\alpha,\beta}F(y) = h(\langle \vec{\theta}, y\rangle) \tag{2.1}$$

for $y \in K$, where

$$h(\vec{\lambda}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(\alpha\vec{u} + \beta\vec{\lambda}) \exp\Big\{-\frac{1}{2}\|\vec{u}\|^2\Big\}\, d\vec{u} \tag{2.2}$$

where $\|\vec{u}\|^2 = \sum_{j=1}^{n} u_j{}^2$ and $d\vec{u} = du_1 \cdots du_n$.

Theorem 2.2. *Let $F, G \in E_\sigma$ be given by (1.7) with corresponding entire functions f and g, respectively. Then the convolution $(F*G)_\alpha$ exists, belongs to E_σ and is given by the formula*

$$(F*G)_\alpha(y) = k(\langle \vec{\theta}, y\rangle) \tag{2.3}$$

for $y \in K$, where

$$k(\vec{\lambda}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f\Big(\frac{\vec{\lambda} + \alpha\vec{u}}{\sqrt{2}}\Big) g\Big(\frac{\vec{\lambda} - \alpha\vec{u}}{\sqrt{2}}\Big) \exp\Big\{-\frac{1}{2}\|\vec{u}\|^2\Big\}\, d\vec{u}. \tag{2.4}$$

Theorem 2.3. *Let $F \in E_\sigma$ be given by (1.7) and let $w \in K$. Then*

$$\delta F(y|w) = p(\langle \vec{\theta}, y\rangle) \tag{2.5}$$

for $y \in K$, where

$$p(\vec{\lambda}) = \sum_{j=1}^{n} \langle \theta_j, w\rangle f_j(\vec{\lambda}). \tag{2.6}$$

Furthermore, as a function of $y \in K$, $\delta F(y|w)$ is an element of E_σ.

Now we state some observations which we use later in this paper. First of all, equation (1.2) implies that

$$\mathcal{F}_{\alpha,\beta}F(y/\sqrt{2}) = \mathcal{F}_{\alpha,\beta/\sqrt{2}}F(y) \tag{2.7}$$

for all $y \in K$. Next, a direct calculation using (1.4), (1.2), (2.5) and (2.7) shows that

$$\begin{aligned} \delta\mathcal{F}_{\alpha,\beta}F(y/\sqrt{2}|w/\sqrt{2}) &= \delta\mathcal{F}_{\alpha,\beta/\sqrt{2}}F(y|w) \\ &= \frac{\beta}{\sqrt{2}}\sum_{j=1}^{n}\langle\theta_j, w\rangle\mathcal{F}_{\alpha,\beta/\sqrt{2}}F_j(y) \end{aligned} \tag{2.8}$$

for all y and w in K. Finally, by similar calculations, we obtain that

$$\mathcal{F}_{\alpha,\beta}(\delta F(\cdot|w))(y/\sqrt{2}) = \frac{\sqrt{2}}{\beta}\delta\mathcal{F}_{\alpha,\beta/\sqrt{2}}F(y|w) \tag{2.9}$$

for all y and w in K, and for all $y \in K$,

$$\left(\mathcal{F}_{\alpha,\beta}F\right)_j(y) = \beta\mathcal{F}_{\alpha,\beta}F_j(y). \tag{2.10}$$

Let

$$A = \{y \in C_0[0,T] : y \text{ is absolutely continuous on } [0,T] \text{ with } y' \in L^2[0,T]\}.$$

We note that if we choose $z \in L_2[0,T]$ and define $w(t) = \int_0^t z(s)ds$ for $t \in [0,T]$, then w is an element of A, $w' = z$ a.e. on $[0,T]$, and for all $v \in L_2[0,T]$, $\langle v, w\rangle = (v, w') = (v, z)$, where $(v, z) = \int_0^T v(s)z(s)ds$.

The following theorem plays a key role throughout this paper. In this theorem the Wiener integral of the first variation of functional F is expressed in terms of the Wiener integral of F multiplied by a linear factor.

