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ON THE PETTIS INTEGRAL OF FUZZY MAPPINGS IN BANACH SPACES

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ABSTRACT. In this paper, we introduce the Pettis integral of fuzzy mappings in Banach spaces using the Pettis integral of closed set-valued mappings. We investigate the relations between the Pettis integral, weak integral and integral of fuzzy mappings in Banach spaces and obtain some properties of the Pettis integral of fuzzy mappings in Banach spaces.

1. Introduction

Several types of integrals of set-valued mappings were studied by Amri and Hess [1], Aumann [2], Papageoriou [4], Wu, Zhang and Wang [6] and others. Integrals of fuzzy mappings are generalizations of integrals of set-valued mappings. Kaleva [3] introduced the integral of fuzzy mappings in $\mathbb{R}^n$ by use of the integral of set-valued mappings in $\mathbb{R}^n$. Xiaoping, Minghu and Ming [7], Xiaoping, Wang and Wu [8] also introduced integrals of fuzzy mappings in Banach spaces by use of Aumann-Pettis and Aumann-Bochner integrals of set-valued mappings. Amri and Hess [1] introduced the Pettis integral of set-valued mappings whose values are closed sets in Banach spaces and established some properties of the integral.

The purpose of this paper is to study the Pettis integral of fuzzy mappings in Banach spaces. We first introduce the Pettis integral of fuzzy mappings in Banach spaces using the Pettis integral of closed set-valued mappings in Banach spaces. And then we investigate the relations between the Pettis integral, weak integral and integral of fuzzy mappings in Banach spaces which were introduced by Xiaoping, Minghu and Ming [7] and obtain some properties of the Pettis integral of fuzzy mappings in Banach spaces.

2. Preliminaries

Throughout this paper, (Ω, Σ, μ) denotes a complete finite measure space and $(X, \|\cdot\|)$ a real separable Banach space with dual X^*. We write

$P_0(X) = \{A : A \text{ is a nonempty subset of } X\}$,

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$P_{b(f)(c)}(X) = \{A \in P_0(X) : A \text{ is bounded (closed)(convex)}\}$,
$P_{wkc}(X) = \{A \in P_0(X) : A \text{ is weakly compact and convex}\}$.

For $A \subseteq X$ and $x^* \in X^*$, let $\sigma(x^*, A) = \sup\{x^*(x) : x \in A\}$, the support function of A.

Let $u : X \to [0,1]$. We denote $[u]^r = \{x \in X : u(x) \geq r\}$ for $r \in (0,1]$. u is called a *generalized fuzzy number* if for each $r \in (0,1]$, $[u]^r \in P_{wkc}(X)$. Let $\mathcal{F}(X)$ denote the set of all generalized fuzzy numbers on X.

For $u, v \in \mathcal{F}(X)$ and $\lambda \in \mathbb{R}$, we define $u + v$ and λu as follows:

$$(u+v)(x) = \sup_{x=y+z} \min(u(y), v(z)),$$

$$(\lambda u)(x) = \begin{cases} u\left(\dfrac{1}{\lambda}x\right), & \lambda \neq 0 \\ \tilde{0}, & \lambda = 0, \text{ where } \tilde{0} = \chi_{\{0\}}. \end{cases}$$

For $u, v \in \mathcal{F}(X)$ and $\lambda \in \mathbb{R}$, $[u+v]^r = [u]^r + [v]^r$ and $[\lambda u]^r = \lambda [u]^r$ for each $r \in (0,1]$. Hence $u + v, \lambda u \in \mathcal{F}(X)$.

For $u, v \in \mathcal{F}(X)$, we define $u \leq v$ as follows:

$$u \leq v \quad \text{if} \quad u(x) \leq v(x) \quad \text{for all } x \in X.$$

For $u, v \in \mathcal{F}(X)$, $u \leq v$ if and only if $[u]^r \subseteq [v]^r$ for each $r \in (0,1]$.

For $A, B \in P_f(X)$, let $H(A,B)$ denote the Hausdorff metric of A and B defined by

$$H(A,B) = \max\left(\sup_{a \in A} d(a,B), \sup_{b \in B} d(b,A)\right),$$

where $d(a,B) = \inf_{b \in B} \|a - b\|$ and $d(b,A) = \inf_{a \in A} \|a - b\|$. Especially,

$$H(A,B) = \sup_{\|x^*\| \leq 1} |\sigma(x^*, A) - \sigma(x^*, B)|$$

whenever A, B are convex sets. Note that $(P_{wkc}(X), H)$ is a complete metric space. The number $\|A\|$ is defined by

$$\|A\| = H(A, \{0\}) = \sup_{x \in A} \|x\|.$$

Define $D : \mathcal{F}(X) \times \mathcal{F}(X) \to [0, +\infty]$ by the equation

$$D(u,v) = \sup_{r \in (0,1]} H([u]^r, [v]^r).$$

Then D is a metric on $\mathcal{F}(X)$.

