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A COMMON FIXED POINT THEOREM IN TWO $\mathcal{M}$-FUZZY METRIC SPACES

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Abstract. In this paper, we give some new definitions of $\mathcal{M}$-fuzzy metric spaces and we prove a common fixed point theorem for six mappings under the condition of compatible mappings of first or second type in two complete $\mathcal{M}$-fuzzy metric spaces.

1. Introduction and preliminaries

The concept of fuzzy sets was introduced initially by Zadeh [20] in 1965. Since then, to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and application. George and Veeramani [6] and Kramosil and Michalek [9] have introduced the concept of fuzzy topological spaces induced by fuzzy metric which have very important applications in quantum particle physics particularly in connections with both string and $\epsilon^{(\infty)}$ theory which were given and studied by El Naschie [2, 3, 4, 5, 17]. Many authors [8, 12, 15] have proved fixed point theorem in fuzzy (probabilistic) metric spaces. Vasuki [18] obtained the fuzzy version of common fixed point theorem which had extra conditions. In fact, Vasuki proved fuzzy common fixed point theorem by a strong definition of Cauchy sequence (see Note 3.13 and Definition 3.15 of [6] also [16, 19]). In this paper, we prove a common fixed point theorem in fuzzy metric spaces for arbitrary t-norms and modified definition of Cauchy sequence in George and Veeramani's sense. There have been a number of generalizations of metric spaces. One such generalization is generalized metric space or D-metric space initiated by Dhage [1] in 1992. He proved some results on fixed points for a self-map satisfying a contraction for complete and bounded D-metric spaces. Rhoades [10] generalized Dhage's contractive condition by increasing the number of factors and proved the existence of unique fixed point of a self-map in D-metric space. Recently, motivated by the concept of compatibility for metric space, Singh and Sharma [14] introduced the concept of D-compatibility of maps in D-metric space and proved some fixed point theorems using a contractive condition.

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In what follows (X, D) will denote a D-metric space, $\mathbb{N}$ the set of all natural numbers, and $\mathbb{R}^+$ the set of all positive real numbers.

Definition 1.1. Let X be a nonempty set. A generalized metric (or D-metric) on X is a function: $D : X^3 \longrightarrow \mathbb{R}^+$ that satisfies the following conditions for each $x, y, z, a \in X$.

(1) $D(x, y, z) \geq 0$,

(2) $D(x, y, z) = 0$ if and only if $x = y = z$,

(3) $D(x, y, z) = D(p\{x, y, z\})$, (symmetry) where p is a permutation function,

(4) $D(x, y, z) \leq D(x, y, a) + D(a, z, z)$.

The pair (X, D) is called a generalized metric (or D-metric) space.

Immediate examples of such a function are

(a) $D(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$,

(b) $D(x, y, z) = d(x, y) + d(y, z) + d(z, x)$.

Here, d is the ordinary metric on X.

(c) If $X = \mathbb{R}^n$ then we define

$$D(x, y, z) = (||x - y||^p + ||y - z||^p + ||z - x||^p)^{\frac{1}{p}}$$

for every $p \in \mathbb{R}^+$.

(d) If $X = \mathbb{R}^+$ then we define

$$D(x, y, z) = \begin{cases} 0 & \text{if } x = y = z, \\ \max\{x, y, z\} & \text{otherwise .} \end{cases}$$

Remark 1.2. In a D-metric space, we prove that $D(x, x, y) = D(x, y, y)$. For

(i) $D(x, x, y) \leq D(x, x, x) + D(x, y, y) = D(x, y, y)$ and similarly

(ii) $D(y, y, x) \leq D(y, y, y) + D(y, x, x) = D(y, x, x)$.

Hence by (i), (ii) we get $D(x, x, y) = D(x, y, y)$.

Let (X, D) be a D-metric space. For $r > 0$ define

$$B_D(x, r) = \{y \in X : D(x, y, y) < r\}$$

Example 1.3. Let $X = \mathbb{R}$. Denote $D(x, y, z) = |x - y| + |y - z| + |z - x|$ for all $x, y, z \in \mathbb{R}$. Thus

$$\begin{aligned} B_D(1, 2) &= \{y \in \mathbb{R} : D(1, y, y) < 2\} \\ &= \{y \in \mathbb{R} : |y - 1| + |y - 1| < 2\} \\ &= \{y \in \mathbb{R} : |y - 1| < 1\} = (0, 2). \end{aligned}$$

Definition 1.4. Let (X, D) be a D-metric space and $A \subset X$.

(1) If for every $x \in A$ there exist $r > 0$ such that $B_D(x, r) \subset A$, then subset A is called open subset of X.

(2) Subset A of X is said to be D-bounded if there exists $r > 0$ such that $D(x, y, y) < r$ for all $x, y \in A$.

(3) A sequence $\{x_n\}$ in X converges to x if and only if $D(x_n, x_n, x) = D(x, x, x_n) \to 0$ as $n \to \infty$. That is for each $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that

$$(*) \qquad \forall n \geq n_0 \Longrightarrow D(x, x, x_n) < \epsilon.$$

This is equivalent with, for each $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that

$$(**) \qquad \forall n, m \geq n_0 \Longrightarrow D(x, x_n, x_m) < \epsilon.$$

Indeed, if have $(*)$, then

$$D(x_n, x_m, x) = D(x_n, x, x_m) \leq D(x_n, x, x) + D(x, x_m, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \varepsilon.$$

Conversely, set $m = n$ in $(**)$ we have $D(x_n, x_n, x) < \epsilon$.