Theorem 2.4. *Let $F \in E_\sigma$ be given by* (1.7) *and $w \in A$, then*

$$\int_{C_0[0,T]} \delta F(x|w)\, m(dx) = \int_{C_0[0,T]} F(x)\langle z, x\rangle\, m(dx) \tag{2.11}$$

where $w(t) = \int_0^t z(s)\, ds$ on $[0,T]$ for some $z \in L_2[0,T]$.

Proof. Let $w(t) = \int_0^t z(s)\, ds$ for some $z \in L^2[0,T]$. Using the Gram-Schmit process we can find an orthonormal set $\{\theta_1, \dots, \theta_n, \theta_{n+1}\}$ with $\theta_{n+1} = \frac{1}{\|z_{n+1}\|}z_{n+1}$, where

$$z_{n+1} = z - \sum_{j=1}^{n}(\theta_j, z)\theta_j.$$

Then by the Wiener integration formula

$$\begin{aligned}
&\int_{C_0[0,T]} F(x)\langle z,x\rangle\, m(dx) \\
&= \int_{C_0[0,T]} f(\langle\vec{\theta},x\rangle)\Big(\sum_{j=1}^{n}(\theta_j,z)\langle\theta_j,x\rangle + \|z_{n+1}\|\langle\theta_{n+1},x\rangle\Big)\, m(dx) \\
&= (2\pi)^{-(n+1)/2}\int_{\mathbb{R}^{n+1}} f(\vec{u})\Big(\sum_{j=1}^{n}(\theta_j,z)u_j + \|z_{n+1}\|u_{n+1}\Big) \\
&\qquad\qquad \times \exp\Big\{-\frac{1}{2}\|\vec{u}\|^2 - \frac{1}{2}u_{n+1}^2\Big\}du_{n+1}d\vec{u}.
\end{aligned}$$

If we evaluate the last integral with respect to u_{n+1}, we obtain

$$\int_{C_0[0,T]} F(x)\langle z,x\rangle\, m(dx) = (2\pi)^{-n/2}\sum_{j=1}^{n}(\theta_j,z)\int_{\mathbb{R}^n} f(\vec{u})u_j\exp\Big\{-\frac{1}{2}\|\vec{u}\|^2\Big\}\, d\vec{u}.$$

On the other hand, since $w \in A \subset K$, by Theorem 2.3

$$\delta F(x|w) = \sum_{j=1}^{n}\langle\theta_j,w\rangle f_j(\langle\vec{\theta},x\rangle) = \sum_{j=1}^{n}(\theta_j,z)f_j(\langle\vec{\theta},x\rangle).$$

Hence by the Wiener integration formula

$$\begin{aligned}
\int_{C_0[0,T]} \delta F(x|w)\, m(dx) &= \sum_{j=1}^{n}(\theta_j,z)\int_{C_0[0,T]} f_j(\langle\vec{\theta},x\rangle)\, m(dx) \\
&= (2\pi)^{-n/2}\sum_{j=1}^{n}(\theta_j,z)\int_{\mathbb{R}^n} f_j(\vec{u})\exp\Big\{-\frac{1}{2}\|u\|^2\Big\}\, d\vec{u}.
\end{aligned}$$

Note that for each $j = 1,\dots,n$, the integration by parts formula yields

$$\begin{aligned}
&\int_{\mathbb{R}} f_j(\vec{u})\exp\Big\{-\frac{1}{2}u_j^2\Big\}\, du_j \\
&= \lim_{b\to\infty}\lim_{a\to-\infty}\Big[f(\vec{u})\exp\Big\{-\frac{1}{2}u_j^2\Big\}\Big]_a^b + \int_{\mathbb{R}} f(\vec{u})u_j\exp\Big\{-\frac{1}{2}u_j^2\Big\}\, du_j.
\end{aligned}$$

But since f is of exponential type, the double limit in the last equation is equal to 0 and so

$$\int_{\mathbb{R}} f_j(\vec{u})\exp\Big\{-\frac{1}{2}u_j^2\Big\}\, du_j = \int_{\mathbb{R}} f(\vec{u})u_j\exp\Big\{-\frac{1}{2}u_j^2\Big\}\, du_j.$$

Hence

$$\int_{C_0[0,T]} \delta F(x|w)\, m(dx) = (2\pi)^{-n/2}\sum_{j=1}^{n}(\theta_j,z)\int_{\mathbb{R}^n} f(\vec{u})u_j\exp\Big\{-\frac{1}{2}\|u\|^2\Big\}\, d\vec{u}$$

and this completes the proof. □

In our next theorem we obtain an integration by parts formula for the products of functionals in E_σ.

