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CROSSED PRODUCTS OF THE FREE GROUP AND SEMIGROUP C^*-ALGEBRAS BY FLOWS

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Dedicated to Professor Hiroshi Takai on his sixtieth birthday

ABSTRACT. We study crossed products of the free group and semigroup C^*-algebras by actions of $\mathbb{R}$, i.e., flows, and estimate and compute their stable rank.

INTRODUCTION

Crossed products of C^*-algebras are of great interest in the C^*-algebra theory (see Blackadar [2] and Pedersen [8]). Given a (noncommutative) C^*-dynamical system $(\mathfrak{A}, \alpha, G)$ as well as a usual dynamical one, its crossed product $\mathfrak{A} \rtimes_\alpha G$ is constructed, where $\mathfrak{A}$ is a C^*-algebra, G is a locally compact group, and α is an action of G on $\mathfrak{A}$, i.e., a homomorphism from G to the automorphism group of $\mathfrak{A}$. If $\mathfrak{A}$ is commutative so that $\mathfrak{A} = C_0(X)$ the C^*-algebra of continuous functions on a locally compact Hausdorff space X vanishing at infinity, then $C_0(X) \rtimes_\alpha G$ corresponds to the (classical) dynamical system (X, α, G), where $\alpha_g(f)(h) = f(\alpha_{g^{-1}}(h))$ for $g \in G$, $f \in C_0(X)$, and $h \in X$.

In the case where $\mathfrak{A}$ is nuclear, i.e., amenable, its crossed products by flows, i.e., $G = \mathbb{R}$ (amenable so that the full and reduced crossed products are the same) have been interested and well studied (see Kishimoto [5] among many others), but the non-nuclear case has not been done well. Although its difficulty has understood to some extent in some senses such as noncommutativity and non-nuclearity, this time we have tried to compute and determined the stable rank (of Rieffel [10]) of the flow crossed products of the important non-nuclear examples such as the full and reduced

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group C^*-algebras of free groups and the full and reduced semigroup C^*-algebras of free semigroups, which are extremely noncommutative.

Let $\mathfrak{A}$ be a unital C^*-algebra. The stable rank of $\mathfrak{A}$ is defined to be the positive smallest integer $n = \mathrm{sr}(\mathfrak{A})$ such that $L_n(\mathfrak{A})$ is dense in $\mathfrak{A}^n$, where $(a_j)_{j=1}^n \in L_n(\mathfrak{A})$ means that there exists $(b_j)_{j=1}^n \in \mathfrak{A}^n$ such that $\sum_{j=1}^n b_j a_j$ is invertible in $\mathfrak{A}$. If no such n, set $\mathrm{sr}(\mathfrak{A}) = \infty$. If $\mathfrak{A}$ is non-unital, its stable rank is defined by that of the unitization of $\mathfrak{A}$ by $\mathbb{C}$. See [10] and [2].

1. Crossed Products of The Free Group C^*-Algebras

Let $C^*(F_2)$ be the full group C^*-algebra of the free group F_2 with two generators. Let U, V be the (universal) unitaries generating $C^*(F_2)$. Define an action α of $\mathbb{R}$ on $C^*(F_2)$ (i.e., a continuous homomorphism from $\mathbb{R}$ to the automorphism group of $C^*(F_2)$) by $\alpha_t(U) = e^{2\pi i t}U$ and $\alpha_t(V) = e^{2\pi i\theta t}V$ for $t \in \mathbb{R}$, where an irrational number $\theta \in \mathbb{R}$ is fixed. Such an action of $\mathbb{R}$ on a C^*-algebra is usually called a flow. To the C^*-dynamical system $(C^*(F_2), \mathbb{R}, \alpha)$ in this sense we can associate the crossed product C^*-algebra denoted by $C^*(F_2) \rtimes_\alpha \mathbb{R}$. See [8] for crossed products of C^*-algebras.

Similarly, let $C_r^*(F_2)$ be the reduced group C^*-algebra of F_2. We can construct its C^*-dynamical system $(C_r^*(F_2), \mathbb{R}, \alpha)$ as above and its crossed product C^*-algebra denoted by $C_r^*(F_2) \rtimes_\alpha \mathbb{R}$.