(4) Sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\epsilon > 0$, there exits $n_0 \in \mathbb{N}$ such that $D(x_n, x_n, x_m) < \epsilon$ for each $n, m \geq n_0$. The D-metric space (X, D) is said to be complete if every Cauchy sequence is convergent.

Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exist $r > 0$ such that $B_D(x, r) \subset A$. Then τ is a topology on X (induced by the D-metric D).

Lemma 1.5. *Let (X, D) be a D-metric space. If $r > 0$, then ball $B_D(x, r)$ with center $x \in X$ and radius r is open ball.*

Proof. Let $z \in B_D(x, r)$, hence $D(x, z, z) < r$. If set $D(x, z, z) = \delta$ and $r' = r - \delta$ then we prove that $B_D(z, r') \subseteq B_D(x, r)$. Let $y \in B_D(z, r')$, by triangular inequality we have $D(x, y, y) = D(y, y, x) \leq D(y, y, z) + D(z, x, x) < r' + \delta = r$. Hence $B_D(z, r') \subseteq B_D(x, r)$. That is ball $B_D(x, r)$ is open ball. □

Lemma 1.6. *Let (X, D) be a D-metric space. If sequence $\{x_n\}$ in X converges to x, then x is unique.*

Proof. Let $x_n \longrightarrow y$ and $y \neq x$. Since $\{x_n\}$ converges to x and y, for each $\epsilon > 0$ there exist $n_1 \in \mathbb{N}$ such that for every $n \geq n_1 \Longrightarrow D(x, x, x_n) < \frac{\epsilon}{2}$ and $n_2 \in \mathbb{N}$ such that for every $n \geq n_2 \Longrightarrow D(y, y, x_n) < \frac{\epsilon}{2}$.

If set $n_0 = \max\{n_1, n_2\}$, then for every $n \geq n_0$ by triangular inequality we have

$$D(x, x, y) \leq D(x, x, x_n) + D(x_n, y, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \varepsilon.$$

Hence $D(x, x, y) = 0$ is a contradiction. So, $x = y$. □

Lemma 1.7. *Let (X, D) be a D-metric space. If sequence $\{x_n\}$ in X is converges to x, then sequence $\{x_n\}$ is a Cauchy sequence.*

Proof. Since $x_n \longrightarrow x$ for each $\epsilon > 0$ there exists

$$n_1 \in \mathbb{N}\text{such that for every}n \geq n_1 \Longrightarrow D(x_n, x_n, x) < \frac{\epsilon}{2}$$

and

$$n_2 \in \mathbb{N}\text{such that for every}m \geq n_2 \Longrightarrow D(x, x_m, x_m) < \frac{\epsilon}{2}.$$

If set $n_0 = \max\{n_1, n_2\}$, then for every $n, m \geq n_0$ by triangular inequality we have

$D(x_n, x_n, x_m) \leq D(x_n, x_n, x) + D(x, x_m, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Hence sequence $\{x_n\}$ is a Cauchy sequence. □

Definition 1.8. A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous t-norm if it satisfies the following conditions

(1) $*$ is associative and commutative,

(2) $*$ is continuous,

(3) $a * 1 = a$ for all $a \in [0,1]$,

(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$.

Two typical examples of continuous t-norm are $a*b = ab$ and $a*b = \min(a,b)$.

Definition 1.9. A 3-tuple $(X, \mathcal{M}, *)$ is called a $\mathcal{M}$-fuzzy metric space if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm, and $\mathcal{M}$ is a fuzzy set on $X^3 \times (0, \infty)$, satisfying the following conditions for each $x, y, z, a \in X$ and $t, s > 0$,

(1) $\mathcal{M}(x, y, z, t) > 0$,

(2) $\mathcal{M}(x, y, z, t) = 1$ if and only if $x = y = z$,

(3) $\mathcal{M}(x, y, z, t) = \mathcal{M}(p\{x, y, z\}, t)$,(symmetry) where p is a permutation function,

(4) $\mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, s) \leq \mathcal{M}(x, y, z, t + s)$,

(5) $\mathcal{M}(x, y, z, .) : (0, \infty) \longrightarrow [0,1]$ is continuous.

Remark 1.10. Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space. We prove that for every $t > 0$, $\mathcal{M}(x, x, y, t) = \mathcal{M}(x, y, y, t)$. Because for each $\epsilon > 0$ by triangular inequality we have

(i) $\mathcal{M}(x, x, y, \epsilon + t) \geq \mathcal{M}(x, x, x, \epsilon) * \mathcal{M}(x, y, y, t) = \mathcal{M}(x, y, y, t)$

(ii) $\mathcal{M}(y, y, x, \epsilon + t) \geq \mathcal{M}(y, y, y, \epsilon) * \mathcal{M}(y, x, x, t) = \mathcal{M}(y, x, x, t)$.

By taking limits of (i) and (ii) when $\epsilon \longrightarrow 0$, we obtain $\mathcal{M}(x, x, y, t) = \mathcal{M}(x, y, y, t)$.

Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space. For $t > 0$, the open ball $B_{\mathcal{M}}(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B_{\mathcal{M}}(x, r, t) = \{y \in X : \mathcal{M}(x, y, y, t) > 1 - r\}.$$

A subset A of X is called open set if for each $x \in A$ there exist $t > 0$ and $0 < r < 1$ such that $B_{\mathcal{M}}(x, r, t) \subseteq A$. A sequence $\{x_n\}$ in X converges to x if and only if $\mathcal{M}(x, x, x_n, t) \longrightarrow 1$ as $n \longrightarrow \infty$, for each $t > 0$. It is called a Cauchy sequence if for each $0 < \epsilon < 1$ and $t > 0$, there exist $n_0 \in \mathbb{N}$ such that $\mathcal{M}(x_n, x_n, x_m, t) > 1 - \epsilon$ for each $n, m \geq n_0$. The $\mathcal{M}$-fuzzy metric $(X, \mathcal{M}, *)$ is said to be complete if every Cauchy sequence is convergent.

Example 1.11. Let X be a nonempty set and D be the D-metric on X. Denote $a * b = a.b$ for all $a, b \in [0,1]$. For each $t \in]0, \infty[$, define

$$\mathcal{M}(x, y, z, t) = \frac{t}{t + D(x, y, z)}$$

for all $x, y, z \in X$. It is easy to see that $(X, \mathcal{M}, *)$ is a $\mathcal{M}$-fuzzy metric space.

Lemma 1.12. *Let $(X, M, *)$ be a fuzzy metric space. If we define $\mathcal{M} : X^3 \times (0, \infty) \longrightarrow [0,1]$ by*

$$\mathcal{M}(x, y, z, t) = M(x, y, t) * M(y, z, t) * M(z, x, t)$$

for every x, y, z in X, then $(X, \mathcal{M}, *)$ is a $\mathcal{M}$-fuzzy metric space.

Proof.
(1) It is easy to see that for every $x, y, z \in X$, $\mathcal{M}(x, y, z, t) > 0\ \forall t > 0$.
(2) $\mathcal{M}(x, y, z, t) = 1$ if and only if $M(x, y, t) = M(y, z, t) = M(z, x, t) = 1$ if and only if $x = y = z$.
(3) $\mathcal{M}(x, y, z, t) = \mathcal{M}(p\{x, y, z\}, t)$, where p is a permutation function.
(4)

$$\begin{aligned} &\mathcal{M}(x, y, z, t+s) \\ &= M(x, y, t+s) * M(y, z, t+s) * M(z, x, t+s) \\ &\geq M(x, y, t) * M(y, a, t) * M(a, z, s) * M(z, a, s) * M(a, x, t) \\ &= \mathcal{M}(x, y, a, t) * M(a, z, s) * M(z, a, s) * M(z, z, s) \\ &= \mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, s) \qquad \text{for every } t, s > 0. \end{aligned}$$

□

Definition 1.13. Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space, then $\mathcal{M}$ is called of *first type* if for every $x, y \in X$ we have

$$\mathcal{M}(x, x, y, t) \geq \mathcal{M}(x, y, z, t)$$

for every $z \in X$.

Also it is called of *second type* if for every $x, y, z \in X$ we have

$$\mathcal{M}(x, y, z, t) = M(x, y, t) * M(y, z, t) * M(z, x, t).$$

Let $a * b = \min(a, b)$ for every $a, b \in [0,1]$ in this case it is easy to see that, if $\mathcal{M}$ is second type then $\mathcal{M}$ is first type.

Example 1.14. If we define $\mathcal{M}(x, y, z, t) = \frac{t}{t + D(x,y,z)}$ where $D(x, y, z) = d(x, y) + d(y, z) + d(x, z)$, or define

$$\mathcal{M}(x, y, z, t) = \begin{cases} 1 & \text{if } x = y = z, \\ \frac{t}{t + \max\{x,y,z\}} & \text{otherwise}, \end{cases}$$

then $\mathcal{M}$ is first type.

If $(X, M, *)$ is a fuzzy metric and $M(x, y, t) = \frac{t}{t + d(x,y)}$, then

$$\mathcal{M}(x, y, z, t) = \frac{t}{t + d(x, y)} * \frac{t}{t + d(y, z)} * \frac{t}{t + d(x, z)}$$

is second type.

Remark 1.15. Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space. If $\mathcal{M}$ is *second* type, sequence $\{x_n\}$ in X converges to x if and only if $\mathcal{M}(x, x, x_n, t) \longrightarrow 1$ or if and only if $M(x, x_n, t) \longrightarrow 1$. For

$$\begin{aligned} \mathcal{M}(x, x, x_n, t) &= M(x, x, t) * M(x, x_n, t) * M(x, x_n, t) \\ &= M(x, x_n, t) * M(x, x_n, t). \end{aligned}$$

2. The main results

Lemma 2.1. *Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space. Then $\mathcal{M}(x, y, z, t)$ is nondecreasing with respect to t, for all x, y, z in X.*

Proof. By Definition 1.9(4) for each $x, y, z, a \in X$ and $t, s > 0$ we have

$$\mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, s) \leq \mathcal{M}(x, y, z, t + s).$$

If set $a = z$ we get $\mathcal{M}(x, y, z, t) * \mathcal{M}(z, z, z, s) \leq \mathcal{M}(x, y, z, t + s)$, that is, $\mathcal{M}(x, y, z, t + s) \geq \mathcal{M}(x, y, z, t)$. □

