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SOME PROPERTIES OF SYMMETRIC BI-(σ,τ)-DERIVATIONS IN NEAR-RINGS

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Abstract. In this paper, we introduce a symmetric bi-(σ,τ)-derivation in a near-ring and generalize some of the results in [5, 6, 8, 9].

1. Introduction

The concept of a symmetric bi-derivation has been introduced by Maksa in [4]. Some recent results on properties of prime rings, semi-prime rings and near-rings with derivations have been investigated in several ways [1, 2, 3, 6, 9, 8, 10]. In [6], Öztürk and Jun have introduced the concept of a symmetric bi-derivation of a near-ring and studied some properties.

In this note, we introduce the concept of symmetric bi-(σ,τ)-derivation of a near-ring and give some properties.

Throughout this paper, N will be a zero-symmetric left near-ring with multiplicative center Z. Recall that a near ring N is 3-prime if $aNb=\{0\}$ implies that $a=0$ or $b=0$. σ and τ will be two near-ring automorphisms of N. For $x,y\in N$, $[x,y]$, $[x,y]_{\sigma,\tau}$ and (x,y) will denote the commutator $xy-yx$, $x\sigma(y)-\tau(y)x$ and $x+y-x-y$ respectively. A mapping $D:N\times N\longrightarrow N$ is said to be symmetric if $D(x,y)=D(y,x)$ for all $x,y\in N$. A mapping $d:N\longrightarrow N$ defined by $d(x)=D(x,x)$ is called the trace of D where $D:N\times N\longrightarrow N$ is a symmetric mapping. It is obvious that, if $D:N\times N\longrightarrow N$ is a symmetric mapping which also bi-additive (i.e., additive in both arguments), then the trace of D satisfies the relation $d(x+y)=d(x)+2D(x,y)+d(y)$ for all $x,y\in N$. A symmetric bi-additive mapping $D:N\times N\longrightarrow N$ is called a symmetric bi-derivation if $D(xy,z)=D(x,y)y+xD(y,z)$ is fulfilled for all $x,y,z\in N$. For the terminology used in near-rings, see [7].

2. Results

The following Lemmas are necessary for the paper.

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Lemma 1 ([4, Lemma 3]). *Let N be a 3-prime near-ring.*

(i) If $z \in Z - \{0\}$, then z is not a zero divisor.

(ii) If Z-$\{0\}$ contains an element z for which $z + z \in Z$, then $(N, +)$ is abelian.

Lemma 2 ([6, Lemma 3.1]). *Let N be a 2-torsion free near-ring, D a symmetric bi-additive mapping on N and d the trace of D. If $d(x) = 0$ for all $x \in N$, then $D = 0$.*

Firstly, we introduce the definition of symmetric bi-(σ, τ)-derivation in a near-ring.

Definition 1. A symmetric bi-additive mapping $D : N \times N \longrightarrow N$ is called a symmetric bi-(σ, τ)-derivation if there exists automorphism $\sigma, \tau : N \longrightarrow N$ such that

$$D(xy, z) = D(x, z)\sigma(y) + \tau(x)D(y, z)$$

for all $x, y, z \in N$.

Note that if $\sigma = 1$ and $\tau = 1$ then D is a symmetric bi-derivation.

Lemma 3. *Let N be a 2-torsion free 3-prime near-ring, D a symmetric bi-(σ, τ)-derivation of N and d the trace of D. If $xd(N) = 0$ for all $x \in N$, then $x = 0$ or $D = 0$.*

Proof. Since $d(y + z) = d(y) + 2D(y, z) + d(z)$ for all $y, z \in N$, multiplying by x from the left hand side, and using the hypothesis we have $xD(y, z) = 0$. Hence replace y by yw, to get $x\tau(y)D(w, z) = 0$ for all $x, y, z, w \in N$. Since τ is an automorphism of N, we get $xND(w, z) = 0$. Again since N is 3-prime near-ring, we have $x = 0$ or $D = 0$. □

Note that in Lemma 3, if D is a nontrivial symmetric bi-(σ, τ)-derivation of N and $xd(N) = 0$ for all $x \in N$ then we get only $x = 0$.

Lemma 4. *Let N be a near-ring. D is a symmetric bi-(σ, τ)-derivation of N if and only if $D(xy, z) = \tau(x)D(y, z) + D(x, z)\sigma(y)$ for all $x, y, z \in N$.*

Proof. Let D be a symmetric bi-(σ, τ)-derivation of N. Since σ is an automorphism, we get for all $x, y, z \in N$,

$$\begin{aligned} D(x(y + y), z) &= D(x, z)\sigma(y + y) + \tau(x)D(y + y, z) \\ &= D(x, z)\sigma(y) + D(x, z)\sigma(y) + \tau(x)D(y, z) + \tau(x)D(y, z) \end{aligned}$$

and

$$\begin{aligned} D(x(y + y), z) &= D(xy + xy, z) = D(xy, z) + D(xy, z) \\ &= D(x, z)\sigma(y) + \tau(x)D(y, z) + D(x, z)\sigma(y) + \tau(x)D(y, z). \end{aligned}$$

Combining the above two equality, we find that

$$D(x, z)\sigma(y) + \tau(x)D(y, z) = \tau(x)D(y, z) + D(x, z)\sigma(y).$$

Hence we have $D(xy, z) = \tau(x)D(y, z) + D(x, z)\sigma(y)$ for all $x, y, z \in N$. Converse can be prove in a similar way. □

