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STOCHASTIC CALCULUS FOR ANALOGUE OF WIENER PROCESS

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ABSTRACT. In this paper, we define an analogue of generalized Wiener measure and investigate its basic properties. We define (Itô type) stochastic integrals with respect to the generalized Wiener process and prove the Itô formula. The existence and uniqueness of the solution of stochastic differential equation associated with the generalized Wiener process is proved. Finally, we generalize the linear filtering theory of Kalman–Bucy to the case of a generalized Wiener process.

1. INTRODUCTION

A *generalized Wiener process* $\{A_t\}_{t\geq 0}$ is an additive process with a mean function α and a variation function β, i.e., it has independent increments and for any $0 \leq s < t$, $A_t - A_s$ is normally distributed with mean $\alpha(t) - \alpha(s)$ and variance $\beta(t) - \beta(s)$, where α is a real-valued function on $\mathbf{R}_+ = [0, \infty)$ and β is a strictly increasing real-valued function on $\mathbf{R}_+$, see [5, 6]. In general, a generalized Wiener process is nonstationary. If the generalized Wiener process $\{A_t\}_{t\geq 0}$ has almost surely continuous sample paths, then α and β are continuous, see [18].

In the (traditional) Itô calculus and Kalman–Bucy filtering theory [11], the standard Wiener process (Brownian motion) was used as a noise process [9, 10, 13, 14]. Recently, the stochastic calculi for several stochastic processes have been developed with wide applications (see [2, 3, 4, 7, 8, 15]). Also, recently, several attempts to solve filtering problems of dynamical systems for other noise processes have been made in [1, 12].

The purpose of this paper is to develop (Itô type) stochastic calculus for a generalized Wiener process (Theorem 3.7 and Theorem 4.1) and to extend the linear

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filtering theory of Kalman–Bucy for the standard Wiener process to the case of a generalized Wiener process (Theorem 5.1).

The system process $\{X(t)\}_{t\geq 0}$ and the observation process $\{Z_t\}_{t\geq 0}$ in our filtering problem satisfy the following stochastic differential equations:

$$\begin{aligned} dX(t) &= F(t)X(t)dt + C(t)dA_t, \qquad X(0) = X_0; \\ dZ(t) &= G(t)X(t)dt + D(t)dW_t, \qquad Z(0) = 0, \end{aligned} \tag{1.1}$$

where $\{A_t\}_{t\geq 0}$ and $\{W_t\}_{t\geq 0}$ are certain generalized Wiener processes with mean functions α_1, α_2 and variation functions β_1, β_2. By choosing mean functions and variance functions the linear filtering problem (1.1) express several types of filtering problem for Wiener process. More precisely, if α_1, α_2, β_1 and β_2 are absolutely continuous, then the generalized Wiener processes $\{A_t\}_{t\geq 0}$ and $\{W_t\}_{t\geq 0}$ can be expressed by

$$dA_t = \alpha_1'(t)dt + \sqrt{\beta_1'(t)}dW_{1,t}, \tag{1.2}$$

$$dW_t = \alpha_2'(t)dt + \sqrt{\beta_2'(t)}dW_{2,t}, \; A_0 = W_0 = 0, \tag{1.3}$$

where $\{W_{1,t}\}_{t\geq 0}$ and $\{W_{2,t}\}_{t\geq 0}$ are standard Wiener processes. Then the problem (1.1) is equivalent to

$$\begin{aligned} dX(t) &= (F(t)X(t) + H(t))\,dt + C'(t)dW_{1,t}, \quad X(0) = X_0; \\ dZ(t) &= (G(t)X(t) + K(t))\,dt + D'(t)dW_{2,t}, \quad Z(0) = 0 \end{aligned} \tag{1.4}$$

for certain functions H, K, C' and D'. The filtering problem (1.4) is a special type of linear nonstationary filtering problem in [13].

This paper is organized as follows: In Section 2 we define an analogue of generalized Wiener measure $\omega_\varphi^{\alpha,\beta}$, and investigate the basic properties of $\omega_\varphi^{\alpha,\beta}$. In Section 3 we study the stochastic integrals with respect to the generalized Wiener process and prove the Itô formula for the generalized Wiener process. In Section 4 we prove the existence and uniqueness of the solution of a stochastic differential equation associated with the generalized Wiener process. In Section 5 we study the linear filtering theory for a generalized Wiener process.

2. Analogue of Generalized Wiener Process

From the analogue of Wiener space $(C[0,T], \mathcal{B}(C[0,T]), \omega_\varphi)$ associate with a complex-valued measure φ on $\mathbf{R}$ [16, 17], we define an *analogue of generalized Wiener* *space* $C[0,T]$ consisting of Borel the σ-algebra $\mathcal{B}(C[0,T])$ and a measure $\omega_\varphi^{\alpha,\beta}$ which is defined by

$$\omega_\varphi^{\alpha,\beta}(E)=\int_B \Psi_{\alpha,\beta}(u_0,u_1,\cdots,u_n)\,d\left(\prod_{j=1}^n m_L\right)(u_1,u_2,\cdots,u_n)d\varphi(u_0),$$

where

$$\Psi_{\alpha,\beta}(u_0,u_1,\cdots,u_n)=\left[\prod_{j=1}^n 2\pi(\beta(t_j)-\beta(t_{j-1}))\right]^{-1/2}$$
$$\times\exp\left\{-\frac{1}{2}\sum_{j=1}^n\frac{[(u_j-\alpha(t_j))-(u_{j-1}-\alpha(t_{j-1}))]^2}{\beta(t_j)-\beta(t_{j-1})}\right\}$$

and $\alpha:[0,T]\longrightarrow\mathbf{R}$ is a function, $\beta:[0,T]\longrightarrow\mathbf{R}_+=[0,\infty)$ is an increasing function and $E\in\mathcal{B}(C[0,T])$ with

$$E=\left\{x\in C[0,T]\mid(x(t_0),x(t_1),\cdots,x(t_n))\in B,B\in\mathcal{B}(\mathbf{R}^{n+1})\right\}.$$

Here $0=t_0<t_1<\cdots<t_n=T$. The completion of $\omega_\varphi^{\alpha,\beta}$ is denoted by the same notation. Then we obtain the following theorem for an analogue of generalized Wiener measure $\omega_\varphi^{\alpha,\beta}$.

Theorem 2.1. (Wiener Integration Formula) *If* $f:\mathbf{R}^{n+1}\to\mathbf{R}$ *is a Borel measurable function, then the following equality holds:*

$$\int_{C[0,T]} f(x(t_0),x(t_1),\cdots,x(t_n))d\omega_\varphi^{\alpha,\beta}(x)$$
$$=\int_{\mathbf{R}^{n+1}} f(u_0,u_1,\cdots,u_n)\Psi_{\alpha,\beta}(u_0,u_1,\cdots,u_n)$$
$$d\left(\prod_{j=1}^n m_L\right)(u_1,u_2,\cdots,u_n)d\varphi(u_0).$$

The proof is obvious by applying the change of variable formula.