Definition 2.2. Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space. $\mathcal{M}$ is said to be continuous function on $X^3 \times (0, \infty)$ if

$$\lim_{n\to\infty} \mathcal{M}(x_n, y_n, z_n, t_n) = \mathcal{M}(x, y, z, t).$$

Whenever a sequence $\{(x_n, y_n, z_n, t_n)\}$ in $X^3 \times (0, \infty)$ converges to a point $(x, y, z, t) \in X^3 \times (0, \infty)$ i.e.,

$$\lim_{n\to\infty} x_n = x, \ \lim_{n\to\infty} y_n = y, \ \lim_{n\to\infty} z_n = z \text{ and } \lim_{n\to\infty} \mathcal{M}(x, y, z, t_n) = \mathcal{M}(x, y, z, t).$$

Lemma 2.3. *Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space. Then $\mathcal{M}$ is continuous function on $X^3 \times (0, \infty)$.*

Proof. Let $x, y, z \in X$ and $t > 0$, and let $(x'_n, y'_n, z'_n, t'_n)_n$ be a sequence in $X^3 \times (0, \infty)$ that converges to (x, y, z, t). Since $(\mathcal{M}(x'_n, y'_n, z'_n, t'_n))_n$ is a sequence in $(0, 1]$, there is a subsequence $(x_n, y_n, z_n, t_n)_n$ of sequence $(x'_n, y'_n, z'_n, t'_n)_n$ such that sequence $(\mathcal{M}(x_n, y_n, z_n, t_n))_n$ converges to some point of $[0, 1]$. Fix $\delta > 0$ such that $\delta < \frac{t}{2}$. Then, there is $n_0 \in \mathbb{N}$ such that $|t - t_n| < \delta$ for every $n \geq n_0$. Hence,

$$\begin{aligned} & \mathcal{M}(x_n, y_n, z_n, t_n) \\ \geq\ & \mathcal{M}(x_n, y_n, z_n, t - \delta) \geq \mathcal{M}(x_n, y_n, z, t - \frac{4\delta}{3}) * \mathcal{M}(z, z_n, z_n, \frac{\delta}{3}) \\ \geq\ & \mathcal{M}(x_n, z, y, t - \frac{5\delta}{3}) * \mathcal{M}(y, y_n, y_n, \frac{\delta}{3}) * \mathcal{M}(z, z_n, z_n, \frac{\delta}{3}) \\ \geq\ & \mathcal{M}(z, y, x, t - 2\delta) * \mathcal{M}(x, x_n, x_n, \frac{\delta}{3}) * \mathcal{M}(y, y_n, y_n, \frac{\delta}{3}) * \mathcal{M}(z, z_n, z_n, \frac{\delta}{3}) \end{aligned}$$

and

$$\begin{aligned} & \mathcal{M}(x,y,z,t+2\delta) \\ \geq\ & \mathcal{M}(x,y,z,t_n+\delta) \geq \mathcal{M}(x,y,z_n,t_n+\frac{2\delta}{3}) * \mathcal{M}(z_n,z,z,\frac{\delta}{3}) \\ \geq\ & \mathcal{M}(x,z_n,y_n,t_n+\frac{\delta}{3}) * \mathcal{M}(y_n,y,y,\frac{\delta}{3}) * \mathcal{M}(z_n,z,z,\frac{\delta}{3}) \\ \geq\ & \mathcal{M}(z_n,y_n,x_n,t_n) * \mathcal{M}(x_n,x,x,\frac{\delta}{3}) * \mathcal{M}(y_n,y,y,\frac{\delta}{3}) * \mathcal{M}(z_n,z,z,\frac{\delta}{3}) \end{aligned}$$

for all $n \geq n_0$. By taking limits when $n \longrightarrow \infty$, we obtain

$$\lim_{n\to\infty} \mathcal{M}(x_n,y_n,z_n,t_n) \geq \mathcal{M}(x,y,z,t-2\delta) * 1 * 1 * 1 = \mathcal{M}(x,y,z,t-2\delta)$$

and

$$\mathcal{M}(x,y,z,t+2\delta) \geq \lim_{n\to\infty} \mathcal{M}(x_n,y_n,z_n,t_n)1 * 1 * 1 = \lim_{n\to\infty} \mathcal{M}(x_n,y_n,z_n,t_n),$$

respectively. So, by continuity of the function $t \longmapsto \mathcal{M}(x,y,z,t)$, we immediately deduce that

$$\lim_{n\to\infty} \mathcal{M}(x_n,y_n,z_n,t_n) = \mathcal{M}(x,y,z,t).$$

Therefore $\mathcal{M}$ is continuous on $X^3 \times (0,\infty)$. □

Henceforth, we assume that $*$ is a continuous t-norm on [0,1] such that for every $\mu \in (0,1)$, there is a $\lambda \in (0,1)$ such that

$$\overbrace{(1-\lambda) * (1-\lambda) * \cdots * (1-\lambda)}^{n} \geq 1-\mu$$

Lemma 2.4. *Let $(X,\mathcal{M},*)$ be a $\mathcal{M}$-fuzzy metric space. If we define $E_{\lambda,\mathcal{M}}$: $X^3 \to \mathbb{R}^+ \cup \{0\}$ by*

$$E_{\lambda,\mathcal{M}}(x,y,z) = \inf\{t > 0 \ : \ \mathcal{M}(x,y,z,t) > 1-\lambda\}$$