The norm $\|u\|$ of $u \in \mathcal{F}(X)$ is defined by

$$\|u\| = D(u, \tilde{0}) = \sup_{r \in (0,1]} H([u]^r, \{0\}) = \sup_{r \in (0,1]} \|[u]^r\|.$$

The mapping $F : \Omega \to P_f(X)$ is called a *set-valued mapping*. F is said to be *scalarly measurable* if for every $x^* \in X^*$, the real-valued function $\sigma(x^*, F)$ is measurable. F is said to be *measurable* if $F^{-1}(A) = \{\omega \in \Omega : F(\omega) \cap A \neq \emptyset\} \in \Sigma$ for every $A \in P_f(X)$.

Let $F : \Omega \to P_f(X)$. Then the following statements are equivalent [4]:

(1) $F : \Omega \to P_f(X)$ is measurable;

(2) $F^{-1}(U) = \{\omega \in \Omega : F(\omega) \cap U \neq \emptyset\} \in \Sigma$ for every open subset U of X;

(3) $\mathrm{Gr}(F) = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\} \in \Sigma \times \mathcal{B}(X)$, where $\mathcal{B}(X)$ is the Borel σ-algebra of X;

(4) (Castaing representation) there exists a sequence $\{f_n\}$ of measurable functions $f_n : \Omega \to X$ such that $F(\omega) = cl\{f_n(\omega)\}$ for all $\omega \in \Omega$.

Note that if $F : \Omega \to P_f(X)$ is measurable then $F : \Omega \to P_f(X)$ is scalarly measurable.

$F : \Omega \to P_f(X)$ is said to be *weakly integrably bounded* if the real-valued function $|x^*F| : \Omega \to \mathbb{R}, |x^*F|(\omega) = \sup\{|x^*(x)| : x \in F(\omega)\}$ is integrable for every $x^* \in X^*$. $F : \Omega \to P_f(X)$ is said to be *integrably bounded* if there exists an integrable real-valued function h such that for each $\omega \in \Omega$, $\|x\| \leq h(\omega)$ for all $x \in F(\omega)$.

$f : \Omega \to X$ is called a *measurable selector* of $F : \Omega \to P_f(X)$ if f is measurable and $f(\omega) \in F(\omega)$ for every $\omega \in \Omega$. A measurable selector f of F is called a *Pettis* (resp., *Bochner*) *integrable selector* of F if f is Pettis (resp., Bochner) integrable. We denote by S_{wF} (resp., S_F) the set of all Pettis (resp., Bochner) integrable selectors of F.

Given $F : \Omega \to P_f(X)$ and $A \in \Sigma$, the *Aumann-Pettis* (resp., *Aumann-Bochner*) *integral* of F is defined by

$$(w) \int_A F d\mu = \left\{ (P) \int_A f d\mu : f \in S_{wF} \right\}$$

$$\left(\text{resp., } \int_A F d\mu = \left\{ \int_A f d\mu : f \in S_F \right\} \right).$$

$F : \Omega \to P_f(X)$ is said to be *Aumann-Pettis* (resp., *Aumann-Bochner*) *integrable* if $S_{wF} \neq \emptyset$ (resp., $S_F \neq \emptyset$).

A measurable extended real-valued function f is said to be *quasi-integrable* if either f^+ or f^- is integrable.

$F : \Omega \to P_f(X)$ is said to be *scalarly integrable* (resp., *scalarly quasi-integrable*) if for every $x^* \in X^*$, $\sigma(x^*, F)$ is integrable (resp., quasi-integrable). $F : \Omega \to P_f(X)$ is said to be *scalarly uniformly integrable* if the set $\{\sigma(x^*, F) : x^* \in B_{X^*}\}$ is uniformly integrable, where B_{X^*} is the closed unit ball of X^*.

A measurable set-valued mapping $F : \Omega \to P_{fc}(X)$ is said to be *Pettis integrable* if it satisfies the following two conditions [1]:

(1) $F : \Omega \to P_{fc}(X)$ is scalarly quasi-integrable,

(2) for every $A \in \Sigma$ there exists $C_A(F) = C_A \in P_{fc}(X)$ such that

$$\sigma(x^*, C_A) = \int_A \sigma(x^*, F) d\mu$$

for every $x^* \in X^*$. In this case $C_A(F) = (P)\int_A F d\mu$ is called the *Pettis integral* of F over A. If $\mathcal{C}$ is a subspace of $P_{fc}(X)$, we say that the set-valued mapping $F : \Omega \to \mathcal{C}$ is *Pettis integrable in* $\mathcal{C}$ if $C_A \in \mathcal{C}$ for each $A \in \Sigma$.