Example 2.2. Suppose φ has a normal distribution with mean $\alpha(0)$ and variance $\beta(0)$. Then we have

(1) For $0\le s\le T$, $\int_{C[0,T]} x(s)d\omega_\varphi^{\alpha,\beta}(x)=\alpha(s)$.

(2) For $0\le s_1,s_2\le T$, $\int_{C[0,T]} x(s_1)x(s_2)d\omega_\varphi^{\alpha,\beta}(x)=\beta(s_1\wedge s_2)+\alpha(s_1)\alpha(s_2)$. In particular, $\int_{C[0,T]} x(s_1)^2d\omega_\varphi^{\alpha,\beta}(x)=\beta(s_1)+\alpha(s_1)^2$.

(3) For $0 \le s_1 < s_2 \le s_3 < s_4 \le T$, $x(s_2) - x(s_1)$ and $x(s_4) - x(s_3)$ are independent.

Example 2.3. Let $0 \le s \le T$. We define the random variables A_s and B_s on $C[0,T]$ by

$$A_s(x) = x(s) - x(0), \qquad B_s(x) = A_s(x) - (\alpha(s) - \alpha(0)),$$

respectively. Then A_s and B_s are generalized Wiener processes (or generalized Brownian motion) such that

$$A_s \sim N(\alpha(s) - \alpha(0), \beta(s) - \beta(0))$$

and

$$B_s \sim N(0, \beta(s) - \beta(0)).$$

By direct computation, we have

$$\begin{aligned} \mathbf{E}[|A_t - A_s|^4] = 3(\beta(t) - \beta(s))^2 + 6(\alpha(t) - \alpha(s))^2(\beta(t) - \beta(s)) \\ + (\alpha(t) - \alpha(s))^4, \qquad 0 \le s,t \le T. \end{aligned} \tag{2.5}$$

If there exist $\frac{1}{2} < \delta < 1$, $\frac{1}{4} < \gamma < 1$ and positive constants C_1, C_2 such that

$$|\beta(t) - \beta(s)| \le C_1|t-s|^\delta, \qquad |\alpha(t) - \alpha(s)| \le C_2|t-s|^\gamma, \qquad 0 \le s,t \le T,$$

then, by Kolomogolov's continuity theorem, $\{A_t\}_{t\ge 0}$ has a continuous version. More generally, if α is continuous and β is a monotone increasing continuous function, then the process $\{A_t\}$ has an equivalent continuous process, for the proof we refer to [18].

3. Stochastic Integral and Itô Formula

Let $\mathcal{A}_t$ be the σ-algebra generated by $\{A_s\,;\, 0 \le s \le t\}$. From now on, we assume that α is continuous, of bounded variation function and β is a monotone increasing continuous function.

Definition 3.1. Let $\mathfrak{M}_{\mathrm{G}} = \mathfrak{M}_{\mathrm{G}}[0,T]$ be the class of functions

$$f : [0,T] \times C[0,T] \longrightarrow \mathbf{R}$$

such that

(1) the map $(t,x) \mapsto f(t,x)$ is $\mathcal{B}([0,T]) \times \mathcal{B}(C[0,T])$-measurable,
(2) for each $0 \le t \le T$, $f(t,\cdot)$ is $\mathcal{A}_t$-measurable,
(3) $\mathbf{E}\left[\int_0^T f(t,x)^2 d\left(\beta(t) + |\alpha|(t)\right)\right] < \infty$.

A function $\phi \in \mathfrak{M}_{\mathrm{G}}$ is called an *elementary function* if it has the form

$$\phi(t,x) = \sum_j e_j(x)\mathbf{1}_{[t_j,t_{j+1})}(t), \tag{3.6}$$

where e_j is $\mathcal{A}_{t_j}$-measurable. For the elementary function ϕ given as in (3.6), we define

$$\int_0^T \phi(t,x)dA_t(x) = \sum_j e_j(x)\left(A_{t_{j+1}} - A_{t_j}\right), \tag{3.7}$$

$$\int_0^T \phi(t,x)dB_t(x) = \sum_j e_j(x)\left(B_{t_{j+1}} - B_{t_j}\right), \tag{3.8}$$

$$\int_0^T \phi(t,x)d\alpha(t) = \sum_j e_j(x)\left(\alpha(t_{j+1}) - \alpha(t_j)\right).$$

Then it is clearly from Example 2.3 that

$$\int_0^T \phi(t,x)dA_t(x) = \int_0^T \phi(t,x)dB_t(x) + \int_0^T \phi(t,x)d\alpha(t). \tag{3.9}$$

The integral defined as in (3.7) is called the *stochastic integral* of ϕ with respect to the generalized Wiener process $\{A_t\}_{t\geq 0}$.

Lemma 3.2. *If $\phi(t,x)$ is an elementary bounded function, then we have*

$$\mathbf{E}\left[\left(\int_0^T \phi(t,x)dA_t(x)\right)^2\right] \leq 2\left(\mathbf{E}\left[\int_0^T \phi(t,x)^2 d\beta(t)\right]\right.$$
$$\left. + V_0^T(\alpha)\mathbf{E}\left[\int_0^T \phi(t,x)^2 d|\alpha|(t)\right]\right),$$

where $V_0^T(\alpha)$ is the total variation of α over $[0,T]$.

Proof. By direct computation we prove that

$$\mathbf{E}\left[\left(\int_0^T \phi(t,x)dB_t(x)\right)^2\right] = \mathbf{E}\left[\int_0^T \phi(t,x)^2 d\beta(t)\right]. \tag{3.10}$$

On the other hand, by Cauchy-Schwartz inequality, we have

$$\left(\int_0^T \phi(t,x)d\alpha(t)\right)^2 \leq \left(\int_0^T \phi(t,x)^2 d|\alpha|(t)\right)\left(\int_0^T d|\alpha|(t)\right).$$

Therefore, by applying (3.9) we obtain the desired inequality. □

Proposition 3.3. *If $f \in \mathfrak{M}_G[0,T]$, we choose elementary functions $\phi_n \in \mathfrak{M}_G[0,T]$ such that*

$$\mathbf{E}\left[\int_0^T |f-\phi_n|^2 d(\beta+|\alpha|)(t)\right] \longrightarrow 0 \quad as \quad n\to\infty \tag{3.11}$$

Proof. By applying Lemma 3.2, the proof is a simple modification of the arguments used in Chapter 3 in [14]. □

Definition 3.4. Let $f \in \mathfrak{M}_G$. The *stochastic integral* with respect to the process $\{A_t\}_{t\geq 0}$ is defined by

$$\int_0^T f(t,x)dA_t(x) = \lim_{n\to\infty}\int_0^T \phi_n(t,x)dA_t(x),$$

where $\{\phi_n\}$ is the sequence of elementary functions in $\mathfrak{M}_G$ given in Proposition 3.3 and the limit exists in $L^2(\omega_\varphi^{\alpha,\beta})$.