Theorem 2.5. *Let $F, G \in E_\sigma$ be given by (1.7) with corresponding entire functions f and g, respectively. Then for $w \in A$, we have the following integration by parts formula.*

(2.12)
$$\int_{C_0[0,T]} [F(y)\delta G(y|w) + \delta F(y|w)G(y)]m(dy) = \int_{C_0[0,T]} F(y)G(y)\langle z, y\rangle m(dy),$$

where $w(t) = \int_0^t z(s)\,ds$ for some $z \in L_2[0,T]$.

Proof. Define $H(y) = F(y)G(y)$ for $y \in K$ and let $h(\vec{\lambda}) = f(\vec{\lambda})g(\vec{\lambda})$. Then $H \in E_\sigma$ and

$$\begin{aligned}\delta H(y|w) &= \sum_{j=1}^{n}\langle\theta_j, w\rangle f_j(\langle\vec{\theta}, y\rangle)g(\langle\vec{\theta}, y\rangle) + f(\langle\vec{\theta}, y\rangle)\sum_{j=1}^{n}\langle\theta_j, w\rangle g_j(\langle\vec{\theta}, y\rangle)\\ &= \delta F(y|w)G(y) + F(y)\delta G(y|w).\end{aligned}$$

Thus equation (2.12) follows from Theorem 2.4. □

By choosing $G = F$ in Theorem 2.5, we obtain the following corollary.

Corollary 2.6. *Let $F \in E_\sigma$ be given by (1.7). Then for each $w \in A$,*

$$\int_{C_0[0,T]} F(y)\delta F(y|w)m(dy) = \frac{1}{2}\int_{C_0[0,T]} [F(y)]^2\langle z, y\rangle m(dy), \tag{2.13}$$

where $w(t) = \int_0^t z(s)\,ds$ for some $z \in L_2[0,T]$.

As we saw in Theorem 2.3 above if F belongs to E_σ, then $\delta F(y|w_1)$ also belongs to E_σ as a function of y. Thus if we replace $G(y)$ with $\delta F(y|w_1)$ in Theorem 2.5, then we have the following corollary.

Corollary 2.7. *Let $F \in E_\sigma$ be given by (1.7). Then for each $w_1, w_2 \in A$,*

$$\begin{aligned}&\int_{C_0[0,T]} [F(y)\delta^2 F(\cdot|w_1)(y|w_2) + \delta F(y|w_2)\delta F(y|w_1)]m(dy)\\ &= \int_{C_0[0,T]} F(y)\delta F(y|w_1)\langle z_2, y\rangle m(dy),\end{aligned} \tag{2.14}$$

where $w_i(t) = \int_0^t z_i(s)\,ds$ for some $z_i \in L_2[0,T], i = 1, 2$.

As we saw in Theorem 2.1 above if G belongs to E_σ, then $\mathcal{F}_{\alpha,\beta}G$ also belongs to E_σ. Thus if we replace G with $\mathcal{F}_{\alpha,\beta}G$ in Theorem 2.5, then we have the following corollary.

Corollary 2.8. *Let $F, G \in E_\sigma$ be given as in Theorem 2.5. Then for each $w \in A$,*

$$\begin{aligned}&\int_{C_0[0,T]} [F(y)\delta\mathcal{F}_{\alpha,\beta}G(y|w) + \delta F(y|w)\mathcal{F}_{\alpha,\beta}G(y)]\, m(dy)\\ &= \int_{C_0[0,T]} F(y)\mathcal{F}_{\alpha,\beta}G(y)\langle z, y\rangle m(dy),\end{aligned} \tag{2.15}$$

where $w(t) = \int_0^t z(s)ds$ for some $z \in L_2[0,T]$.