for every $\lambda \in (0,1)$, then

(i) *for each $\mu \in (0,1)$ there exists $\lambda \in (0,1)$ such that*

$$\begin{aligned} & E_{\mu,\mathcal{M}}(x_1,x_1,x_n) \\ \leq\ & E_{\lambda,\mathcal{M}}(x_1,x_1,x_2) + E_{\lambda,\mathcal{M}}(x_2,x_2,x_3) + \cdots + E_{\lambda,\mathcal{M}}(x_{n-1},x_{n-1},x_n) \end{aligned}$$

for any $x_1, x_2, ..., x_n \in X$,

(ii) *The sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent in $\mathcal{M}$-fuzzy metric space $(X,\mathcal{M},*)$ if and only if $E_{\lambda,\mathcal{M}}(x_n,x_n,x) \to 0$. Also the sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy sequence if and only if it is Cauchy with $E_{\lambda,\mathcal{M}}$.*

Proof. (i). For every $\mu \in (0,1)$, we can find a $\lambda \in (0,1)$ such that

$$\overbrace{(1-\lambda) * (1-\lambda) * \cdots * (1-\lambda)}^{n} \geq 1-\mu$$

by triangular inequality we have

$$\begin{aligned}
&\mathcal{M}(x_1,x_1,x_n,E_{\lambda,\mathcal{M}}(x_1,x_1,x_2)+E_{\lambda,\mathcal{M}}(x_2,x_2,x_3)+\cdots\\
&+E_{\lambda,\mathcal{M}}(x_{n-1},x_{n-1},x_n)+n\delta)\\
\geq\ &\mathcal{M}(x_1,x_1,x_2,E_{\lambda,\mathcal{M}}(x_1,x_1,x_2)+\delta)*\cdots\\
&*\mathcal{M}(x_{n-1},x_{n-1},x_n,E_{\lambda,\mathcal{M}}(x_{n-1},x_{n-1},x_n)+\delta)\\
\geq\ &\overbrace{(1-\lambda)*(1-\lambda)*\cdots*(1-\lambda)}^{n}\geq 1-\mu
\end{aligned}$$

for very $\delta>0$, which implies that

$$\begin{aligned}
&E_{\mu,\mathcal{M}}(x_1,x_1,x_n)\\
\leq\ &E_{\lambda,\mathcal{M}}(x_1,x_1,x_2)+E_{\lambda,\mathcal{M}}(x_2,x_2,x_3)+\cdots+E_{\lambda,\mathcal{M}}(x_{n-1},x_{n-1},x_n)+n\delta.
\end{aligned}$$

Since $\delta>0$ is arbitrary, we have

$$\begin{aligned}
&E_{\mu,\mathcal{M}}(x_1,x_1,x_n)\\
\leq\ &E_{\lambda,\mathcal{M}}(x_1,x_1,x_2)+E_{\lambda,\mathcal{M}}(x_2,x_2,x_3)+\cdots+E_{\lambda,\mathcal{M}}(x_{n-1},x_{n-1},x_n).
\end{aligned}$$

(ii). Note that since $\mathcal{M}$ is continuous in its third place and

$$E_{\lambda,\mathcal{M}}(x,x,y)=\inf\{t>0\ :\ \mathcal{M}(x,x,y,t)>1-\lambda\}.$$

Hence, we have

$$\mathcal{M}(x_n,x,x,\eta)>1-\lambda\Longleftrightarrow E_{\lambda,\mathcal{M}}(x_n,x,x)<\eta$$

for every $\eta>0$. □

Lemma 2.5. *Let $(X,\mathcal{M},*)$ be a $\mathcal{M}$-fuzzy metric space. If*

$$\mathcal{M}(x_n,x_n,x_{n+1},t)\geq\mathcal{M}(x_0,x_0,x_1,k^nt)$$

for some $k>1$ and for every $n\in\mathbb{N}$. Then sequence $\{x_n\}$ is a Cauchy sequence.

Proof. For every $\lambda\in(0,1)$ and $x_n,x_{n+1}\in X$, we have

$$\begin{aligned}
E_{\lambda,\mathcal{M}}(x_n,x_n,x_{n+1}) &= \inf\{t>0\ :\ \mathcal{M}(x_n,x_n,x_{n+1},t)>1-\lambda\}\\
&\leq \inf\{t>0\ :\ \mathcal{M}(x_0,x_0,x_1,k^nt)>1-\lambda\}\\
&= \inf\{\frac{t}{k^n}>0\ :\ \mathcal{M}(x_0,x_0,x_1,t)>1-\lambda\}\\
&= \frac{1}{k^n}\inf\{t>0\ :\ \mathcal{M}(x_0,x_0,x_1,t)>1-\lambda\}\\
&= \frac{1}{k^n}E_{\lambda,\mathcal{M}}(x_0,x_0,x_1).
\end{aligned}$$

By Lemma 2.4, for every $\mu \in (0,1)$ there exists $\lambda \in (0,1)$ such that

$$\begin{aligned} &E_{\mu,\mathcal{M}}(x_n, x_n, x_m) \\ &\leq E_{\lambda,\mathcal{M}}(x_n, x_n, x_{n+1}) + E_{\lambda,\mathcal{M}}(x_{n+1}, x_{n+1}, x_{n+2}) + \cdots + E_{\lambda,\mathcal{M}}(x_{m-1}, x_{m-1}, x_m) \\ &\leq \frac{1}{k^n} E_{\lambda,\mathcal{M}}(x_0, x_0, x_1) + \frac{1}{k^{n+1}} E_{\lambda,\mathcal{M}}(x_0, x_0, x_1) + \cdots + \frac{1}{k^{m-1}} E_{\lambda,\mathcal{M}}(x_0, x_0, x_1) \\ &= E_{\lambda,\mathcal{M}}(x_0, x_0, x_1) \sum_{j=n}^{m-1} \frac{1}{k^j} \longrightarrow 0. \end{aligned}$$

Hence sequence $\{x_n\}$ is Cauchy sequence. □

A class of implicit relation

Let Φ denotes a family of mappings such that each $\phi \in \Phi$, $\phi : [0,1] \longrightarrow [0,1]$, such that ϕ is continuous and $\phi(s) > s$ for every $s \in [0,1)$.