Example 3.5. Put $\phi_n(s,x) = \sum A_j(x)\mathbf{1}_{[t_j,t_{j+1})}(s)$, where $A_j = A_{t_j}$. Then we have

$$\begin{aligned}&\mathbf{E}\left[\int_0^t (\phi_n - A_s)^2 d(\beta+|\alpha|)(s)\right]\\ &= \sum_j \int_{t_j}^{t_{j+1}} \left[(\beta(s)-\beta(t_j)+(\alpha(s)-\alpha(t_j))^2)\right] d(\beta+|\alpha|)(s)\\ &\to 0 \quad \text{as} \quad n\to\infty.\end{aligned}$$

Hence we have

$$\int_0^t A_s dA_s = \lim_{n\to\infty}\int_0^t \phi_n dA_s = \lim_{\Delta t_j\to 0}\sum_j A_j\Delta A_j = \frac{1}{2}A_t^2 - \frac{1}{2}\sum_j(\Delta A_j)^2.$$

On the other hand, by applying (2.5), for $a_n = \sum_j \left(\Delta\beta(t_j) + (\Delta\alpha(t_j))^2\right)$ we have

$$\mathbf{E}\left[\left(\sum_j(\Delta A_j)^2 - a_n\right)^2\right] \longrightarrow 0 \text{ in } L^2(\omega_\varphi^{\alpha,\beta}) \quad \text{as } n\longrightarrow\infty.$$

Finally, we prove that

$$\int_0^t A_s dA_s = \lim_{n\to\infty}\int_0^t \phi_n dA_s = \frac{1}{2}A_t^2 - \frac{1}{2}\left(\beta(t)-\beta(0)\right). \tag{3.12}$$

Proposition 3.6. *For any $f \in \mathfrak{M}_G[0,T]$ we have*

$$\mathbf{E}\left[\left(\int_0^T f(t,x)dA_t(x)\right)^2\right]$$

$$\leq 2\mathbf{E}\left[\int_0^T f(t,x)^2 d\beta(t)\right] + 2V_0^T(\alpha)\mathbf{E}\left[\int_0^T f(t,x)^2 d|\alpha|(t)\right].$$

Proof. By applying Lemma 3.2 and Definition 3.4, the proof is straightforward. □

A *diffusion process* for a generalized Wiener process $\{A_t\}_{t\geq 0}$ is a stochastic process X_t of the form

$$X_t = X_0 + \int_0^t u(s,x)ds + \int_0^t v(s,x)dA_s,$$

where $v \in \mathfrak{M}_G[0,T]$ and u is $\mathcal{A}_t$-adapted with

$$E\left[\int_0^T u(s,x)^2 ds\right] < \infty.$$

Theorem 3.7. (Itô Formula) *Let X_t be a diffusion process given by*

$$dX_t = udt + vdA_t. \tag{3.13}$$

Let $g \in C^2(\mathbf{R}_+ \times \mathbf{R})$. Then $Y_t = g(t,X_t)$ is again a diffusion process and satisfies the following Stochastic differential equation:

$$dY_t = \frac{\partial g}{\partial t}(t,X_t)dt + \frac{\partial g}{\partial x}(t,X_t)dX_t + \frac{1}{2}\frac{\partial^2 g(t,X_t)}{\partial x^2}(dX_t)^2, \tag{3.14}$$

where $(dX_t)^2 = (dX_t)(dX_t)$ is computed according to the rules:

$$dt\cdot dt = dt\cdot dA_t = dA_t\cdot dt = 0, \qquad dA_t\cdot dA_t = d\beta(t).$$

Proof. The proof is a simple modification of the arguments used in the proof of Theorem 4.1.2 in [14]. Here we prove only the case of g, $\partial g/\partial t$, $\partial g/\partial x$, $\partial^2 g/\partial x^2$ are bounded, and u, v are elementary. From (3.13) and (3.14), we have

$$g(t,X_t) - g(0,X_0) = \int_0^t \left(\frac{\partial g}{\partial t} + \frac{\partial g}{\partial x}v + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}v^2\beta'(s)\right)ds + \int_0^t \frac{\partial g}{\partial x}vdA_s.$$

Therefore, $Y_t = g(t,X_t)$ is a diffusion process. By applying Taylor's theorem we have

$$g(t,X_t) = g(0,X_0) + \sum_j \left(\frac{\partial g}{\partial t}\Delta t_j + \frac{\partial g}{\partial x}\Delta X_{t_j} + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(\Delta X_{t_j})^2 + Z_j + R_j\right)$$

with

$$Z_j = \frac{1}{2}\frac{\partial^2 g}{\partial t^2}(\Delta t_j)^2 + \frac{1}{2}\frac{\partial^2 g}{\partial t\partial x}\Delta X_{t_j}\Delta t_j + \frac{1}{2}\frac{\partial^2 g}{\partial x\partial t}\Delta t_j\Delta X_{t_j},$$

where $\Delta t_j = t_{j+1} - t_j$, $\Delta X_{t_j} = X_{t_{j+1}} - X_{t_j}$ and $R_j = o(|\Delta t_j|^2, |\Delta X_{t_j}|^3)$ for any j. If $\Delta t_j \to 0$, we can easily show that $Z_j \to 0$ and

$$\sum_j \frac{\partial g}{\partial t}\Delta t_j \longrightarrow \int_0^t \frac{\partial g}{\partial t}(s, X_s)ds, \quad \sum_j \frac{\partial g}{\partial x}\Delta X_j \longrightarrow \int_0^t \frac{\partial g}{\partial x}(s, X_s)dX_s \tag{3.15}$$

in $L^2(\omega_\varphi^{\alpha,\beta})$. On the other hand, since u and v are elementary, we have

$$\sum_j \frac{\partial^2 g}{\partial x^2}(\Delta X_{t_j})^2 = \sum_j \frac{\partial^2 g}{\partial x^2}\left(u_j^2(\Delta t_j)^2 + 2u_j v_j \Delta t_j \Delta A_{t_j} + v_j^2(\Delta A_{t_j})^2\right).$$

Also, if $\Delta t_j \to 0$, we can easily show that

$$\mathbf{E}\left[\left(\sum_j \frac{\partial^2 g}{\partial x^2}u_j^2(\Delta t_j)^2\right)^2\right] \to 0, \quad \mathbf{E}\left[\left(\sum_j \frac{\partial^2 g}{\partial x^2}u_j v_j \Delta t_j \Delta A_j\right)^2\right] \to 0.$$

By direct computation we prove that

$$\mathbf{E}\left[\left(\sum_j e_j\left[(\Delta A_j)^2 - \Delta\beta(t_j)\right]\right)^2\right] \to 0$$

as $\Delta t_j \longrightarrow 0$, where $e_j = e(t_j)$, $e(t) = \left(\partial^2 g(t, X_t)/\partial x^2\right) v(t,x)^2$. Hence, we prove that

$$\sum_j \frac{\partial^2 g}{\partial x^2}(\Delta X_j)^2 \to \int_0^t \frac{\partial^2 g}{\partial x^2}(s, X_s)v(s,x)^2 d\beta(s) \tag{3.16}$$

in $L^2(\omega_\varphi^{\alpha,\beta})$ as $\Delta t_j \longrightarrow 0$. Finally, since $\sum_j R_j \to 0$, we prove that

$$\begin{aligned} g(t, X_t) - g(0, X_0) = &\int_0^t \frac{\partial g}{\partial t}(s, X_s)ds + \int_0^t \frac{\partial g}{\partial x}(s, X_s)dX_s \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2 g}{\partial x^2}(s, X_s)v(s,x)^2 d\beta(s) \end{aligned}$$

which is equivalent to (3.14). □

By applying Itô formula, we solve Example 3.5.