3. Results

A mapping $\tilde{F} : \Omega \to \mathcal{F}(X)$ is called a *fuzzy mapping* in a Banach space X. In this case $\tilde{F}^r : \Omega \to P_{wkc}(X)$ defined by $\tilde{F}^r(\omega) = [\tilde{F}(\omega)]^r$ is a set-valued mapping for each $r \in (0,1]$.

A fuzzy mapping $\tilde{F} : \Omega \to \mathcal{F}(X)$ is said to be *measurable* if $\tilde{F}^r : \Omega \to P_{wkc}(X)$ is measurable for each $r \in (0,1]$.

Definition 3.1 ([7]). A fuzzy mapping $\tilde{F} : \Omega \to \mathcal{F}(X)$ is said to be *weakly integrable* if for each $A \in \Sigma$ there exists $u_A \in \mathcal{F}(X)$ such that $[u_A]^r = (w)\int_A \tilde{F}^r d\mu$ for each $r \in (0,1]$. In this case $u_A = (w)\int_A \tilde{F} d\mu$ is called the *weak integral* of $\tilde{F}$ over A.

A fuzzy mapping $\tilde{F} : \Omega \to \mathcal{F}(X)$ is said to be *integrable* if for each $A \in \Sigma$ there exists $u_A \in \mathcal{F}(X)$ such that $[u_A]^r = \int_A \tilde{F}^r d\mu$ for each $r \in (0,1]$. In this case $u_A = \int_A \tilde{F} d\mu$ is called the *integral* of $\tilde{F}$ over A.

Definition 3.2. A measurable fuzzy mapping $\tilde{F} : \Omega \to \mathcal{F}(X)$ is said to be *Pettis integrable* if for each $A \in \Sigma$ there exists $u_A \in \mathcal{F}(X)$ such that $[u_A]^r = (P)\int_A \tilde{F}^r d\mu$ for each $r \in (0,1]$. In this case $u_A = (P)\int_A \tilde{F} d\mu$ is called the *Pettis integral* of $\tilde{F}$ over A.

Theorem 3.3. *Let $\tilde{F} : \Omega \to \mathcal{F}(X)$ and $\tilde{G} : \Omega \to \mathcal{F}(X)$ be Pettis integrable and $\lambda \geq 0$. Then*

(1) *$\tilde{F} + \tilde{G}$ is Pettis integrable and for each $A \in \Sigma$*

$$(P)\int_A (\tilde{F} + \tilde{G}) d\mu = (P)\int_A \tilde{F} d\mu + (P)\int_A \tilde{G} d\mu,$$

(2) *$\lambda\tilde{F}$ is Pettis integrable and for each $A \in \Sigma$*

$$(P)\int_A \lambda\tilde{F} d\mu = \lambda(P)\int_A \tilde{F} d\mu.$$

Proof. (1) Let $\tilde{F} : \Omega \to \mathcal{F}(X)$ and $\tilde{G} : \Omega \to \mathcal{F}(X)$ be Pettis integrable and let $A \in \Sigma$. Then there exist $u_A, v_A \in \mathcal{F}(X)$ such that $[u_A]^r = (P)\int_A \tilde{F}^r d\mu$,

$[v_A]^r = (P)\int_A \tilde{G}^r d\mu$ for each $r \in (0,1]$. Therefore

$$\sigma(x^*, [u_A]^r) = \int_A \sigma(x^*, \tilde{F}^r) d\mu, \quad \sigma(x^*, [v_A]^r) = \int_A \sigma(x^*, \tilde{G}^r) d\mu$$

for each $r \in (0,1]$ and $x^* \in X^*$. Hence we have

$$\begin{aligned}\sigma(x^*, [u_A + v_A]^r) &= \sigma(x^*, [u_A]^r + [v_A]^r) \\ &= \sigma(x^*, [u_A]^r) + \sigma(x^*, [v_A]^r) \\ &= \int_A \sigma(x^*, \tilde{F}^r) du + \int_A \sigma(x^*, \tilde{G}^r) d\mu \\ &= \int_A \Big(\sigma(x^*, \tilde{F}^r) + \sigma(x^*, \tilde{G}^r)\Big) d\mu \\ &= \int_A \sigma(x^*, \tilde{F}^r + \tilde{G}^r) d\mu \\ &= \int_A \sigma(x^*, (\tilde{F} + \tilde{G})^r) d\mu\end{aligned}$$

for each $r \in (0,1]$ and $x^* \in X^*$. Thus $[u_A + v_A]^r = (P)\int_A (\tilde{F} + \tilde{G})^r d\mu$ for each $r \in (0,1]$. Hence $\tilde{F} + \tilde{G}$ is Pettis integrable and for each $A \in \Sigma$

$$(P)\int_A (\tilde{F} + \tilde{G}) d\mu = u_A + v_A = (P)\int_A \tilde{F} d\mu + (P)\int_A \tilde{G} d\mu.$$