We first check the following which might be known to specialists:

Proposition 1.1. *The crossed product $C^*(F_2) \rtimes_\alpha \mathbb{R}$ is not simple, but the crossed product $C_r^*(F_2) \rtimes_\alpha \mathbb{R}$ is simple.*

Proof. By universality we have an onto $*$-homomorphism from $C^*(F_2)$ to $C_r^*(F_2)$. By construction this extends from $C^*(F_2) \rtimes_\alpha \mathbb{R}$ to $C_r^*(F_2) \rtimes_\alpha \mathbb{R}$. Suppose that $C^*(F_2) \rtimes_\alpha \mathbb{R} \cong C_r^*(F_2) \rtimes_\alpha \mathbb{R}$. Then their dual crossed product C^*-algebras by dual actions of $\mathbb{R}$ must be isomorphic, which implies that $C^*(F_2) \otimes \mathbb{K} \cong C_r^*(F_2) \otimes \mathbb{K}$ by Takai duality (see Takai [11] and cf. [2] and [8]), where $\mathbb{K}$ is the C^*-algebra of compact operators, from which it follows that $C^*(F_2) \cong C_r^*(F_2)$ by cutting down by a minimal projection of $\mathbb{K}$. This contradicts with them non-isomorphic.

The simplicity of $C_r^*(F_2) \rtimes_\alpha \mathbb{R}$ follows from the norm minimality and simplicity of $C_r^*(F_2)$ and the action minimality of α, which is deduced from the minimality on the product space(s) of the C^*-algebras generated by U and V respectively (cf. Akemann and Lee [1]). □

Theorem 1.2. *We obtain* $\mathrm{sr}(C_r^*(F_2) \rtimes_\alpha \mathbb{R}) = 1$.

Proof. Since $F_2 \cong \mathbb{Z} * \mathbb{Z}$ the free product of $\mathbb{Z}$, $C_r^*(F_2)$ is isomorphic to the reduced free product C^*-algebra $C^*(\mathbb{Z}) *_{\mathbb{C},r} C^*(\mathbb{Z})$. This is isomorphic to $C(\mathbb{T}) *_{\mathbb{C},r} C(\mathbb{T})$ via the Fourier transform. Since each $C(\mathbb{T})$ is invariant under the action α of $\mathbb{R}$, the corresponding crossed products of the form $C(\mathbb{T}) \rtimes_\alpha \mathbb{R}$ are C^*-subalgebras of $C_r^*(F_2) \rtimes_\alpha \mathbb{R}$, and they generate $C_r^*(F_2) \rtimes_\alpha \mathbb{R}$. By the imprimitivity theorem (of Green [4]),

$$C(\mathbb{T}) \rtimes_\alpha \mathbb{R} \cong C(\mathbb{R}/\mathbb{Z}) \rtimes_\alpha \mathbb{R} \cong C^*(\mathbb{Z}) \otimes \mathbb{K}(L^2(\mathbb{R}/\mathbb{Z})) \cong C(\mathbb{T}) \otimes \mathbb{K},$$

where $\mathbb{K}(L^2(\mathbb{R}/\mathbb{Z}))$ is the C^*-algebra of compact operators on the Hilbert space $L^2(\mathbb{R}/\mathbb{Z})$, and $\mathbb{K}$ is the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space. It is deduced from this splitting into tensor products that there exists a quotient map from the (minimal) tensor product $(C(\mathbb{T}) *_{\mathbb{C},r} C(\mathbb{T})) \otimes (\mathbb{K} * \mathbb{K})$ to $C_r^*(F_2) \rtimes_\alpha \mathbb{R}$. Since $\mathbb{K}$ is an inductive limit of $n \times n$ matrix algebras $M_n(\mathbb{C})$ over $\mathbb{C}$, we have

$$\mathbb{K} * \mathbb{K} \cong (\varinjlim M_n(\mathbb{C})) * (\varinjlim M_n(\mathbb{C})) \cong \varinjlim (M_n(\mathbb{C}) * M_n(\mathbb{C}))$$