Example 3.8. Let $X_t = A_t$. Then by applying the Itô formula to the process $Y_t = A_t^2/2$, we have

$$dY_t = A_t dA_t + \frac{1}{2}d\beta(t)$$

which is equivalent to (3.12).

4. Stochastic Differential Equations

For each function $f : [0,T] \times C[0,T] \to \mathbf{R}$ satisfying the conditions (1) in Definition 3.1 and $f(t,\cdot)$ is $\mathcal{B}_t$(the σ-algebra generated by $\{B_s\,;\,0 \le s \le t\}$)-measurable for each $0 \le t \le T$ with

$$\mathbf{E}\left[\int_0^T f(t,x)^2 d\beta(t)\right] < \infty,$$

the integral $M_t \equiv \int_0^t f(s,x)dB_s(x)$ is well-defined as following:

$$\int_0^t f(s,x)dB_s(x) = \lim_{n\to\infty}\int_0^t \phi_n(s,x)dB_s(x),$$

where the limit exists in $L^2(\omega_\varphi^{\alpha,\beta})$ and $\int_0^t \phi_n(s,x)dB_s(x)$ is defined as in (3.8) for the elementary function ϕ_n given by

$$\phi_n(t,x) = \sum_j^n e_j(x)\mathbf{1}_{[t_j,t_{j+1})}(t)$$

in which e_j is $\mathcal{B}_{t_j}$-measurable. Then by applying (3.10) we can easily prove that

$$\mathbf{E}\left[\left(\int_0^t f(s,x)dB_s x)\right)^2\right] = \mathbf{E}\left[\int_0^t f(s,x)^2 d\beta(s)\right], \tag{4.17}$$

and the process $\{M_t\}_{0\le t\le T}$ is a martingale with respect to the filtration $\mathcal{B}_t$.

For the notational convenience, the identity function on $\mathbf{R}$ is denoted by ρ.

Theorem 4.1. *Let $T > 0$ and $b(\cdot,\cdot) : [0,T] \times \mathbf{R} \to \mathbf{R}$, $\sigma(\cdot,\cdot) : [0,T] \times \mathbf{R} \to \mathbf{R}$ be measurable functions satisfying*

$$|b(t,x)| + |\sigma(t,x)| \le C(t)(1+|x|), \qquad x \in \mathbf{R}, \quad t \in [0,T] \tag{4.18}$$

and

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le C(t)|x-y|, \qquad x,y \in \mathbf{R}, \quad t \in [0,T] \tag{4.19}$$

for some measurable function $C \in L^\infty[0,T]$. Let Z be a random variable which is independent of the σ-algebra $\mathcal{A}_\infty$ generated by $\{A_s\,;\,s \ge 0\}$ and such that $\mathbf{E}[|Z|^2] < \infty$. Then the stochastic differential equation

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dA_t, \qquad X_0 = Z, \quad 0 \le t \le T \tag{4.20}$$

has the a unique t-continuous solution X_t with the property that X_t is adapted to the filtration $\mathcal{A}_t^Z$ generated by Z and $\mathcal{A}_t$ and

$$\mathbf{E}\left[\int_0^T |X_t|^2\, d(\rho+\beta+|\alpha|)(t)\right] < \infty. \tag{4.21}$$

Proof. Uniqueness. Let X_t and Y_t be solutions of (4.20) satisfying (4.21). Put $a(s) = b(s, X_s) - b(s, Y_s)$ and $\gamma(s) = \sigma(s, X_s) - \sigma(s, Y_s)$. Then by applying (4.19) and Proposition 3.6 we have

$$\begin{aligned}\mathbf{E}[|X_t - Y_t|^2] &= \mathbf{E}\left[\left(\int_0^t a(s)ds + \int_0^t \gamma(s)dA_s\right)^2\right] \\ &\le 2\mathbf{E}\left[\left(\int_0^t a(s)ds\right)^2\right] + 2\mathbf{E}\left[\left(\int_0^t \gamma(s)dA_s\right)^2\right] \\ &\le \|C\|_\infty^2 \int_0^t \mathbf{E}[|X_s - Y_s|^2] d\left(2T\rho + 4\beta + 4V_0^T(\alpha)|\alpha|\right)(s).\end{aligned}$$

Therefore, by the Gronwall's inequality we prove that

$$\mathbf{E}[|X_t - Y_t|^2] = 0, \qquad 0 \le t \le T. \tag{4.22}$$

Hence $X_t(x) = Y_t(x)$ a.e. x in $C[0, T]$.

Existence. Let's define $Y_t^{(0)} = X_0$ and $Y_t^{(k)}$ inductively as following:

$$Y_t^{(k+1)} = X_0 + \int_0^t b(s, Y_s^{(k)})ds + \int_0^t \sigma(s, Y_s^{(k)})dA_s. \tag{4.23}$$

In fact, for any $k = 0, 1, 2, \cdots$, by applying Proposition 3.6 and (4.18), we have

$$\begin{aligned}&\mathbf{E}\left[\left|Y_t^{(k+1)}\right|^2\right] \\ \le& 3\mathbf{E}(|X_0|^2) + 3\mathbf{E}\left[\left(\int_0^t b(s, Y_s^{(k)})ds\right)^2\right] + 3\mathbf{E}\left[\left(\int_0^t \sigma(s, Y_s^{(k)})dA_s\right)^2\right] \\ \le& 3\mathbf{E}(|X_0|^2) + 3t\mathbf{E}\left[\int_0^t b(s, Y_s^{(k)})^2 ds\right] + 6\mathbf{E}\left[\int_0^t \sigma(s, Y_s^{(k)})^2 d\beta(s)\right] \\ &+ 6V_0^T(\alpha)\mathbf{E}\left[\int_0^t \sigma(s, Y_s^{(k)})^2 d|\alpha|(s)\right]\end{aligned}$$

and so we have

$$\begin{aligned}
&\mathbf{E}\left[\left|Y_t^{(k+1)}\right|^2\right] \\
&\le 3\mathbf{E}[|X_0|^2] + 3t\|C\|_\infty^2 \left(\int_0^t \mathbf{E}[(1+|Y_s^{(k)}|^2)]ds\right) \\
&\quad + 6\|C\|_\infty^2 \left(\int_0^t \mathbf{E}[(1+|Y_s^{(k)}|^2)]d\beta(s)\right) \\
&\quad + 6V_0^T(\alpha)\|C\|_\infty^2 \left(\int_0^t \mathbf{E}[(1+|Y_s^{(k)}|^2)]d|\alpha|(s)\right) \\
&\le 3\mathbf{E}[|X_0|^2] + 3K_1 \left(\int_0^t (1+\mathbf{E}[|Y_s^{(k)}|^2])d(\rho+\beta+|\alpha|)(s)\right),
\end{aligned}$$