(2) Let $\tilde{F} : \Omega \to \mathcal{F}(X)$ be Pettis integrable and $\lambda \geq 0$. Then for each $A \in \Sigma$ there exists $u_A \in \mathcal{F}(X)$ such that $[u_A]^r = (P)\int_A \tilde{F}^r d\mu$ for each $r \in (0,1]$. Since $\sigma(x^*, [\lambda u_A]^r) = \sigma(x^*, \lambda[u_A]^r) = \lambda\sigma(x^*, [u_A]^r)$ for each $r \in (0,1]$ and $x^* \in X^*$, using the same method as (1) we obtain that $\lambda\tilde{F}$ is Pettis integrable and for each $A \in \Sigma$

$$(P)\int_A \lambda\tilde{F} d\mu = \lambda(P)\int_A \tilde{F} d\mu.$$

□

Theorem 3.4. *Let $\tilde{F} : \Omega \to \mathcal{F}(X)$ and $\tilde{G} : \Omega \to \mathcal{F}(X)$ be Pettis integrable. Then*

(1) *if $\tilde{F}(\omega) \leq \tilde{G}(\omega)$ μ-a.e., then $(P)\int_A \tilde{F} d\mu \leq (P)\int_A \tilde{G} d\mu$ for each $A \in \Sigma$;*

(2) *if $\tilde{F}(\omega) = \tilde{G}(\omega)$ μ-a.e., then $(P)\int_A \tilde{F} d\mu = (P)\int_A \tilde{G} d\mu$ for each $A \in \Sigma$.*

Proof. (1) Since $\tilde{F} : \Omega \to \mathcal{F}(X)$ and $\tilde{G} : \Omega \to \mathcal{F}(X)$ are Pettis integrable, for each $A \in \Sigma$ there exist $u_A, v_A \in \mathcal{F}(X)$ such that $u_A = (P)\int_A \tilde{F} d\mu$, $v_A = (P)\int_A \tilde{G} d\mu$. If $\tilde{F}(\omega) \leq \tilde{G}(\omega)$ μ-a.e., then $\tilde{F}^r(\omega) \leq \tilde{G}^r(\omega)$ μ-a.e. for each

$r \in (0,1]$. By [1, Proposition 4.1], $[u_A]^r = (P)\int_A \tilde{F}^r d\mu \subseteq (P)\int_A \tilde{G}^r d\mu = [v_A]^r$ for each $r \in (0,1]$ and $A \in \Sigma$. Thus $(P)\int_A \tilde{F} d\mu \leq (P)\int_A \tilde{G} d\mu$ for each $A \in \Sigma$.

(2) The proof is similar to (1). □

A fuzzy mapping $\tilde{F} : \Omega \to \mathcal{F}(X)$ is said to be *weakly integrably bounded* if $\tilde{F}^r : \Omega \to P_{wkc}(X)$ is weakly integrably bounded for each $r \in (0,1]$ [5].

A fuzzy mapping $\tilde{F} : \Omega \to \mathcal{F}(X)$ is said to be *scalarly integrable* (resp., *scalarly uniformly integrable*) if $\tilde{F}^r : \Omega \to P_{wkc}(X)$ is scalarly integrable (resp., scalarly uniformly integrable) for each $r \in (0,1]$.

Theorem 3.5. *If $\tilde{F} : \Omega \to \mathcal{F}(X)$ is measurable, weakly integrably bounded and scalarly uniformly integrable, then $\tilde{F} : \Omega \to \mathcal{F}(X)$ is Pettis integrable.*

Proof. Let $A \in \Sigma$. Since $\tilde{F} : \Omega \to \mathcal{F}(X)$ is measurable and scalarly uniformly integrable, by [1, Theorem 5.4] $\tilde{F}^r : \Omega \to P_{wkc}(X)$ is Pettis integrable in $P_{wkc}(X)$ for each $r \in (0,1]$. Thus $M_r = (P)\int_A \tilde{F}^r d\mu \in P_{wkc}(X)$ for each $r \in (0,1]$. For $r_1, r_2 \in (0,1]$ with $r_1 < r_2$, $\tilde{F}^{r_1}(\omega) \supseteq \tilde{F}^{r_2}(\omega)$ for each $\omega \in \Omega$. By [1, Proposition 4.1] $M_{r_1} = (P)\int_A \tilde{F}^{r_1} d\mu \supseteq (P)\int_A \tilde{F}^{r_2} d\mu = M_{r_2}$. Let $r \in (0,1]$ and $\{r_n\}$ be a sequence in $(0,1]$ such that $r_1 \leq r_2 \leq r_3 \leq \cdots$ and $\lim_{n\to\infty} r_n = r$. Then $\tilde{F}^r(\omega) = \cap_{n=1}^{\infty} \tilde{F}^{r_n}(\omega)$ for each $\omega \in \Omega$. By [7, Lemma 4.2] $\lim_{n\to\infty} \sigma(x^*, \tilde{F}^{r_n}(\omega)) = \sigma(x^*, \tilde{F}^r(\omega))$ for each $\omega \in \Omega$ and $x^* \in X^*$. For each $n \in \mathbb{N}$, $|\sigma(x^*, \tilde{F}^{r_n}(\omega))| \leq |x^* \tilde{F}^{r_1}|(\omega)$ for each $\omega \in \Omega$ and $x^* \in X^*$. Since $\tilde{F} : \Omega \to \mathcal{F}(X)$ is weakly integrably bounded, by Lebesgue dominated convergence theorem we have