(see Pedersen [9]). Furthermore,

$$\begin{aligned}(C(\mathbb{T}) *_{\mathbb{C},r} C(\mathbb{T})) \otimes (\mathbb{K} * \mathbb{K}) &\cong (C(\mathbb{T}) *_{\mathbb{C},r} C(\mathbb{T})) \otimes \varinjlim (M_n(\mathbb{C}) * M_n(\mathbb{C})) \\ &\cong \varinjlim [(C(\mathbb{T}) *_{\mathbb{C},r} C(\mathbb{T})) \otimes (M_n(\mathbb{C}) * M_n(\mathbb{C}))] \\ &\cong \varinjlim (\mathfrak{B}_n *_{(C(\mathbb{T}) *_{\mathbb{C},r} C(\mathbb{T})) \otimes \mathbb{C}, r} \mathfrak{B}_n),\end{aligned}$$

where $\mathfrak{B}_n *_{(C(\mathbb{T}) *_{\mathbb{C},r} C(\mathbb{T})) \otimes \mathbb{C}, r} \mathfrak{B}_n$ is the reduced amalgamated free product C^*-algebra of $\mathfrak{B}_n$ over $(C(\mathbb{T}) *_{\mathbb{C},r} C(\mathbb{T})) \otimes \mathbb{C}$, with $\mathfrak{B}_n = (C(\mathbb{T}) *_{\mathbb{C},r} C(\mathbb{T})) \otimes M_n(\mathbb{C})$. Since $C(\mathbb{T}) *_{\mathbb{C},r} C(\mathbb{T})$ has stable rank one (Dykema, Haagerup and Rørdam [3]), we have

$$\mathrm{sr}((C(\mathbb{T}) *_{\mathbb{C},r} C(\mathbb{T})) \otimes M_n(\mathbb{C})) = 1$$

by Rieffel [10, Theorem 6.1]. It follows by an application of [3] that

$$\mathrm{sr}(((C(\mathbb{T}) *_{\mathbb{C},r} C(\mathbb{T})) \otimes M_n(\mathbb{C})) *_r ((C(\mathbb{T}) *_{\mathbb{C},r} C(\mathbb{T})) \otimes M_n(\mathbb{C}))) = 1,$$

where note that $M_n(\mathbb{C}) \cong \mathbb{C}^n \rtimes \mathbb{Z}_n$ the crossed product of $\mathbb{C}^n$ by the cyclic group $\mathbb{Z}_n$ with the action permutation. This implies that $\mathfrak{B}_n *_{(C(\mathbb{T}) *_{\mathbb{C},r} C(\mathbb{T})) \otimes \mathbb{C}, r} \mathfrak{B}_n$ has stable rank one. Indeed, any element of the canonical dense part (generated by $\mathfrak{B}_n$ and (distinct) $\mathfrak{B}_n$ with $(C(\mathbb{T}) *_{\mathbb{C},r} C(\mathbb{T})) \otimes \mathbb{C}$ identified) in $\mathfrak{B}_n *_{(C(\mathbb{T}) *_{\mathbb{C},r} C(\mathbb{T})) \otimes \mathbb{C}, r} \mathfrak{B}_n$ can be lifited to an element of that of $\mathfrak{B}_n *_{\mathbb{C},r} \mathfrak{B}_n$. Therefore, $(C(\mathbb{T}) *_{\mathbb{C},r} C(\mathbb{T})) \otimes (\mathbb{K} * \mathbb{K})$ has stable rank one, which implies the conclusion because the stable rank is preserved under taking quotients (see [10, Theorem 4.3]). □

Theorem 1.3. *We obtain* $\mathrm{sr}(C^*(F_2) \rtimes_\alpha \mathbb{R}) = \infty$.