where $K_1 = \|C\|_\infty^2 \max\{3T, 6, 6V_0^T(\alpha)\}$. Therefore, we can easily see that for any $k = 0, 1, 2, \cdots$, $\sigma(s, Y_s^{(k)}) \in \mathfrak{M}_{\mathrm{G}}[0,T]$ and so the integrals in the right hand side of (4.23) is well-defined. On the other hand, by applying Proposition 3.6 and (4.18), we have

$$\begin{aligned}
&\mathbf{E}\left[\left|Y_t^{(1)} - Y_t^{(0)}\right|^2\right] \\
&\le 2\mathbf{E}\left[\left(\int_0^t b(s,X_0)ds\right)^2\right] + 2\mathbf{E}\left[\left(\int_0^t \sigma(s,X_0)dA_s\right)^2\right] \\
&\le 2T\mathbf{E}\left[\int_0^t b(s,X_0)^2 ds\right] + 4\mathbf{E}\left[\int_0^t \sigma(s,X_0)^2 d\beta(s)\right] \\
&\quad + 4V_0^T(\alpha)\mathbf{E}\left[\int_0^t \sigma(s,X_0)^2 d|\alpha|(s)\right] \\
&\le K_2\mathbf{E}[(1+|X_0|^2)]H(t),
\end{aligned}$$

where $K_2 = \|C\|_\infty^2 \max\{2T, 4, 4V_0^T(\alpha)\}$ and $H(t) = \int_0^t d(\rho+\beta+|\alpha|)(s)$. Now, we put

$$a(s,k) = b(s, Y_s^{(k)}) - b(s, Y_s^{(k-1)}), \qquad \gamma(s,k) = \sigma(s, Y_s^{(k)}) - \sigma(s, Y_s^{(k-1)}).$$

Then, by similarly arguments using Proposition 3.6 and (4.19), we have

$$\begin{aligned}
\mathbf{E}\left[\left|Y_t^{(k+1)} - Y_t^{(k)}\right|^2\right] &\le 2T\mathbf{E}\left[\int_0^t a(s,k)^2 ds\right] + 4\mathbf{E}\left[\int_0^t \gamma(s,k)^2 d\beta(s)\right] \\
&\quad + 4V_0^T(\alpha)\mathbf{E}\left[\int_0^t \gamma(s,k)^2 d|\alpha|(s)\right] \\
&\le K_2 \int_0^t \mathbf{E}[|Y_s^{(k)} - Y_s^{(k-1)}|^2]d(\rho+\beta+|\alpha|)(s).
\end{aligned}$$

Therefore, for any $k = 0, 1, 2, \cdots$, we have

$$\mathbf{E}\left[\left|Y_t^{(k+1)} - Y_t^{(k)}\right|^2\right] \leq \mathbf{E}[(1 + |X_0|^2)]\frac{(K_2 H(t))^k}{k!}. \tag{4.24}$$

Note that

$$\sup_{0\leq t\leq T}\left|Y_t^{(k+1)} - Y_t^{(k)}\right| \leq \int_0^T |a(s,k)|ds + \sup_{0\leq t\leq T}\left|\int_0^t \gamma(s,k)dB_s\right| + \int_0^T |\gamma(s,k)|d|\alpha|(s).$$

Hence by the martingale inequality and the Chebychev inequality we have

$$\begin{aligned}
&P\left[\sup_{0\leq t\leq T}\left|Y_t^{(k+1)} - Y_t^{(k)}\right| > 3^{-k}\right] \\
&\leq P\left[\left(\int_0^t |a(s,k)|ds\right)^2 > 3^{-2(k+1)}\right] \\
&\quad + P\left[\sup_{0\leq t\leq T}\left|\int_0^t \gamma(s,k)dB_s\right| > 3^{-(k+1)}\right] \\
&\quad + P\left[\int_0^T |\gamma(s,k)|d|\alpha|(s) > 3^{-(k+1)}\right] \\
&\leq 3^{2(k+1)}T\int_0^T \mathbf{E}[|a(s,k)|^2]ds + 3^{2(k+1)}\int_0^T \mathbf{E}[|\gamma(s,k)|^2]d\beta(s) \\
&\quad + 3^{2(k+1)}V_0^T(\alpha)\int_0^T \mathbf{E}[|\gamma(s,k)|^2]d|\alpha|(s) \\
&\leq 3^{2(k+1)}K_3\|C\|_\infty^2\int_0^T \mathbf{E}[|Y_s^{(k)} - Y_s^{(k-1)}|^2]d(\rho + \beta + |\alpha|)(s) \\
&\leq K_3\mathbf{E}[(1 + |X_0|^2)]\frac{3^{2(k+1)}K_2^{k-1}H(T)^k}{k!},
\end{aligned}$$

where $K_3 = \|C\|_\infty^2 \max\{T, 1, V_0^T(\alpha)\}$. Therefore, by the Borel–Cantelli lemma, we prove that

$$P\left[\sup_{0\leq t\leq T}\left|Y_t^{(k+1)} - Y_t^{(k)}\right| > 3^{-k} \quad \text{for infintely many } k\right] = 0.$$

Hence, for almost all x there exists $k_0 = k_0(x)$ such that

$$\sup_{0\leq t\leq T}\left|Y_t^{(k+1)}(x) - Y_t^{(k)}(x)\right| \leq 3^{-k} \quad \text{for } k \geq k_0.$$

Therefore, the sequence

$$Y_t^{(n)}(x) = Y_t^{(0)}(x) + \sum_{k=0}^{n-1} \left(Y_t^{(k+1)}(x) - Y_t^{(k)}(x) \right)$$

is uniformly convergent in $[0,T]$ for almost all x and the limit is denoted by $X_t(x)$. Then, since $Y_t^{(n)}(x)$ is t-continuous for any n and almost all x, X_t is t-continuous for almost all x. Moreover, since $Y_t^{(n)}$ is $\mathcal{A}_t^Z$ measurable for any $0 \le t \le T$ and any n, X_t is $\mathcal{A}_t^Z$ measurable. Next, we note that for $m > n \ge 0$ we have by (4.24)

$$\begin{aligned} \mathbf{E}\left[\left| Y_t^{(m)} - Y_t^{(n)} \right|^2 \right]^{1/2} &\le \sum_{k=n}^{m-1} \left\| Y_t^{(k+1)} - Y_t^{(k)} \right\|_{L^2(\omega_\varphi^{\alpha,\beta})} \\ &\le \sqrt{\mathbf{E}[(1+|X_0|^2)]} \sum_{k=n}^{\infty} \left[\frac{(K_2 H(T))^k}{k!} \right]^{1/2} \\ &\to 0 \end{aligned}$$