By replacing F and G by $\mathcal{F}_{\alpha,\beta}F$ and $\mathcal{F}_{\alpha,\beta}G$, respectively, in Theorem 2.5, we obtain the following corollary.

Corollary 2.9. *Let $F, G \in E_\sigma$ be as in Theorem 2.5. Then for each $w \in A$,*

$$\begin{aligned}&\int_{C_0[0,T]} [\mathcal{F}_{\alpha,\beta}F(y)\delta\mathcal{F}_{\alpha,\beta}G(y|w) + \delta\mathcal{F}_{\alpha,\beta}F(y|w)\mathcal{F}_{\alpha,\beta}G(y)]m(dy)\\ &= \int_{C_0[0,T]} \mathcal{F}_{\alpha,\beta}F(y)\mathcal{F}_{\alpha,\beta}G(y)\langle z, y\rangle m(dy),\end{aligned} \tag{2.16}$$

where $w(t) = \int_0^t z(s)ds$ for some $z \in L_2[0,T]$.

3. Various integration formulas and examples

In this section we establish various integration formulas involving integral transforms, convolution products and first variations. Furthermore we give some examples to illustrate the integration formulas in this paper.

In [7], Kim, Kim and Skoug established various relationships holding among $\mathcal{F}_{\alpha,\beta}F$, $\mathcal{F}_{\alpha,\beta}G$, $(F * G)_\alpha$, δF and δG. From these relationships and the results in Section 2 above, we can establish various integration formulas.

From Theorem 2.4 above we know that the Wiener integral of the first variation of functional $F \in E_\sigma$ is expressed in terms of the Wiener integral of F multiplied by a linear factor. On the other hand, some of the formulas, for example, Formulas 3.3, 3.5, 4.1, 4.2, and 5.2 in [7] give us the expressions of the first variation of various functionals. Hence it is easy to obtain the following formulas (3.1) through (3.7) below. We just state the formulas without proofs.

Let $w \in A$ with $w(t) = \int_0^t z(s)ds$ for some $z \in L_2[0,T]$ throughout this section. The paper [7] was concerned with the class E_0. But as commented in Remark 5.6 of that paper, all the formulas in [7] still true for functionals in E_σ. Hence we will assume that $F \in E_\sigma$ in Formula 3.1 through Formula 3.6 and Corollary 3.7 below.

Formula 3.1. From Formula 3.3 of [7], we have

$$\beta \int_{C_0[0,T]} \mathcal{F}_{\alpha,\beta}\delta F(\cdot|w)(y)m(dy) = \int_{C_0[0,T]} \mathcal{F}_{\alpha,\beta}F(y)\langle z, y\rangle m(dy). \tag{3.1}$$

Formula 3.2. From Formula 3.5 of [7], we have

$$\begin{aligned}(3.2)\qquad &\sum_{j=1}^{n}\frac{\langle\theta_j,w\rangle}{\sqrt{2}}\int_{C_0[0,T]}[(F_j*G)_\alpha(y)+(F*G_j)_\alpha(y)]\,m(dy)\\ &=\int_{C_0[0,T]}(F*G)_\alpha(y)\langle z,y\rangle m(dy)\end{aligned}$$

and if $F=G$,

(3.3)

$$\sqrt{2}\sum_{j=1}^{n}\langle\theta_j,w\rangle\int_{C_0[0,T]}(F*F_j)_\alpha(y)m(dy)=\int_{C_0[0,T]}(F*F)_\alpha(y)\langle z,y\rangle m(dy).$$