Proof. Note that $C^*(F_2)$ is isomorphic to the unital full free product C^*-algebra $C^*(\mathbb{Z}) *_\mathbb{C} C^*(\mathbb{Z})$. This is isomorphic to $C(\mathbb{T}) *_\mathbb{C} C(\mathbb{T})$. We use the same methods for the proof of Theorem 1.2. Since $C^*(F_2)$ has stable rank ∞ (see [10, Theorem 6.7]), so does $C^*(F_2) \otimes M_n(\mathbb{C})$ (see [10, Theorem 6.1]). Also, it is shown that the free product of $C^*(F_2) \otimes M_n(\mathbb{C})$ and its amalgam over $C^*(F_2) \otimes \mathbb{C}$ have stable rank ∞. It follows that $C^*(F_2) \otimes (\mathbb{K} * \mathbb{K})$ has stable rank ∞. Therefore, the conclusion is deduced by considering lifting from $C^*(F_2) \rtimes_\alpha \mathbb{R}$.

See also the proof of Theorem 2.2 given below. It is shown that $\mathbb{K} * \mathbb{K}$ has stable rank ∞. Note that there exists an onto $*$-homomorphism from $C^*(F_2)$ to $\mathbb{C}$. This implies that there exists an onto $*$-homomorphism from $C^*(F_2) \otimes (\mathbb{K} * \mathbb{K})$ to $\mathbb{K} * \mathbb{K}$. □

Let $C^*(F_n)$ be the full group C^*-algebra of the free group F_n with n generators. Let U_j $(1 \leq j \leq n)$ be the (universal) unitaries generating $C^*(F_n)$. Define an action α of $\mathbb{R}$ on $C^*(F_n)$ by $\alpha_t(U_j) = e^{2\pi i\theta_j t}U_j$ for $t \in \mathbb{R}$, where $\theta_j \in \mathbb{R}$ are rationally independent and fixed. To the C^*-dynamical system $(C^*(F_n), \mathbb{R}, \alpha)$ in this sense we can associate the crossed product C^*-algebra denoted by $C^*(F_n) \rtimes_\alpha \mathbb{R}$.

Similarly, let $C_r^*(F_n)$ be the reduced group C^*-algebra of F_n. We can construct its C^*-dynamical system $(C_r^*(F_n), \mathbb{R}, \alpha)$ as above and its crossed product C^*-algebra denoted by $C_r^*(F_n) \rtimes_\alpha \mathbb{R}$.

Proposition 1.4. *The crossed product* $C^*(F_n) \rtimes_\alpha \mathbb{R}$ *is not simple, but the crossed product* $C_r^*(F_n) \rtimes_\alpha \mathbb{R}$ *is simple.*

Proof. This is proved by the same method as in Proposition 1.1. □

Theorem 1.5. *We obtain* $\mathrm{sr}(C_r^*(F_n) \rtimes_\alpha \mathbb{R}) = 1$.

Proof. Since $F_n \cong *^n\mathbb{Z}$ the n-fold free product of $\mathbb{Z}$, $C_r^*(F_n)$ is isomorphic to the reduced n-fold free product C^*-algebra $*^n_{\mathbb{C},r}C^*(\mathbb{Z})$. This is isomorphic to $*^n_{\mathbb{C},r}C(\mathbb{T})$ via the Fourier transform. Since each $C(\mathbb{T})$ is invariant under the action α of $\mathbb{R}$, the corresponding crossed products of the form $C(\mathbb{T}) \rtimes_\alpha \mathbb{R}$ are C^*-subalgebras of $C_r^*(F_n) \rtimes_\alpha \mathbb{R}$, and they generate $C_r^*(F_n) \rtimes_\alpha \mathbb{R}$. By imprimitivity theorem, $C(\mathbb{T}) \rtimes_\alpha \mathbb{R} \cong C(\mathbb{T}) \otimes \mathbb{K}$ as shown in Theorem 1.2. It is deduced from this splitting into tensor products that there exists a quotient map from the (minimal) tensor product