$$\lim_{n\to\infty} \sigma(x^*, M_{r_n}) = \lim_{n\to\infty} \int_A \sigma(x^*, \tilde{F}^{r_n}) d\mu = \int_A \sigma(x^*, \tilde{F}^r) d\mu = \sigma(x^*, M_r)$$

for each $x^* \in X^*$. By [7, Lemma 4.2] $M_r = \cap_{n=1}^{\infty} M_{r_n}$. Let $M_0 = X$. Then by [7, Lemma 4.1] there exists $u_A \in \mathcal{F}(X)$ such that $[u_A]^r = M_r = (P)\int_A \tilde{F}^r d\mu$ for each $r \in (0,1]$. Hence $\tilde{F} : \Omega \to \mathcal{F}(X)$ is Pettis integrable. □

Theorem 3.6. *If $\tilde{F} : \Omega \to \mathcal{F}(X)$ is scalarly integrable and Pettis integrable, then $\tilde{F} : \Omega \to \mathcal{F}(X)$ is scalarly uniformly integrable.*

Proof. If $\tilde{F} : \Omega \to \mathcal{F}(X)$ is scalarly integrable and Pettis integrable, then for each $r \in (0,1]$ $\tilde{F}^r : \Omega \to P_{wkc}(X)$ is measurable and scalarly integrable and for each $A \in \Sigma$ there exists $u_A \in \mathcal{F}(X)$ such that $[u_A]^r = (P)\int_A \tilde{F}^r d\mu$ for each $r \in (0,1]$. Thus $\tilde{F}^r : \Omega \to P_{wkc}(X)$ is Pettis integrable for each $r \in (0,1]$. By

[1, Theorem 5.4] $\tilde{F}^r : \Omega \to P_{wkc}(X)$ is scalarly uniformly integrable for each $r \in (0,1]$. Hence $\tilde{F} : \Omega \to \mathcal{F}(X)$ is scalarly uniformly integrable. □

Remark 3.7. If $\tilde{F} : \Omega \to \mathcal{F}(X)$ is weakly integrably bounded, then $\tilde{F} : \Omega \to \mathcal{F}(X)$ is scalarly integrable. But the converse is not true.

We can obtain the following corollary from Theorem 3.5, Theorem 3.6 and Remark 3.7.

Corollary 3.8. *Let $\tilde{F} : \Omega \to \mathcal{F}(X)$ be measurable and weakly integrably bounded. Then $\tilde{F} : \Omega \to \mathcal{F}(X)$ is Pettis integrable if and only if $\tilde{F} : \Omega \to \mathcal{F}(X)$ is scalarly uniformly integrable.*

$\tilde{F} : \Omega \to \mathcal{F}(X)$ is said to be *integrably bounded* if there exists an integrable real-valued function h such that for each $\omega \in \Omega$, $||x|| \leq h(\omega)$ for all $x \in \tilde{F}^0(\omega)$, where $\tilde{F}^0(\omega) = cl\left(\cup_{0<r\leq 1}\tilde{F}^r(\omega)\right)$. If $\tilde{F} : \Omega \to \mathcal{F}(X)$ is integrably bounded, then $\tilde{F} : \Omega \to \mathcal{F}(X)$ is weakly integrably bounded.