as $n \to \infty$. Therefore, $\{Y_t^{(n)}\}$ converges to a limit Y_t in $L^2(\omega_\varphi^{\alpha,\beta})$. Then a subsequence of $Y_t^{(n)}(x)$ converges x-pointwise to $Y_t(x)$ and so we have $Y_t = X_t$ almost surely. In particular, X_t satisfy (4.21) and is adapted to the filtration $\mathcal{A}_t^Z$ generated by Z and $\mathcal{A}_t$. Since $Y_t^{(n)}(x) \to X_t(x)$ uniformly in $t \in [0,T]$ for almost all x as $n \to \infty$, by Fatou lemma we have

$$\begin{aligned} &\mathbf{E}\left[\int_0^T |X_t - Y_t^{(n)}|^2 d(\rho + \beta + |\alpha|)(t) \right] \\ &\le \limsup_{m \to \infty} \mathbf{E}\left[\int_0^T |Y_t^{(m)} - Y_t^{(n)}|^2 d(\rho + \beta + |\alpha|)(t) \right] \\ &\to 0 \end{aligned}$$

as $n \to \infty$. It follows by Proposition 3.6 and (4.19) that

$$\int_0^t \sigma(s, Y_s^{(n)}) dA_s \to \int_0^t \sigma(s, X_s) dA_s$$

and by the Hölder inequality that

$$\int_0^t b(s, Y_s^{(n)}) ds \to \int_0^t b(s, X_s) ds$$

in $L^2(\omega_\varphi^{\alpha,\beta})$. Therefore, taking the limit of (4.23) as $n \to \infty$ we prove that

$$X_t = X_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dA_s$$

which is equivalent to the stochastic differential equation (4.20). □

Example 4.2. Consider the following stochastic differential equation:

$$dX_t = bX_t dt + \sigma X_t dA_t,$$

where b and σ are some constants. By applying the Itô formula (3.14), we have

$$X_t = X_0 e^{\sigma A_t + bt - \frac{1}{2}\sigma^2(\beta(t) - \beta(0))}$$

which is called the *geometric generalized Wiener process.*

5. STOCHASTIC LINEAR FILTERING

Suppose that the system process $\{X_t\}_{t\geq 0}$ and the observation process $\{Z_t\}_{t\geq 0}$ satisfy the following stochastic differential equations:

$$\begin{aligned} dX_t &= F(t)X_t dt + C(t)dA_t; \\ dZ_t &= G(t)X_t dt + D(t)dW_t, \qquad Z_0 = 0, \end{aligned} \tag{5.25}$$

where F, G, C and D are real valued functions and bounded on bounded intervals. Moreover, $D \neq 0$ and $1/D$ is bounded on bounded intervals. We assume that $\{A_t\}_{t\geq 0}$ and $\{W_t\}_{t\geq 0}$ are independent generalized Wiener processes of which the mean functions and the variance functions are α_1, α_2 and β_1, β_2, respectively, where α_1, α_2 are of bounded variation functions and β_1, β_2 are absolutely continuous with $\beta_2'(t) \geq 1$ for any $t > 0$. We assume that $|\alpha_2|$ is absolutely continuous with respect to β_2 with $d|\alpha_2|(t)/d\beta_2(t) \geq 1$ for any $t > 0$. We also assume that X_0 is normally distributed and independent of $\{A_t\}, \{W_t\}$.

Let $\mathcal{G}_t$ be the σ-algebra generated by $\{Z_s; 0 \leq s \leq t\}$ and let

$$\mathcal{Z}_t = \{Y\,;\, Y \in L^2(\omega_\varphi^{\alpha,\beta}) \text{ and } Y \text{ is } \mathcal{G}_t\text{–measurable}\}.$$

Then *filtering problem* is to find $\widehat{X}_t$ satisfying that

(1) $\widehat{X}_t$ is $\mathcal{G}_t$-measurable;

(2) $\widehat{X}_t$ is the best approximation of X_t in the following sense:

$$\mathbf{E}\left[\left|X_t - \widehat{X}_t\right|^2\right] = \inf\{\mathbf{E}\left[|X_t - Y|^2\right]\,;\, Y \in \mathcal{Z}_t\}.$$

Now, we solve the filtering problem by using similar arguments in [14]. Let $\mathcal{L}_t = \mathcal{L}_t(Z)$ be the closure in $L^2(\omega_\varphi^{\alpha,\beta})$ of

$$\{c_0 + c_1 Z_{t_1} + \cdots + c_k Z_{t_k}\,;\, 0 \leq t_i \leq t,\ c_j \in \mathbf{R}\}$$

and let $P_{\mathcal{L}_t}$ be the orthogonal projection from $L^2(\omega_\varphi^{\alpha,\beta})$ onto $\mathcal{L}_t$. Then from Theorem 6.1.2 and Lemma 6.2.2 in [14] we have

$$\widehat{X}_t = P_{\mathcal{Z}_t}(X_t) = \mathbf{E}[X_t|\mathcal{G}_t] = P_{\mathcal{L}_t}(X_t). \tag{5.26}$$

Note that $dW_t = d\alpha_2(t) + dB_{2,t}$ for some generalized Wiener process $\{B_{2,t}\}_{t\geq 0}$ with mean 0 and variance function β_2. Therefore, for any $f \in L^2([0,T], d\beta_2(t))$, since $\{X_t\}$ and $\{B_{2,t}\}$ are independent, by (3.9), (5.25) and the by Itô isometry we have

$$\begin{aligned}\mathbf{E}\left[\left(\int_0^T f(t)dZ_t\right)^2\right] &= \mathbf{E}\left[\left(\int_0^T f(t)G(t)X_t dt + \int_0^T f(t)D(t)d\alpha_2(t)\right)^2\right] \\ &\quad + \left[\int_0^T f(t)^2 D(t)^2 d\beta_2(t)\right].\end{aligned}$$

On the other hand, since $\beta_2'(t) \geq 1$ and $d|\alpha_2|(t)/d\beta_2(t) \geq 1$, by the Cauchy-Schwartz inequality we have

$$\mathbf{E}\left[\left(\int_0^T f(t)G(t)X_t dt + \int_0^T f(t)D(t)d\alpha_2(t)\right)^2\right] \leq A_1 \int_0^T f(t)^2\, d\beta_2(t)$$

for some constant A_1 which is independent of f. Therefore, we obtain that

$$A_0 \int_0^T f(t)^2\, d\beta_2(t) \leq \mathbf{E}\left[\left(\int_0^T f(t)dZ_t\right)^2\right] \leq A_2 \int_0^T f(t)^2\, d\beta_2(t)$$

for some constant A_0 and A_2 which are independent of f, in fact, $D(t)^2 \geq A_0$ for any $0 \leq t \leq T$. Then we have

$$\mathcal{L}_T(Z) = \left\{c_0 + \int_0^T f(t)dZ_t\,;\ c_0 \in \mathbf{R} \text{ and } f \in L^2([0,T], d\beta_2(t))\right\} \tag{5.27}$$

The proof is similar to the proof of Lemma 6.2.4 in [14].