Theorem 2.6. *Let $(X, \mathcal{M}, *)$ and $(Y, \mathcal{N}, \diamond)$ be two complete $\mathcal{M}$ and $\mathcal{N}$-fuzzy metric spaces, respectively where $\mathcal{M}$ and $\mathcal{N}$ are first or second type. If A, B, C be three mappings of X to Y and T, S, R be three mappings of Y to X such that satisfies the following conditions:*

(i) *$\mathcal{M}(SAx, TBx', RCx'', t) \geq \phi(\mathcal{M}(x, x', x'', k_1 t))$, for every $x, x', x'' \in X$ some $k_1 > 1$ and $\phi \in \Phi$,*

(ii) *$\mathcal{N}(CTy, ARy', BSy'', t) \geq \psi(\mathcal{N}(y, y', y'', k_2 t))$, for every $y, y', y'' \in Y$ some $k_2 > 1$ and $\psi \in \Phi$.*

If at least A, B, C, T, S or R be continuous mapping, then there exist a unique point $z \in X$ and $w \in Y$, such that $SAz = TBz = RCz = z$ and $ARw = BSw = CTw = w$. Moreover,

$$Sw = Tw = Rw = z \qquad\qquad Az = Bz = Cz = w.$$

Proof. Let $x_0 \in X$ be an arbitrary point in X, define

$$Ax_0 = y_1, \quad Sy_1 = x_1, \quad Bx_1 = y_2, \quad Ty_2 = x_2, \quad Cx_2 = y_3, \quad \text{and} \quad Ry_3 = x_3.$$

So by induction, for $n = 0, 1, 2, \ldots$ we have

$$Ax_{3n} = y_{3n+1}, Sy_{3n+1} = x_{3n+1}, Bx_{3n+1} = y_{3n+2},$$

$$Ty_{3n+2} = x_{3n+2}, Cx_{3n+2} = y_{3n+3}, Ry_{3n+3} = x_{3n+3}.$$

Now, we prove that $\{x_n\}$ and $\{y_n\}$ are a Cauchy sequence in X and Y respectively. Let

$$d_n(t) = \mathcal{M}(x_n, x_{n+1}, x_{n+2}, t).$$

Now, for $3n$, we get

$$\begin{aligned}
&d_{3n}(t)\\
&= \mathcal{M}(x_{3n}, x_{3n+1}, x_{3n+2}, t)\\
&= \mathcal{M}(Ry_{3n}, Sy_{3n+1}, Ty_{3n+2}, t)\\
&= \mathcal{M}(RCx_{3n-1}, SAx_{3n}, TBx_{3n+1}, t)\\
&= \mathcal{M}(SAx_{3n}, TBx_{3n+1}, RC_{3n-1}, t)\\
&\geq \phi(\mathcal{M}(x_{3n}, x_{3n+1}, x_{3n-1}, k_1 t)\\
&\geq \mathcal{M}(x_{3n-1}, x_{3n}, x_{3n+1}, k_1 t)\\
&= d_{3n-1}(k_1 t).
\end{aligned}$$

For $3n+1$, we have

$$\begin{aligned}
d_{3n+1}(t) &= \mathcal{M}(x_{3n+1}, x_{3n+2}, x_{3n+3}, t) = \mathcal{M}(Sy_{3n+1}, Ty_{3n+2}, Ry_{3n+3}, t)\\
&= \mathcal{M}(SAx_{3n}, TBx_{3n+1}, RCx_{3n+2}, t)\\
&\geq \phi(\mathcal{M}(x_{3n}, x_{3n+1}, x_{3n+2}, k_1 t))\\
&= \mathcal{M}(x_{3n}, x_{3n+1}, x_{3n+2}, k_1 t) = d_{3n}(k_1 t).
\end{aligned}$$

Also, for $3n+2$, we get

$$\begin{aligned}
d_{3n+2}(t) &= \mathcal{M}(x_{3n+2}, x_{3n+3}, x_{3n+4}, t)\\
&= \mathcal{M}(Ty_{3n+2}, Ry_{3n+3}, Sy_{3n+4}, t)\\
&= \mathcal{M}(TBx_{3n+1}, RCx_{3n+2}, SAx_{3n+3}, t)\\
&\geq \phi(\mathcal{M}(x_{3n+1}, x_{3n+2}, x_{3n+3}, k_1 t))\\
&= \mathcal{M}(x_{3n+1}, x_{3n+2}, x_{3n+3}, k_1 t) = d_{3n+1}(k_1 t).
\end{aligned}$$

Hence for every $n \in \mathbb{N}$ we have $d_n(t) \geq d_{n-1}(k_1 t)$. That is,

$$\begin{aligned}
d_n(t) &= \mathcal{M}(x_n, x_{n+1}, x_{n+1}, t)\\
&\geq \mathcal{M}(x_{n-1}, x_n, x_{n+1}, k_1 t) \geq \cdots \geq \mathcal{M}(x_0, x_1, x_2, k_1^n t).
\end{aligned}$$

Since $\mathcal{M}$ is a first or second type, hence by Remark 1.15 $\{x_n\}$ is Cauchy and the completeness of X, $\{x_n\}$ converges to z in X. That is, $\lim_{n\to\infty} x_n = z$.