$(*^n_{\mathbb{C},r}C(\mathbb{T}))\otimes(*^n\mathbb{K})$ to $C^*_r(F_n)\rtimes_\alpha\mathbb{R}$. It follows that

$$\begin{aligned}*^n\,\mathbb{K}&\cong(\cdots((\mathbb{K}*\mathbb{K})*\mathbb{K})\cdots)*\mathbb{K}\\ &\cong(\cdots(((\varinjlim M_n(\mathbb{C}))*(\varinjlim M_n(\mathbb{C})))*(\varinjlim M_n(\mathbb{C}))\cdots)*\mathbb{K}\\ &\cong(\cdots(((\varinjlim(M_n(\mathbb{C})*M_n(\mathbb{C})))*(\varinjlim M_n(\mathbb{C}))\cdots)*\mathbb{K}\\ &\cong(\cdots((\varinjlim(M_n(\mathbb{C})*M_n(\mathbb{C})*M_n(\mathbb{C}))\cdots)*(\varinjlim M_n(\mathbb{C}))\cong\varinjlim(*^nM_n(\mathbb{C})).\end{aligned}$$

Furthermore,

$$\begin{aligned}(*^n_{\mathbb{C},r}C(\mathbb{T}))\otimes(*^n\mathbb{K})&\cong(*^n_{\mathbb{C},r}C(\mathbb{T}))\otimes\varinjlim(*^nM_n(\mathbb{C}))\\ &\cong\varinjlim[(*^n_{\mathbb{C},r}C(\mathbb{T}))\otimes(*^nM_n(\mathbb{C}))]\\ &\cong\varinjlim[*^n_{(*^n_{\mathbb{C},r}C(\mathbb{T}))\otimes\mathbb{C},r}((*^n_{\mathbb{C},r}C(\mathbb{T}))\otimes M_n(\mathbb{C}))],\end{aligned}$$

where $*^n_{(*^n_{\mathbb{C},r}C(\mathbb{T}))\otimes\mathbb{C},r}((*^n_{\mathbb{C},r}C(\mathbb{T}))\otimes M_n(\mathbb{C}))$ is the n-fold reduced amalgamated free product C^*-algebra of $(*^n_{\mathbb{C},r}C(\mathbb{T}))\otimes M_n(\mathbb{C})$ over $(*^n_{\mathbb{C},r}C(\mathbb{T}))\otimes\mathbb{C}$. Since $*^n_{\mathbb{C},r}C(\mathbb{T})$ has stable rank one (Dykema, Haagerup and Rørdam [3]), we have $\mathrm{sr}((*^n_{\mathbb{C},r}C(\mathbb{T}))\otimes M_n(\mathbb{C}))=1$ by Rieffel [10, Theorem 6.1]. It follows by an application of [3] that $\mathrm{sr}(*^n_r((*^n_{\mathbb{C},r}C(\mathbb{T}))\otimes M_n(\mathbb{C})))=1$, which implies that $*^n_{(*^n_{\mathbb{C},r}C(\mathbb{T}))\otimes\mathbb{C},r}((*^n_{\mathbb{C},r}C(\mathbb{T}))\otimes M_n(\mathbb{C}))$ has stable rank one. Therefore, $(*^n_{\mathbb{C},r}C(\mathbb{T}))\otimes(*^n\mathbb{K})$ has stable rank one, which implies the conclusion because the stable rank is preserved under taking quotients. □

Theorem 1.6. *We obtain* $\mathrm{sr}(C^*(F_n)\rtimes_\alpha\mathbb{R})=\infty$.

Proof. Note that $C^*(F_n)$ is isomorphic to the unital n-fold full free product C^*-algebra $*^n_{\mathbb{C}}C^*(\mathbb{Z})$. This is isomorphic to $*^n_{\mathbb{C}}C(\mathbb{T})$. We use the same methods for the proof of Theorem 1.5. The conclusion follows from the same reasoning as in the proof of Theorem 1.3. □