Now, we define the *innovation process* N_t as following:

$$N_t = Z_t - \int_0^t \widehat{GX}_s ds - \int_0^t D(s)d\alpha_2(s), \qquad \widehat{GX}_s = P_{\mathcal{L}_s}(G(s)X_s) = G(s)\widehat{X}_s$$

which is equivalent to the differential form:

$$\begin{aligned}dN_t &= G(t)(X_t - \widehat{X}_t)dt - D(t)d\alpha_2(t) + D(t)dW_t \\ &= G(t)(X_t - \widehat{X}_t)dt + D(t)dB_{2,t}.\end{aligned}$$

Then we can show that $\{N_t\}_{t\geq 0}$ is a Gaussian process and N_t has orthogonal increments. Moreover,

$$\mathbf{E}[N_t^2] = \int_0^t D^2(s)d\beta_2(s), \qquad \mathcal{L}_t(N) = \mathcal{L}_t(Z), \qquad t \geq 0,$$

for the proof, see the proof of Lemma 6.2.5 in [14].

Define process $\{R_t\}_{t\geq 0}$ by

$$dR_t = \frac{1}{D(t)} dN_t, \qquad t \geq 0, \quad R_0 = 0.$$

Then $\{R_t\}_{t\geq 0}$ is a generalized Wiener process with mean 0 and variance function β_2, i.e., for any $s, t \geq 0$, R_t has continuous sample paths, R_t has orthogonal increments, R_t is Gaussian random variable and

$$\mathbf{E}[R_t] = 0, \qquad \mathbf{E}[R_t R_s] = \min\{\beta(s), \beta(t)\} - \beta(0).$$

Since $\mathcal{L}_t(N) = \mathcal{L}_t(R)$, we have

$$\widehat{X}_t = P_{\mathcal{L}_t(R)}(X_t).$$

Therefore, by (5.27) for each $t \geq 0$

$$\widehat{X}_t = c_0(t) + \int_0^t g(s) dR_s \tag{5.28}$$

for some $g \in L^2([0,t], d\beta_2(s))$ and $c_0(t) \in \mathbf{R}$. Hence by (5.26),

$$c_0(t) = \mathbf{E}[\widehat{X}_t] = \mathbf{E}[X_t].$$

Since $X_t - \widehat{X}_t$ and $\int_0^t f(s) dR_s$ are orthogonal for all $f \in L^2([0,t], d\beta_2(s))$, by Itô isometry we have

$$\mathbf{E}\left[X_t \int_0^t f(s) dR_s\right] = \mathbf{E}\left[\widehat{X}_t \int_0^t f(s) dR_s\right] = \int_0^t g(s) f(s) d\beta_2(s).$$

In particular, by taking $f = \mathbf{1}_{[0,r]}$, $0 \leq r \leq t$, we have

$$g(r) = \frac{1}{\beta_2'(r)} \frac{\partial}{\partial r} \int_0^r g(s) ds = \frac{1}{\beta_2'(r)} \frac{\partial}{\partial r} \mathbf{E}\left[X_t R_r\right].$$

Therefore, by (5.28)

$$\widehat{X}_t = \mathbf{E}[X_t] + \int_0^t \frac{1}{\beta_2'(r)} \frac{\partial}{\partial s} \mathbf{E}[X_t R_s] dR_s. \tag{5.29}$$

On the other hand, from (5.25) by using the Itô formula we obtain that

$$X_t = \exp\left(\int_0^t F(s) ds\right) X_0 + \int_0^t \exp\left(\int_s^t F(u) du\right) C(s) dA_s \tag{5.30}$$

More generally, for any $0 \leq r \leq t$ we have

$$X_t = \exp\left(\int_r^t F(s) ds\right) X_r + \int_r^t \exp\left(\int_s^t F(u) du\right) C(s) dA_s \tag{5.31}$$

Theorem 5.1. *The solution* $\widehat{X}_t = \mathbf{E}[X_t|\mathcal{G}_t]$ *of the linear filtering problem* (5.25) *satisfies the following stochastic differential equation:*

$$d\widehat{X}_t = C(t)d\alpha_1(t) - \frac{G(t)S(t)}{\beta_2'(t)D(t)}d\alpha_2(t) + \left(F(t) - \frac{G^2(t)S(t)}{\beta_2'(t)D^2(t)}\right)\widehat{X}_t dt$$
$$+ \frac{G(t)S(t)}{\beta_2'(t)D^2(t)}dZ_t, \tag{5.32}$$

where $\widehat{X}_0 = \mathbf{E}[X_0]$ *and* $S(t) = \mathbf{E}[(X_t - \widehat{X}_t)^2]$ *satisfies the following Riccati equation:*

$$\frac{dS(t)}{dt} = C^2(t)\beta_1'(t) + 2F(t)S(t) - \frac{G(t)^2}{\beta_2'(t)D(t)^2}S(t)^2, \tag{5.33}$$
$$S(0) = \mathbf{E}[(X_0 - \mathbf{E}[X_0])^2].$$

Proof. To obtain the stochastic differential equation (5.32) for $\widehat{X}_t$, we start with the formula from (5.29)

$$\widehat{X}_t = \mathbf{E}[X_t] + \int_0^t h(s,t)dR_s, \qquad h(s,t) = \frac{1}{\beta_2'(s)}\frac{\partial}{\partial s}\mathbf{E}[X_t R_s], \tag{5.34}$$

where

$$R_s = \int_0^s \frac{G(r)}{D(r)}\widetilde{X}_r dr + B_{2,s}, \qquad \widetilde{X}_r = X_r - \widehat{X}_r. \tag{5.35}$$

On the other hand, since X_t and $B_{2,t}$ are independent and $\mathbf{E}[B_{2,t}] = 0$, from (5.35) we obtain

$$\mathbf{E}[X_t R_s] = \int_0^s \frac{G(r)}{D(r)}\mathbf{E}[X_t\widetilde{X}_r]dr. \tag{5.36}$$

Since

$$\mathbf{E}\left[\left(\int_r^t \exp\left(\int_s^t F(u)du\right)C(s)dA_s\right)\widetilde{X}_r\right] = 0$$

and

$$\mathbf{E}\left[\widehat{X}_r\widetilde{X}_r\right] = 0, \qquad 0 \le r \le T,$$

by using the formula (5.31) for X_t we have

$$\mathbf{E}[X_t\widetilde{X}_r] = \exp\left(\int_r^t F(v)dv\right)\mathbf{E}[X_r\widetilde{X}_r] = \exp\left(\int_r^t F(v)dv\right)S(r), \tag{5.37}$$

where $S(r) = \mathbf{E}[\widetilde{X}_r^2] = \mathbf{E}\left[\left(X_r - \widehat{X}_r\right)^2\right]$. Therefore, by (5.36) and (5.37)

$$\mathbf{E}[X_t R_s] = \int_0^s \frac{G(r)}{D(r)}\exp\left(\int_r^t F(v)dv\right)S(r)dr.$$