Lemma 5. *Let N be a near-ring, D a symmetric bi-(σ,τ)-derivation of N and d the trace of D. Then, for all $x,y,z,w \in N$,*

$$(i)\ \left[D(x,z)\sigma(y)+\tau(x)D(y,z)\right]w = D(x,z)\sigma(y)w+\tau(x)\ D(y,z)w,$$
$$(ii)\ \left[\tau(x)D(y,z)+D(x,z)\sigma(y)\right]w = \tau(x)\ D(y,z)w+D(x,z)\sigma(y)w.$$

Proof. Using σ and τ are automorphism and with the technique using in [6, Lemma 3.4], we have the proof. □

Lemma 6. *Let N be a 3-prime near-ring, D a nonzero symmetric bi-(σ,τ)-derivation of N. Then $D(N,N)=0$ for all $x\in N$ implies $x=0$ and $xD(N,N)$ $=0$ implies $x=0$.*

Proof. Suppose $D(y,z)x=0$ for all $x,y,z\in N$. Then taking yw instead of y, using Lemma 5 (i) we have for all $x,y,z\in N$,

$$\begin{aligned} 0 &= D(yw,z)x=\left[D(y,z)\sigma(w)+\tau(y)D(w,z)\right]x \\ &= D(y,z)\sigma(w)x+\tau(y)D(w,z)x \\ &= D(y,z)\sigma(w)x. \end{aligned}$$

Since σ is an automorphism, we have $D(y,z)Nx=0$ for all $x,y,z\in N$. Since N is a 3-prime near ring and D is nontrivial, this implies $x=0$. If $xD(N,N)=0$ then for all $y,w,z\in N$, we have

$$\begin{aligned} 0 &= xD(yw,z)=x(D(y,z)\sigma(w)+\tau(y)D(w,z)) \\ &= xD(y,z)\sigma(w)+x\tau(y)D(w,z) \\ &= x\tau(y)D(w,z) \end{aligned}$$

and with similar argument in the above we have $x=0$. □

Theorem 1. *Let N be a 3-prime near-ring, D a non-zero symmetric bi-(σ,τ)-derivation of N. If N is 2-torsion free and $D(N,N)\subset Z$, then N is a commutative ring.*

Proof. Since $D(N,N)\subset Z$ and D is a non-zero, there exists non-zero elements $x,y\in N$ such that $D(x,y)\in Z-\{0\}$. Then $D(x,y+y)=D(x,y)+D(x,y)\in Z$ and hence $(N,+)$ is abelian by Lemma 1. $D(x,y)\in Z$ for all $x,y\in N$, implies that $zD(x,y)=D(x,y)z$ for all $z\in N$. Hence replace xw with x to get

$$z(D(x,y)\sigma(w)+\tau(x)D(w,y))=(D(x,y)\sigma(w)+\tau(x)D(w,y))z.$$

By Lemma 5 (i) and $D(N,N)\subset Z$, we get

$$D(x,y)z\sigma(w)+D(w,y)z\tau(x)=D(x,y)\sigma(w)z+\tau(x)D(w,y)z$$

or $(N,+)$ abelian we have for all $x,y,z,w\in N$,

$$D(x,y)\left[z,\sigma(w)\right]=D(w,y)\left[z,\tau(x)\right].$$

Taking $D(u,v)$ instead of $\sigma(w)$ for all $u,v\in N$ and since $D(u,v)\in Z$ we have

$$D(D(u,v),y)\left[z,\tau(x)\right]=0$$

for all $x, y, z, u, v \in N$. Hence by Lemma 3 we have $[z, \tau(x)] = 0$ for all $x, z \in N$. So N is a commutative ring since τ is an automorphism. □

Theorem 2. *Let N be a 3-prime near-ring, D a nontrivial symmetric bi-(σ,τ)-derivation of N and d the trace of D. If $[x, d(x)]_{\sigma,\tau} = 0$ then $(N, +)$ is commutative.*

Proof. Since D is a symmetric bi-(σ,τ)-derivation we have

$$\begin{aligned} D(x(x+y),x) &= D(x,x)\sigma(x+y) + \tau(x)D(x+y,x) \\ &= d(x)\sigma(x) + d(x)\sigma(y) + \tau(x)d(x) + \tau(x)D(y,x) \end{aligned}$$

and

$$\begin{aligned} D(x(x+y),x) &= D(x^2+xy,x) = D(x^2,x) + D(xy,x) \\ &= d(x)\sigma(x) + \tau(x)d(x) + d(x)\sigma(y) + \tau(x)D(y,x). \end{aligned}$$

Combining the above equality we have

$$d(x)\sigma(y) + \tau(x)d(x) = \tau(x)d(x) + d(x)\sigma(y).$$

Since $[d(x), x]_{\sigma,\tau} = 0$, we have

$$d(x)\sigma(y) + d(x)\sigma(x) - d(x)\sigma(y) - d(x)\sigma(x) = 0$$

or

$$d(x)(\sigma(y) + \sigma(x) - \sigma(y) - \sigma(x)) = 0.$$

Hence we have $d(x)(\sigma(x), \sigma(y)) = 0$ for all $x, y \in N$. From Lemma 3 and since σ is an automorphism we get that $(N, +)$ is abelian. □

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