Formula 3.3. From Formula 4.1 of [7], we have

(3.4)

$$\begin{aligned}&\int_{C_0[0,T]}\Big(\mathcal{F}_{\alpha,\beta/\sqrt{2}}F(y)\delta\mathcal{F}_{\alpha,\beta/\sqrt{2}}G(y|w)+\delta\mathcal{F}_{\alpha,\beta/\sqrt{2}}F(y|w)\mathcal{F}_{\alpha,\beta/\sqrt{2}}G(y)\Big)m(dy)\\ &=\int_{C_0[0,T]}\Big(\mathcal{F}_{\alpha,\beta}F\Big(\frac{y}{\sqrt{2}}\Big)\delta\mathcal{F}_{\alpha,\beta}G\Big(\frac{y}{\sqrt{2}}|\frac{w}{\sqrt{2}}\Big)+\delta\mathcal{F}_{\alpha,\beta}F\Big(\frac{y}{\sqrt{2}}|\frac{w}{\sqrt{2}}\Big)\mathcal{F}_{\alpha,\beta}G\Big(\frac{y}{\sqrt{2}}\Big)\Big)m(dy)\\ &=\int_{C_0[0,T]}\mathcal{F}_{\alpha,\beta}(F*G)_\alpha(y)\langle z,y\rangle m(dy)\end{aligned}$$

and if $F=G$,

$$\begin{aligned}(3.5)\qquad &\int_{C_0[0,T]}\mathcal{F}_{\alpha,\beta}\Big[F\Big(\frac{y}{\sqrt{2}}\Big)\Big]\Big[\delta\mathcal{F}_{\alpha,\beta}F\Big(\frac{y}{\sqrt{2}}|\frac{w}{\sqrt{2}}\Big)\Big]\,m(dy)\\ &=\frac{1}{2}\int_{C_0[0,T]}\mathcal{F}_{\alpha,\beta}(F*F)_\alpha(y)\langle z,y\rangle m(dy).\end{aligned}$$

Formula 3.4. From Formula 4.2 of [7], we have

(3.6)

$$\begin{aligned}&\frac{\beta}{\sqrt{2}}\sum_{j=1}^{n}\langle\theta_j,w\rangle\int_{C_0[0,T]}[(\mathcal{F}_{\alpha,\beta}F_j*\mathcal{F}_{\alpha,\beta}G)_\alpha(y)+(\mathcal{F}_{\alpha,\beta}F*\mathcal{F}_{\alpha,\beta}G_j)_\alpha(y)]m(dy)\\ &=\int_{C_0[0,T]}(\mathcal{F}_{\alpha,\beta}F*\mathcal{F}_{\alpha,\beta}G)_\alpha(y)\langle z,y\rangle m(dy).\end{aligned}$$

Formula 3.5. From Formula 5.2 of [7] we have,

$$\begin{aligned}(3.7)\qquad &\int_{C_0[0,T]}\mathcal{F}_{\alpha,\beta}\left(\delta F(\cdot|w)G(\cdot)+F(\cdot)\delta G(\cdot|w)\right)\Big(\frac{y}{\sqrt{2}}\Big)m(dy)\\ &=\frac{\sqrt{2}}{\beta}\int_{C_0[0,T]}(\mathcal{F}_{\alpha,\beta}F*\mathcal{F}_{\alpha,\beta}G)_{i\alpha/\beta}(y)\langle z,y\rangle m(dy).\end{aligned}$$

Using the equations (2.7) through (2.9) we obtain the following integration formula for the Wiener integral of the integral transform with respect to the first argument of the variation.

Formula 3.6. For $F \in E_\sigma$ we have

(3.8)

$$\int_{C_0[0,T]} \mathcal{F}_{\alpha,\beta/\sqrt{2}}(\delta F(\cdot|w))(y)m(dy) = \sum_{j=1}^{n} \langle \theta_j, w\rangle \int_{C_0[0,T]} \mathcal{F}_{\alpha,\beta/\sqrt{2}}F_j(y)m(dy).$$

Proof. By (2.8) and Theorem 2.4 we have

$$\frac{\beta}{\sqrt{2}}\sum_{j=1}^{n} \langle \theta_j, w\rangle \int_{C_0[0,T]} \mathcal{F}_{\alpha,\beta/\sqrt{2}}F_j(y)m(dy) = \int_{C_0[0,T]} \mathcal{F}_{\alpha,\beta/\sqrt{2}}F(y)\langle z, y\rangle m(dy).$$

Similarly by (2.9) and Theorem 2.4 we have

$$\frac{\beta}{\sqrt{2}}\int_{C_0[0,T]} \mathcal{F}_{\alpha,\beta/\sqrt{2}}(\delta F(\cdot|w))\Big(\frac{y}{\sqrt{2}}\Big)m(dy) = \int_{C_0[0,T]} \mathcal{F}_{\alpha,\beta/\sqrt{2}}F(y)\langle z, y\rangle m(dy).$$

Thus we have the above formula (3.8). □

We next obtain an integration formula for functionals which is a product of elements of E_σ by some linear factors.