Lemma 3.9. *If $F : \Omega \to P_{wkc}(X)$ and $G : \Omega \to P_{wkc}(X)$ are integrably bounded and Pettis integrable in $P_{wkc}(X)$, then $H(F,G)$ is integrable and*

$$H\left((P)\int_\Omega F d\mu, (P)\int_\Omega G d\mu\right) \leq \int_\Omega H(F,G) d\mu.$$

Proof. Since F and G are measurable, there exist Castaing representations $\{f_n\}$ and $\{g_n\}$ for F and G. Since f_n and g_n are measurable for all $n \in \mathbb{N}$,

$$H(F(\omega), G(\omega)) = \max\left(\sup_{n\geq 1}\inf_{k\geq 1} ||f_n(\omega) - g_k(\omega)||, \sup_{n\geq 1}\inf_{k\geq 1} ||g_n(\omega) - f_k(\omega)||\right)$$

is measurable. Since F and G are integrably bounded, there exist integrable real-valued functions h_1 and h_2 such that for each $\omega \in \Omega$, $||x|| \leq h_1(\omega)$ for all $x \in F(\omega)$ and $||x|| \leq h_2(\omega)$ for all $x \in G(\omega)$. Hence we have

$$H(F(\omega), G(\omega)) \leq H(F(\omega), \{0\}) + H(G(\omega), \{0\}) \leq h_1(\omega) + h_2(\omega)$$

for each $\omega \in \Omega$. Therefore $H(F,G)$ is integrable and by [1, Proposition 2.2] we have

$$\begin{aligned} &H\left((P)\int_\Omega F d\mu, (P)\int_\Omega G d\mu\right) \\ = &\sup_{||x^*||\leq 1}\left|\sigma\left(x^*, (P)\int_\Omega F d\mu\right) - \sigma\left(x^*, (P)\int_\Omega G d\mu\right)\right| \end{aligned}$$

$$= \sup_{\|x^*\|\le 1} \left| \int_\Omega \sigma(x^*, F) d\mu - \int_\Omega \sigma(x^*, G) d\mu \right|$$
$$\le \sup_{\|x^*\|\le 1} \int_\Omega |\sigma(x^*, F) - \sigma(x^*, G)| d\mu$$
$$\le \int_\Omega \sup_{\|x^*\|\le 1} |\sigma(x^*, F) - \sigma(x^*, G)| d\mu$$
$$= \int_\Omega H(F, G) d\mu \ .$$

□

Theorem 3.10. *If $\tilde{F} : \Omega \to \mathcal{F}(X)$ and $\tilde{G} : \Omega \to \mathcal{F}(X)$ are integrably bounded and Pettis integrable, then $D(\tilde{F}, \tilde{G})$ is integrable and*

$$D\left((P)\int_\Omega \tilde{F} d\mu, (P)\int_\Omega \tilde{G} d\mu\right) \le \int_\Omega D(\tilde{F}, \tilde{G}) d\mu.$$

Proof. Since $\tilde{F}$ and $\tilde{G}$ are measurable, there exist Castaing representations $\{f_n^r\}$ and $\{g_n^r\}$ for $\tilde{F}^r$ and $\tilde{G}^r$ for each $r \in (0,1]$. Since f_n^r and g_n^r are measurable for all $n \in \mathbb{N}$,

$$H(\tilde{F}^r(\omega), \tilde{G}^r(\omega)) = \max\left(\sup_{n\ge 1} \inf_{k\ge 1} \|f_n^r(\omega) - g_k^r(\omega)\|, \sup_{n\ge 1} \inf_{k\ge 1} \|g_n^r(\omega) - f_k^r(\omega)\|\right)$$

is measurable for each $r \in (0,1]$. Hence

$$D(\tilde{F}(\omega), \tilde{G}(\omega)) = \sup_{k\ge 1} H(\tilde{F}^{r_k}(\omega), \tilde{G}^{r_k}(\omega))$$

is measurable, where $\{r_k : k \in \mathbb{N}\}$ is dense in $(0,1]$. Since $\tilde{F}$ and $\tilde{G}$ are integrably bounded, there exist integrable real-valued functions h_1 and h_2 such that for each $\omega \in \Omega$, $\|x\| \le h_1(\omega)$ for all $x \in \tilde{F}^0(\omega)$ and $\|x\| \le h_2(\omega)$ for all $x \in \tilde{G}^0(\omega)$. Hence we have

$$D(\tilde{F}(\omega), \tilde{G}(\omega)) \le D(\tilde{F}(\omega), \tilde{0}) + D(\tilde{G}(\omega), \tilde{0}) \le h_1(\omega) + h_2(\omega)$$

for each $\omega \in \Omega$. Therefore $D(\tilde{F}, \tilde{G})$ is integrable and by Lemma 3.9

$$H\left((P)\int_\Omega \tilde{F}^r d\mu, (P)\int_\Omega \tilde{G}^r d\mu\right) \le \int_\Omega H(\tilde{F}^r, \tilde{G}^r) d\mu$$

for each $r \in (0,1]$. Hence we have

$$D\left((P)\int_\Omega \tilde{F} d\mu, (P)\int_\Omega \tilde{G} d\mu\right)$$
$$= \sup_{r\in(0,1]} H\left(\left[(P)\int_\Omega \tilde{F} d\mu\right]^r, \left[(P)\int_\Omega \tilde{G} d\mu\right]^r\right)$$

$$
\begin{aligned}
&= \sup_{r\in(0,1]} H\left((P)\int_\Omega \tilde{F}^r d\mu, (P)\int_\Omega \tilde{G}^r d\mu\right)\\
&\le \sup_{r\in(0,1]} \int_\Omega H(\tilde{F}^r,\tilde{G}^r)d\mu\\
&\le \int_\Omega \sup_{r\in(0,1]} H(\tilde{F}^r,\tilde{G}^r)d\mu\\
&= \int_\Omega D(\tilde{F},\tilde{G})d\mu.
\end{aligned}
$$