Let

$$L_n(t) = \mathcal{N}(y_n, y_{n+1}, y_{n+2}, t).$$

Now, for $3n$, we get

$$\begin{aligned}
L_{3n}(t) &= \mathcal{N}(y_{3n}, y_{3n+1}, y_{3n+2}, t) = \mathcal{N}(Cx_{3n-1}, Ax_{3n}, Bx_{3n+1}, t)\\
&= \mathcal{N}(CTy_{3n-1}, ARy_{3n}, BSy_{3n+1}, t) = \mathcal{N}(SAx_{3n}, TBx_{3n+1}, RC_{3n-1}, t)\\
&\geq \psi(\mathcal{N}(y_{3n-1}, y_{3n}, y_{3n+1}, k_2 t)\\
&\geq \mathcal{N}(y_{3n-1}, y_{3n}, y_{3n+1}, k_2 t) = L_{3n-1}(k_2 t).
\end{aligned}$$

For $3n+1$, we have

$$\begin{aligned} L_{3n+1}(t) &= \mathcal{N}(y_{3n+1}, y_{3n+2}, y_{3n+3}, t) = \mathcal{N}(Ax_{3n}, Bx_{3n+1}, Cx_{3n+2}, t) \\ &= \mathcal{N}(ARy_{3n}, BSy_{3n+1}, CTy_{3n+2}, t) \\ &\geq \psi(\mathcal{N}(y_{3n}, y_{3n+1}, y_{3n+2}, k_2t)) \\ &= \mathcal{N}(y_{3n}, y_{3n+1}, y_{3n+2}, k_2t) = L_{3n}(k_2t). \end{aligned}$$

Also, for $3n+2$, we get

$$\begin{aligned} L_{3n+2}(t) &= \mathcal{N}(y_{3n+2}, y_{3n+3}, y_{3n+4}, t) \\ &= \mathcal{N}(Bx_{3n+2}, Cx_{3n+3}, Ax_{3n+4}, t) \\ &= \mathcal{N}(BSy_{3n+1}, CTy_{3n+2}, ARy_{3n+3}, t) \\ &\geq \psi(\mathcal{N}(y_{3n+1}, y_{3n+2}, y_{3n+3}, k_2t)) \\ &= \mathcal{N}(y_{3n+1}, y_{3n+2}, y_{3n+3}, k_2t) = L_{3n+1}(k_2t). \end{aligned}$$

Hence for every $n \in \mathbb{N}$ we have $L_n(t) \geq L_{n-1}(k_2t)$. That is,

$$\begin{aligned} L_n(t) &= \mathcal{N}(y_n, y_{n+1}, y_{n+1}, t) \\ &\geq \mathcal{N}(y_{n-1}, y_n, y_{n+1}, k_2t) \geq \cdots \geq \mathcal{M}(y_0, y_1, y_2, k_2^n t). \end{aligned}$$

Since $\mathcal{N}$ is a first or second type, hence by Remark 1.15 $\{y_n\}$ is Cauchy and the completeness of Y, $\{y_n\}$ converges to w in Y. That is, $\lim_{n\to\infty} y_n = w$.

Let A is continuous, hence $\lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} Ax_{3n} = A \lim_{n\to\infty} x_{3n} = Az = w$. Now, we prove that $SAz = z$. For by (i), we have

$$\mathcal{M}(SAz, TBx_{3n+1}, RCx_{3n+2}, t) \geq \phi(\mathcal{M}(z, x_{3n+1}, x_{3n+2}, k_1t))$$

On making $n \longrightarrow \infty$ we get

$$\mathcal{M}(SAz, z, z, t) \geq \phi(\mathcal{M}(z, z, z, k_1t)) = \phi(1) = 1.$$

Thus $Sw = SAz = z$. Now, we prove that $Bz = w$ for

$$\mathcal{N}(CTy_{3n-1}, ARy_{3n}, BSw, t) \geq \psi(\mathcal{N}(y_{3n-1}, y_{3n}, w, k_2t)).$$

Thus

$$\mathcal{N}(y_{3n}, y_{3n+1}, BSw, t) \geq \psi(\mathcal{N}(y_{3n-1}, y_{3n}, w, k_2t)).$$

As $n \longrightarrow \infty$ we have

$$\mathcal{N}(w, w, BSw, t) \geq \psi(\mathcal{N}(w, w, w, k_2t)) = \psi(1) = 1.$$

Therefore, $BSw = Bz = w$. Again, replacing y by y_{3n-1}, y' by w and y'' by w in (i), we have

$$\mathcal{N}(CTy_{3n-1}, ARw, BSw, t) = \mathcal{N}(y_{3n}, ARw, BSw, t) \geq \psi(\mathcal{N}(y_{3n-1}, w, w, k_2t)).$$