Corollary 3.7. *Let k be a natural number and let $z_j \in L_2[0,T]$ for $j = 1, 2, \ldots, k+1$. Let $F \in E_\sigma$ and let*

$$F^{[k]}(y) = F^{[k-1]}(y)\langle z_k, y\rangle = F(y)\prod_{j=1}^{k}\langle z_j, y\rangle.$$

Then we have the following integral equation.

(3.9)

$$\begin{aligned}&\int_{C_0[0,T]} F^{[k+1]}(y)m(dy)\\ &= \int_{C_0[0,T]} \delta F^{[k-1]}(y|w_{k+1})\langle z_k, y\rangle m(dy) + \langle z_k, w_{k+1}\rangle \int_{C_0[0,T]} F^{[k-1]}(y)m(dy),\end{aligned}$$

where $F^{[0]} = F$ and $w_{k+1}(t) = \int_0^t z_{k+1}(s)ds$.

Proof. To prove this theorem we simply take the first variation of the $F^{[k]}(y) = F^{[k-1]}(y)\langle z_k, y\rangle$. Now we have

$$\begin{aligned}\delta F^{[k]}(y|w_{k+1}) &= \frac{\partial}{\partial t}(F^{[k-1]}(y + tw_{k+1})\langle z_k, y + tw_{k+1}\rangle)|_{t=0}\\ &= \delta F^{[k-1]}(y)\langle z_k, y\rangle + F^{[k-1]}(y)\langle z_k, w_{k+1}\rangle.\end{aligned}$$

Hence we have

$$\begin{aligned}&\int_{C_0[0,T]} \delta F^{[k]}(y|w_{k+1})m(dy)\\ &= \int_{C_0[0,T]} \delta F^{[k-1]}(y)\langle z_k, y\rangle m(dy) + \langle z_k, w_{k+1}\rangle \int_{C_0[0,T]} F^{[k-1]}(y)m(dy).\end{aligned}$$

But by Theorem 2.4,

$$\begin{aligned}\int_{C_0[0,T]} \delta F^{[k]}(y|w_{k+1})m(dy) &= \int_{C_0[0,T]} \delta F^{[k]}\langle z_{k+1}, y\rangle m(dy) \\ &= \int_{C_0[0,T]} F^{[k+1]}(y)m(dy)\end{aligned}$$

and this completes the proof. □

We finish this section by giving some examples for the illustration of the integration by parts formulas.

Example 3.8. Let $F(y) = \sum_{j=1}^{n} \langle \theta_j, y\rangle$ which is an element of E_σ, then we have $\delta F(y|w) = F(w)$, where $w(t) = \int_0^t z(s)ds \in A$. Thus we can obtain

$$\begin{aligned}\int_{C_0[0,T]} \delta F(y|w)m(dy) &= \int_{C_0[0,T]} F(w)m(dy) = \sum_{j=1}^{n}\langle \theta_j, w\rangle \\ &= \sum_{j=1}^{n}\int_0^T \theta_j(s)z(s)ds.\end{aligned}$$

Since the constant functional $G \equiv 1$ belongs to E_σ and its first variation equals to zero, Theorem 2.5 yields the following.

$$\sum_{j=1}^{n}\int_{C_0[0,T]} \langle \theta_j, y\rangle\langle z, y\rangle m(dy) = \sum_{j=1}^{n}\int_0^T \theta_j(s)z(s)ds. \tag{3.10}$$