□

Theorem 3.11. *If a measurable fuzzy mapping $\tilde{F}:\Omega\to\mathcal{F}(X)$ is integrable, then $\tilde{F}:\Omega\to\mathcal{F}(X)$ is weakly integrable and $\int_A \tilde{F}d\mu = (w)\int_A \tilde{F}d\mu$ for each $A\in\Sigma$.*

Proof. If $\tilde{F}:\Omega\to\mathcal{F}(X)$ is integrable, then for each $A\in\Sigma$ there exists $u_A\in\mathcal{F}(X)$ such that $u_A=\int_A \tilde{F}d\mu$. Thus $[u_A]^r=\int_A \tilde{F}^r d\mu$ for each $r\in(0,1]$. Since $\tilde{F}^r:\Omega\to P_{wkc}(X)$ is Aumann-Bochner integrable for each $r\in(0,1]$, $\tilde{F}^r:\Omega\to P_{wkc}(X)$ is Aumann-Pettis integrable and $w\text{-}cl\int_A \tilde{F}^r d\mu = w\text{-}cl\,(w)\int_A \tilde{F}^r d\mu$ for each $r\in(0,1]$ by [1, Proposition 3.12]. Since $[u_A]^r=\int_A \tilde{F}^r d\mu\in P_{wkc}(X)$ for each $r\in(0,1]$, $\int_A \tilde{F}^r d\mu = w\text{-}cl\int_A \tilde{F}^r d\mu = w\text{-}cl\,(w)\int_A \tilde{F}^r d\mu \supseteq (w)\int_A \tilde{F}^r d\mu$ for each $r\in(0,1]$. Generally, $\int_A \tilde{F}^r d\mu \subseteq (w)\int_A \tilde{F}^r d\mu$ for each $r\in(0,1]$. Hence $[u_A]^r=\int_A \tilde{F}^r d\mu = (w)\int_A \tilde{F}^r d\mu$ for each $r\in(0,1]$. Thus $u_A=(w)\int_A \tilde{F}d\mu$. Therefore $\tilde{F}:\Omega\to\mathcal{F}(X)$ is weakly integrable and $\int_A \tilde{F}d\mu=(w)\int_A \tilde{F}d\mu$ for each $A\in\Sigma$. □

Theorem 3.12. *If a measurable fuzzy mapping $\tilde{F}:\Omega\to\mathcal{F}(X)$ is weakly integrable, then $\tilde{F}:\Omega\to\mathcal{F}(X)$ is Pettis integrable and $(w)\int_A \tilde{F}d\mu=(P)\int_A \tilde{F}d\mu$ for each $A\in\Sigma$.*

Proof. If $\tilde{F}:\Omega\to\mathcal{F}(X)$ is weakly integrable, then for each $A\in\Sigma$ there exists $u_A\in\mathcal{F}(X)$ such that $u_A=(w)\int_A \tilde{F}d\mu$. Thus $[u_A]^r=(w)\int_A \tilde{F}^r d\mu$ for each $r\in(0,1]$. Since $\tilde{F}^r:\Omega\to P_{wkc}(X)$ is measurable and Aumann-Pettis integrable for each $r\in(0,1]$, by [1, Theorem 3.7] $\tilde{F}^r:\Omega\to P_{wkc}(X)$ is Pettis integrable for each $r\in(0,1]$ and by [7, Lemma 4.3] $\sigma\left(x^*,(w)\int_A \tilde{F}^r d\mu\right)=$

$\int_A \sigma(x^*, \tilde{F}^r)d\mu$ for each $x^* \in X^*$ and $r \in (0,1]$. Hence $[u_A]^r = (w)\int_A \tilde{F}^r d\mu$ $= (P)\int_A \tilde{F}^r d\mu$ for each $r \in (0,1]$. Thus $u_A = (P)\int_A \tilde{F}d\mu$. Therefore $\tilde{F} : \Omega \to \mathcal{F}(X)$ is Pettis integrable and $(w)\int_A \tilde{F}d\mu = (P)\int_A \tilde{F}d\mu$ for each $A \in \Sigma$. □

We can obtain the following two corollaries from Theorem 3.10, Theorem 3.11 and Theorem 3.12