On making $n \longrightarrow \infty$ we get

$$\mathcal{N}(w, ARw, w, t) \geq \psi(\mathcal{N}(w, w, w, k_2t)) = \psi(1) = 1.$$

Thus $ARw = w$. So

$$\mathcal{N}(CTw, ARw, BSw, t) \geq \psi(\mathcal{N}(w, w, w, k_2t)) = 1.$$

Therefore, $CTw = ARw = BSw = w$. Again, replacing x by z, x' by z and x'' by x_{3n+1} in (i), we have

$$\mathcal{M}(RCz, SAz, TBx_{3n+1}, t) \geq \phi(\mathcal{M}(z, z, x_{3n+1}, k_1 t)).$$

On making $n \longrightarrow \infty$ we get

$$\mathcal{M}(RCz, z, z, t) \geq \phi(\mathcal{M}(z, z, z, k_1 t)) = 1.$$

Therefore, $RCz = z$. Now, we prove that $TBz = z$ for

$$\mathcal{M}(RCz, SAz, TBz, t) \geq \phi(\mathcal{M}(z, z, z, k_1 t)) = 1.$$

That is, $TBz = Tw = z$. Hence

$$TBz = RCz = SAz = z.$$

Now, we have $Cz = CTw = w$. So $Rw = RCz = z$. Hence

$$TAz = RCz = SAz = z \quad \text{and} \quad CTw = ARw = BSw = w.$$

Therefore

$$Az = Bz = Cz = w \quad \text{and} \quad Sw = Tw = Rw = z.$$

Uniqueness, let z' be another common fixed point of A, B, C. If $\mathcal{M}(z, z, z', t) < 1$, then

$$\begin{aligned} \mathcal{M}(z, z, z', t) &= \mathcal{M}(TAz, RCz, SAz', t)) \geq \phi(\mathcal{M}(z, z, z', k_1 t)) \\ &> \mathcal{M}(z, z, z', k_1 t) \end{aligned}$$

is a contradiction. Therefore, $z = z'$ is the unique common fixed point of self-maps A, B, C. Similarly we prove that w is unique. Let w' be another common fixed point of R, S, T. If $\mathcal{N}(w, w, w', t) < 1$, then

$$\begin{aligned} \mathcal{N}(w, w, w', t) &= \mathcal{N}(CTw, ARw, BSw', t)) \geq \psi(\mathcal{N}(w, w, w', k_2 t)) \\ &> \mathcal{N}(w, w, w', k_2 t) \end{aligned}$$

is a contradiction. Therefore, $w = w'$ is the unique common fixed point of self-maps T, R, S. □

Example 2.7. Let $X = [0, 1]$, $Y = [1, 2]$. If $S, T, R : [1, 2] \longmapsto [0, 1]$ defined

$$Ty = \begin{cases} 1 & \text{if } y \text{ is rational,} \\ 0 & \text{if } y \text{ is irrational.} \end{cases} \quad Ry = \begin{cases} 1 & \text{if } y \text{ is rational,} \\ \frac{1}{2} & \text{if } y \text{ is irrational.} \end{cases}$$

$$Sy = \begin{cases} 1 & \text{if } y \text{ is rational,} \\ \frac{1}{3} & \text{if } y \text{ is irrational.} \end{cases}$$

Moreover, if $A, B, C : [0, 1] \longmapsto [1, 2]$, defined $Ax = 2$ and

$$Bx = \begin{cases} 2 & \text{if } x \text{ is rational,} \\ 1 & \text{if } x \text{ is irrational.} \end{cases} \quad Cx = \begin{cases} 2 & \text{if } x \text{ is rational,} \\ \frac{3}{2} & \text{if } x \text{ is irrational.} \end{cases}$$

Let $\mathcal{M}, \mathcal{N}, \phi$ and ψ be choice, such that A, B, C and T, R, S satisfying in the above theorem. Then it is easy to see that,$A1 = B1 = C1 = 2$ and $T2 = S2 = R2 = 1$. Hence

$$BS2 = AR2 = CT2 = 2 \text{ and } TB1 = SA1 = RC1 = 1.$$

Corollary 2.8. *Let $(X, \mathcal{M}, *)$ and $(Y, \mathcal{N}, \diamond)$ be two complete $\mathcal{M}$ and $\mathcal{N}$-fuzzy metric spaces, respectively where $\mathcal{M}$ and $\mathcal{N}$ are first or second type. If f be a mapping of X to Y and g be a mapping of Y to X such that satisfies the following conditions:*

(i) $\mathcal{M}(gfx, gfx', gfx'', t) \geq \phi(\mathcal{M}(x, x', x'', k_1 t))$, *for every* $x, x', x'' \in X$ *some* $k_1 > 1$ *and* $\phi \in \Phi$,

(ii) $\mathcal{N}(fgy, fgy', fgy'', t) \geq \psi(\mathcal{N}(y, y', y'', k_2 t))$, *for every* $y, y', y'' \in Y$ *some* $k_2 > 1$ *and* $\psi \in \Phi$.

If at least f or g be continuous mapping, then there exist a unique point $z \in X$ and $w \in Y$,such that $gfz = z$ and $fgw = w$. Moreover,

$$gw = z \qquad\qquad fz = w.$$

Proof. It is enough set $A = B = C = f$ and $R = S = T = g$ in Theorem 2.6. □

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