2. Crossed Products of The Free Semigroup C^*-Algebras by $\mathbb{R}$

Let $C^*(\mathbb{N}*\mathbb{N})$ be the full group C^*-algebra of the free semigroup $\mathbb{N}*\mathbb{N}$ with two generators. Let S,T be the (universal) isometries generating $C^*(\mathbb{N}*\mathbb{N})$. Define an action α of $\mathbb{R}$ on $C^*(\mathbb{N}*\mathbb{N})$ (i.e., a continuous homomorphism from $\mathbb{R}$ to the automorphism group of $C^*(\mathbb{N}*\mathbb{N})$) by $\alpha_t(S)=e^{2\pi it}S$ and $\alpha_t(T)=e^{2\pi i\theta t}T$ for $t\in\mathbb{R}$, where an irrational number $\theta\in\mathbb{R}$ is fixed. To the C^*-dynamical system $(C^*(\mathbb{N}*\mathbb{N}),\mathbb{R},\alpha)$ in this sense we can associate the crossed product C^*-algebra denoted by $C^*(\mathbb{N}*\mathbb{N})\rtimes_\alpha\mathbb{R}$.

Similarly, let $C_r^*(\mathbb{N} * \mathbb{N})$ be the reduced semigroup C^*-algebra of $\mathbb{N} * \mathbb{N}$. We can construct its C^*-dynamical system $(C_r^*(\mathbb{N}*\mathbb{N}), \mathbb{R}, \alpha)$ as above and its crossed product C^*-algebra denoted by $C_r^*(\mathbb{N} * \mathbb{N}) \rtimes_\alpha \mathbb{R}$.

Proposition 2.1. *The crossed product $C^*(\mathbb{N} * \mathbb{N}) \rtimes_\alpha \mathbb{R}$ is not simple.*

Proof. This follows from the same argument as in Proposition 1.1. □

Theorem 2.2. *The crossed product $C_r^*(\mathbb{N} * \mathbb{N}) \rtimes_\alpha \mathbb{R}$ is not simple, and we obtain* $\mathrm{sr}(C_r^*(\mathbb{N} * \mathbb{N}) \rtimes_\alpha \mathbb{R}) = \infty$.

Proof. The C^*-algebra $C_r^*(\mathbb{N} * \mathbb{N})$ is isomorphic to the reduced free product C^*-algebra $C^*(\mathbb{N}) *_{\mathbb{C},r} C^*(\mathbb{N})$. Also, $C^*(\mathbb{N})$ is just the Toeplitz algebra generated by a proper isometry (see Murphy [6]). Since each $C^*(\mathbb{N})$ is invariant under the action α of $\mathbb{R}$, the corresponding crossed products of the form $C^*(\mathbb{N}) \rtimes_\alpha \mathbb{R}$ are C^*-subalgebras of $C_r^*(\mathbb{N} * \mathbb{N}) \rtimes_\alpha \mathbb{R}$, and they generate $C_r^*(\mathbb{N} * \mathbb{N}) \rtimes_\alpha \mathbb{R}$.

Recall that $C^*(\mathbb{N})$ has the following exact sequence:

$$0 \to \mathbb{K} \to C^*(\mathbb{N}) \to C(\mathbb{T}) \to 0.$$

Moreover, since $\mathbb{K}$ is invariant under the action α of $\mathbb{R}$, we have

$$0 \to \mathbb{K} \rtimes_\alpha \mathbb{R} \to C^*(\mathbb{N}) \rtimes_\alpha \mathbb{R} \to C(\mathbb{T}) \rtimes_\alpha \mathbb{R} \to 0.$$

Also, $\mathbb{K} \rtimes_\alpha \mathbb{R} \cong \mathbb{K} \otimes C^*(\mathbb{R}) \cong \mathbb{K} \otimes C_0(\mathbb{R})$ because the action α on $\mathbb{K}$ is in fact an adjoint action by an implemented unitary. Furthermore, the imprimitivity theorem implies $C(\mathbb{T}) \rtimes_\alpha \mathbb{R} \cong C(\mathbb{T}) \otimes \mathbb{K}$ as shown in Theorem 1.2. It is deduced from this decomposition that there exists a short exact sequence

$$0 \to (\mathbb{K} * C^*(\mathbb{N}) + C^*(\mathbb{N}) * \mathbb{K}) \otimes C_0(\mathbb{R}) \to C_r^*(\mathbb{N} * \mathbb{N}) \rtimes_\alpha \mathbb{R} \to Q \to 0,$$