Corollary 3.13. *If measurable fuzzy mappings $\tilde{F} : \Omega \to \mathcal{F}(X)$ and $\tilde{G} : \Omega \to \mathcal{F}(X)$ are integrably bounded and integrable, then $D(\tilde{F}, \tilde{G})$ is integrable and*

$$D\left(\int_\Omega \tilde{F}d\mu, \int_\Omega \tilde{G}d\mu\right) \leq \int_\Omega D(\tilde{F}, \tilde{G})d\mu.$$

Corollary 3.14. *If measurable fuzzy mappings $\tilde{F} : \Omega \to \mathcal{F}(X)$ and $\tilde{G} : \Omega \to \mathcal{F}(X)$ are integrably bounded and weakly integrable, then $D(\tilde{F}, \tilde{G})$ is integrable and*

$$D\left((w)\int_\Omega \tilde{F}d\mu, (w)\int_\Omega \tilde{G}d\mu\right) \leq \int_\Omega D(\tilde{F}, \tilde{G})d\mu.$$

Theorem 3.15. *Let X contain no copy of c_0 and let $\tilde{F}_n : \Omega \to \mathcal{F}(X)$ $(n \in \mathbb{N})$ and $\tilde{F} : \Omega \to \mathcal{F}(X)$ be measurable fuzzy mappings such that*

$$\lim_{n\to\infty} D(\tilde{F}_n(\omega), \tilde{F}(\omega)) = 0$$

on Ω. If there exists an integrable real-valued function h such that $\|\tilde{F}_n^0(\omega)\| \leq h(\omega)$ on Ω for each $n \in \mathbb{N}$, then $\tilde{F} : \Omega \to \mathcal{F}(X)$ is weakly integrable and

$$\lim_{n\to\infty} D\left((w)\int_\Omega \tilde{F}_n d\mu, (w)\int_\Omega \tilde{F}d\mu\right) = 0.$$

Proof. Since $\lim_{n\to\infty} D(\tilde{F}_n(\omega), \tilde{F}(\omega)) = 0$ on Ω, for each $\epsilon > 0$ and $\omega \in \Omega$ there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow D(\tilde{F}_n(\omega), \tilde{F}(\omega)) < \epsilon$. For some $n \in \mathbb{N}$ with $n \geq N$,

$$\begin{aligned}\|\tilde{F}^0(\omega)\| &= D(\tilde{F}(\omega), \tilde{0}) \leq D(\tilde{F}(\omega), \tilde{F}_n(\omega)) + D(\tilde{F}_n(\omega), \tilde{0}) \\ &< \|\tilde{F}_n^0(\omega)\| + \epsilon \leq h(\omega) + \epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $\|\tilde{F}^0(\omega)\| \leq h(\omega)$ on Ω. Thus $\tilde{F}_n : \Omega \to \mathcal{F}(X)$ $(n \in \mathbb{N})$ and $\tilde{F} : \Omega \to \mathcal{F}(X)$ are integrably bounded and so weakly integrably bounded. By [7, Theorem 4.5], $\tilde{F}_n : \Omega \to \mathcal{F}(X)$ $(n \in \mathbb{N})$ and $\tilde{F} : \Omega \to \mathcal{F}(X)$ are weakly integrable. By Corollary 3.14 and Lebesgue dominated convergence theorem,

$$D\left((w)\int_\Omega \tilde{F}_n d\mu, (w)\int_\Omega \tilde{F}d\mu\right) \leq \int_\Omega D(\tilde{F}_n, \tilde{F})d\mu \to 0 \quad \text{as} \quad n \to \infty.$$

Thus $\lim_{n\to\infty} D\left((w)\int_\Omega \tilde{F}_n d\mu, (w)\int_\Omega \tilde{F}d\mu\right) = 0.$ □

We can obtain the following corollary from Theorem 3.12 and Theorem 3.15.

Corollary 3.16. *Let X contain no copy of c_0 and let $\tilde{F}_n : \Omega \to \mathcal{F}(X)$ $(n \in \mathbb{N})$ and $\tilde{F} : \Omega \to \mathcal{F}(X)$ be measurable fuzzy mappings such that*

$$\lim_{n\to\infty} D(\tilde{F}_n(\omega), \tilde{F}(\omega)) = 0$$

on Ω. If there exists an integrable real-valued function h such that $\|\tilde{F}_n^0(\omega)\| \leq h(\omega)$ on Ω for each $n \in \mathbb{N}$, then $\tilde{F} : \Omega \to \mathcal{F}(X)$ is Pettis integrable and

$$\lim_{n\to\infty} D\left((P)\int_\Omega \tilde{F}_n d\mu, (P)\int_\Omega \tilde{F} d\mu\right) = 0.$$

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