Example 3.9. Let $G(y) = \exp\{\sum_{j=1}^{n}\langle \theta_j, y\rangle\}$ which is an element of E_σ, then we have

$$\delta G(y|w) = \sum_{j=1}^{n}\langle \theta_j, w\rangle \exp\left\{\sum_{j=1}^{n}\langle \theta_j, y\rangle\right\} = \sum_{j=1}^{n}\langle \theta_j, w\rangle G(y),$$

where $w(t) = \int_0^t z(s)ds \in A$. Hence by the Wiener integration formula

$$\begin{aligned}\int_{C_0[0,T]} \delta G(y|w)m(dy) &= \sum_{j=1}^{n}\langle \theta_j, w\rangle \int_{C_0[0,T]} \sum_{j=1}^{n} \exp\{\langle \theta_j, y\rangle\}m(dy) \\ &= e^{n/2}\sum_{j=1}^{n}\langle \theta_j, w\rangle.\end{aligned}$$

From Theorem 2.5 we obtain the following Wiener integral.

$$\int_{C_0[0,T]} \exp\left\{\sum_{j=1}^{n}\langle \theta_j, y\rangle\right\}\langle z, y\rangle m(dy) = e^{n/2}\sum_{j=1}^{n}\langle \theta_j, w\rangle. \tag{3.11}$$

Example 3.10. Let $H(y) = \sum_{j=1}^{n} [\langle\theta_j, y\rangle]^2$ which is an element of E_σ, then we have,

$$\delta H(y|w) = 2\sum_{j=1}^{n}\langle\theta_j, w\rangle\langle\theta_j, y\rangle = 2\sum_{j=1}^{n}(\theta_j, z)\langle\theta_j, y\rangle,$$

where $w(t) = \int_0^t z(s)ds \in A$. Thus we can obtain the following by Wiener integration formula.

$$\int_{C_0[0,T]} \delta H(y|w)m(dy) = 2\sum_{j=1}^{n}(\theta_j, z)\int_{C_0[0,T]}\langle\theta_j, y\rangle m(dy) = 0.$$

By Theorem 2.5 we have the following.

$$\sum_{j=1}^{n}\int_{C_0[0,T]}[\langle\theta_j, y\rangle]^2\langle z, y\rangle m(dy) = 0. \tag{3.12}$$

Example 3.11. Let $L(y) = \left[\sum_{j=1}^{n}\langle\theta_j, y\rangle\right]^2$ which is an element of E_σ, then we have,

$$\delta\mathcal{F}_{\alpha,\beta}L(y|w) = 2\beta^2\sum_{j=1}^{n}\langle\theta_j, w\rangle\sum_{j=1}^{n}\langle\theta_j, y\rangle.$$

From the Wiener integration formula, we have the followings.

$$\int_{C_0[0,T]}\delta\mathcal{F}_{\alpha,\beta}L(y|w)m(dy) = 2\beta^2\sum_{j=1}^{n}\langle\theta_j, w\rangle\int_{C_0[0,T]}\sum_{j=1}^{n}\langle\theta_j, y\rangle m(dy) = 0$$

and

$$\begin{aligned}
&\int_{C_0[0,T]}\mathcal{F}_{\alpha,\beta}L(y)\langle z, y\rangle m(dy)\\
&= \int_{C_0[0,T]}\left[n\alpha^2 + [\beta\sum_{j=1}^{n}\langle\theta_j, y\rangle]^2\right]\langle z, y\rangle\ m(dy)\\
&= 0 + \beta^2\int_{C_0[0,T]}\left[\sum_{j=1}^{n}\langle\theta_j, y\rangle\right]^2\langle z, y\rangle\ m(dy).
\end{aligned}$$

From the Corollary 2.8 we have the following.

$$\int_{C_0[0,T]}\left[\sum_{j=1}^{n}\langle\theta_j, y\rangle\right]^2\langle z, y\rangle\ m(dy) = 0. \tag{3.13}$$

Examples 3.8 through 3.11 are interesting to note that we can obtain the Wiener integrals on the left hand side of (3.10) through (3.13) by using Theorem

2.5 or Corollary 2.8 rather than direct calculation using Wiener integration formula.

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