and there exists a quotient map from the (minimal) tensor product $(C(\mathbb{T}) *_{\mathbb{C},r} C(\mathbb{T})) \otimes (\mathbb{K} * \mathbb{K})$ to the quotient C^*-algebra Q. It follows that $C_r^*(\mathbb{N} * \mathbb{N}) \rtimes_\alpha \mathbb{R}$ is not simple. Furthermore, we have

$$0 \to \mathbb{K} * \mathbb{K} \to \mathbb{K} * C^*(\mathbb{N}) + C^*(\mathbb{N}) * \mathbb{K} \to \mathbb{K} * C(\mathbb{T}) + C(\mathbb{T}) * \mathbb{K} \to 0,$$

which implies that

$$\begin{aligned} 0 &\to (\mathbb{K} * \mathbb{K}) \otimes C_0(\mathbb{R}) \\ &\to (\mathbb{K} * C^*(\mathbb{N}) + C^*(\mathbb{N}) * \mathbb{K}) \otimes C_0(\mathbb{R}) \to (\mathbb{K} * C(\mathbb{T}) + C(\mathbb{T}) * \mathbb{K}) \otimes C_0(\mathbb{R}) \to 0. \end{aligned}$$

Note that $(\mathbb{K}*\mathbb{K})\otimes C_0(\mathbb{R})$ has $\mathbb{K}*\mathbb{K}$ as a quotient, and $\mathbb{K}*\mathbb{K} \cong \varinjlim(M_n(\mathbb{C})*M_n(\mathbb{C}))$. Also, $M_n(\mathbb{C}) \cong \mathbb{C}^n \rtimes \mathbb{Z}_n$, where the action of $\mathbb{Z}_n$ on $\mathbb{C}^n$ is the permutation. It follows that

$$M_n(\mathbb{C}) * M_n(\mathbb{C}) \cong (\mathbb{C}^n \rtimes \mathbb{Z}_n) * (\mathbb{C}^n \rtimes \mathbb{Z}_n) \cong (\mathbb{C}^n * \mathbb{C}^n) \rtimes (\mathbb{Z}_n * \mathbb{Z}_n).$$

Since $\mathbb{C}^n$ has the trivial $*$-homomorphism to $\mathbb{C}$, it induces a $*$-homomorphism from $\mathbb{C}^n * \mathbb{C}^n$ to $\mathbb{C}$. Since the unit of $\mathbb{C}^n * \mathbb{C}^n$ is invariant under the action of $\mathbb{Z}_n * \mathbb{Z}_n$, there exists an onto $*$-homomorphism:

$$(\mathbb{C}^n * \mathbb{C}^n) \rtimes (\mathbb{Z}_n * \mathbb{Z}_n) \to \mathbb{C} \rtimes (\mathbb{Z}_n * \mathbb{Z}_n) \to 0,$$

and $\mathbb{C} \rtimes (\mathbb{Z}_n * \mathbb{Z}_n) \cong C^*(\mathbb{Z}_n * \mathbb{Z}_n)$ is the full group C^*-algebra of the free product $\mathbb{Z}_n * \mathbb{Z}_n$. It is shown in Nagisa [7] that $C^*(\mathbb{Z}_n * \mathbb{Z}_n)$ has stable rank ∞. Since $\mathbb{K}$ is the c_0-direct limit of $M_n(\mathbb{C})$, $\mathbb{K} * \mathbb{K}$ is also the c_0-direct limit of $M_n(\mathbb{C}) * M_n(\mathbb{C})$. Hence by Rieffel [10, Theorem 5.2],

$$\mathrm{sr}(\mathbb{K} * \mathbb{K}) = \sup_n \mathrm{sr}(M_n(\mathbb{C}) * M_n(\mathbb{C})) = \infty.$$

Thus, $\mathrm{sr}((\mathbb{K} * \mathbb{K}) \otimes C_0(\mathbb{R})) = \infty$ by [10, Theorem 4.3]. Therefore, by [10, Theorem 4.4] we obtain the conclusion. □

Corollary 2.3. *We obtain* $\mathrm{sr}(C^*(\mathbb{N} * \mathbb{N}) \rtimes_\alpha \mathbb{R}) = \infty$.