Hence by (5.34) we obtain that

$$h(s,t) = \frac{G(s)S(s)}{\beta_2'(s)D(s)} \exp\left(\int_s^t F(v)dv\right). \tag{5.38}$$

Therefore, by (5.34) and (5.38), we have

$$\begin{aligned} d\widehat{X}_t &= dc_0(t) + h(t,t)dR_t + \left(\int_0^t \frac{\partial}{\partial t} h(s,t)dR_s\right) dt \\ &= dc_0(t) + \frac{G(t)S(t)}{\beta_2'(t)D(t)} dR_t + \left(\int_0^t h(s,t)dR_s\right) F(t)dt \end{aligned}$$

and equivalently, since $dc_0(t) = d\mathbf{E}[X_t] = F(t)c_0(t)dt + C(t)d\alpha_1(t)$ from (5.30),

$$\begin{aligned} d\widehat{X}_t &= dc_0(t) + F(t)\cdot(\widehat{X}_t - c_0(t))dt + \frac{G(t)S(t)}{\beta_2'(t)D(t)} dR_t \\ &= F(t)\widehat{X}_t dt + C(t)d\alpha_1(t) + \frac{G(t)S(t)}{\beta_2'(t)D(t)} dR_t \end{aligned}$$

Finally, since

$$dR_t = \frac{1}{D(t)}\left[dZ_t - G(t)\widehat{X}_t dt - D(t)d\alpha_2(t)\right],$$

we obtain the stochastic differential equation (5.32).

Now, we obtain the equation (5.33). Since $X_t - \widehat{X}_t$ and $\widehat{X}_t$ are orthogonal, from (5.34) we have

$$\begin{aligned} S(t) &= \mathbf{E}\left[\left(X_t - \widehat{X}_t\right)^2\right] \\ &= \mathbf{E}\left[X_t^2\right] - \mathbf{E}\left[\widehat{X}_t^2\right] \\ &= U(t) - \mathbf{E}[X_t]^2 - \int_0^t h(s,t)^2 d\beta_2(s), \end{aligned}$$

where $U(t) = \mathbf{E}[X_t^2]$. On the other hand, from (5.30) we have

$$\begin{aligned} U(t) &= \exp\left(2\int_0^t F(s)ds\right)\mathbf{E}[X_0^2] \\ &\quad + 2\exp\left(\int_0^t F(s)ds\right)\mathbf{E}[X_0]\int_0^t \exp\left(\int_s^t F(u)du\right)C(s)d\alpha_1(s) \\ &\quad + \left[\int_0^t \exp\left(\int_s^t F(u)du\right)C(s)d\alpha_1(s)\right]^2 \\ &\quad + \left[\int_0^t \exp\left(2\int_s^t F(u)du\right)C^2(s)d\beta_1(s)\right]. \end{aligned}$$

Therefore, by direct computation we prove that

$$dU(t) = \left(2F(t)U(t) + C^2(t)\beta_1'(t)\right) dt + 2C(t)\mathbf{E}[X_t]d\alpha_1(t)$$

and

$$\begin{aligned}
&dS(t) \\
&= dU(t) - \left(h(t,t)^2\beta_2'(t) + 2F(t)\int_0^t h(s,t)^2 d\beta_2(s) + 2F(t)\mathbf{E}[X_t]^2\right) dt \\
&\quad - 2C(t)\mathbf{E}[X_t]d\alpha_1(t) \\
&= dU(t) - \left(\frac{G(t)^2}{\beta_2'(t)D(t)^2}S(t)^2 - 2F(t)(S(t) - U(t))\right) dt \\
&\quad - 2C(t)\mathbf{E}[X_t]d\alpha_1(t) \\
&= \left(C^2(t)\beta_1'(t) + 2F(t)S(t) - \frac{G(t)^2}{\beta_2'(t)D(t)^2}S(t)^2\right) dt
\end{aligned}$$

which is equivalent to the equation (5.33). □

References

1. N. U. Ahmed & C. D. Charalambous: Filtering for linear systems driven by fractional Brownian motion. *SIAM J. Control Optim.* **41** (2002), 313–330.
2. E. Alòs, O. Mazet & D. Nualart: Stochastic calculus with respect to Gaussian processes. *Ann. Probab.* **29** (2001), 766–801.
3. Alòs & D. Nualart: Stochastic integration with respect to the fractional Brownian motion. *Stoch. Stoch. Rep.* **75** (2003), 129–152.
4. P. Carmona, L. Coutin & G. Montseny: Stochastic integration with respect to fractional Brownian motion. *Ann. Inst. H. Poincaré Probab. Statist.* **39** (2003), 27–68.
5. S. J. Chang & D. M. Chung: Conditional function space integrals with applications. *Rocky Mountain J. Math.* **26** (1996), 37–62.
6. S. J. Chang, J. G. Choi & D. Skoug: Integration by parts formulas involving generalized Fourier-Feynman transforms on function space. *Trans. Amer. Math. Soc.* **355** (2003), 2925–2948.
7. T. S. Chung & U. C. Ji: Gaussian white noise and applications to finance. preprint, 2003.
8. T. Duncan, Y. Hu & B. Pasik-Duncan: Stochastic calculus for fractional Brownian motion I, Theory. *SIAM J. Control Optim.* **38** (2000), 582–612.
9. G. Kallianpur: *Stochastic Filtering Theory.* Springer, New York, 1980.
10. G. Kallianpur & R. L. Karandikar: *White Noise Theory of Prediction, Filtering and Smoothing.* Gordon & Breach Science Publishers, New York, 1988.

11. R. E. Kalman & R.S. Bucy: New results in linear filtering and prediction theory. *J. Basic. Eng. ASME* **83** (1961), 95 108.
12. M. L. Kleptsyna, P. E. Kloeden & V.V. Anh: Linear filtering with fractional Brownian motion. *Stochastic Anal. Appl.* **16** (1998), 907 914.
13. R. S. Liptser & A.N. Shiryayev: *Statistics of Random Processes.* Springer Verlag, 1974.
14. B. Øksendal: *Stochastic Differential Equations: An Introduction with Applications.* Springer-Verlag, Berlin, 1998.
15. F. Russo & P. Vallois: Stochastic calculus with respect to continuous finite quadratic variation processes. *Stoch. Stoch. Rep.* **70** (2000), 1 40.
16. K. S. Ryu & M. K. Im: A measure-valued analogue of Wiener measure and the measure-valued Feynman-Kac formula. *Trans. Amer. Math. Soc.* **354** (2002), 4921 4951.
17. K. S. Ryu & M. K. Im: An analogue of Wiener measure and its applications. *J. Korean Math. Soc.* **39** (2002), 801 819.
18. J. Yeh: *Stochastic Processes and the Wiener Integral.* Marcel Deckker, New York, 1973.

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