Let $C^*(*^k\mathbb{N})$ be the full group C^*-algebra of the k-fold free semigroup $*^k\mathbb{N}$ with k generators. Let S_j $(1 \leq j \leq k)$ be the (universal) k isometries generating $C^*(*^k\mathbb{N})$. Define an action α of $\mathbb{R}$ on $C^*(*^k\mathbb{N})$ by $\alpha_t(S_j) = e^{2\pi i\theta_j t}S_j$ for $t \in \mathbb{R}$, where $\theta_j \in \mathbb{R}$ are rationally independent and fixed. To the C^*-dynamical system $(C^*(*^k\mathbb{N}), \mathbb{R}, \alpha)$ in this sense we can associate the crossed product C^*-algebra denoted by $C^*(*^k\mathbb{N}) \rtimes_\alpha \mathbb{R}$.

Similarly, let $C_r^*(*^k\mathbb{N})$ be the reduced semigroup C^*-algebra of $*^k\mathbb{N}$. We can construct its C^*-dynamical system $(C_r^*(*^k\mathbb{N}), \mathbb{R}, \alpha)$ as above and its crossed product C^*-algebra denoted by $C_r^*(*^k\mathbb{N}) \rtimes_\alpha \mathbb{R}$.

Proposition 2.4. *The crossed product* $C^*(*^k\mathbb{N}) \rtimes_\alpha \mathbb{R}$ *is not simple.*

Proof. This follows from the same argument as in Proposition 1.1. □

Theorem 2.5. *The crossed product* $C_r^*(*^k\mathbb{N}) \rtimes_\alpha \mathbb{R}$ *is not simple, and we obtain* $\mathrm{sr}(C_r^*(*^k\mathbb{N}) \rtimes_\alpha \mathbb{R}) = \infty$.

Proof. The C^*-algebra $C_r^*(*^k\mathbb{N})$ is isomorphic to the reduced k-fold free product C^*-algebra $*_{\mathbb{C},r}^k C^*(\mathbb{N})$. Since each $C^*(\mathbb{N})$ is invariant under the action α of $\mathbb{R}$, the corresponding crossed products of the form $C^*(\mathbb{N}) \rtimes_\alpha \mathbb{R}$ are C^*-subalgebras of $C_r^*(*^k\mathbb{N}) \rtimes_\alpha \mathbb{R}$, and they generate $C_r^*(*^k\mathbb{N}) \rtimes_\alpha \mathbb{R}$. We use the following decomposition:

$$0 \to \mathbb{K} \rtimes_\alpha \mathbb{R} \to C^*(\mathbb{N}) \rtimes_\alpha \mathbb{R} \to C(\mathbb{T}) \rtimes_\alpha \mathbb{R} \to 0.$$

and $\mathbb{K} \rtimes_\alpha \mathbb{R} \cong \mathbb{K} \otimes C_0(\mathbb{R})$ and $C(\mathbb{T}) \rtimes_\alpha \mathbb{R} \cong C(\mathbb{T}) \otimes \mathbb{K}$, which is shown in Theorem 2.2. It is deduced from this decomposition that $C_r^*(*^k\mathbb{N}) \rtimes_\alpha \mathbb{R}$ has $(*^k\mathbb{K}) \otimes C_0(\mathbb{R})$ as a closed ideal, from which $C_r^*(*^k\mathbb{N}) \rtimes_\alpha \mathbb{R}$ is not simple. This closed ideal has $*^k\mathbb{K}$ as a quotient. Since $\mathbb{K} * \mathbb{K} \cong \varinjlim(M_n(\mathbb{C}) * M_n(\mathbb{C}))$, we have

$$*^k\mathbb{K} \cong \varinjlim(*^k M_n(\mathbb{C}))$$

as shown before. Also, $M_n(\mathbb{C}) \cong \mathbb{C}^n \rtimes \mathbb{Z}_n$. It follows that

$$*^k M_n(\mathbb{C}) \cong *^k(\mathbb{C}^n \rtimes \mathbb{Z}_n) \cong (*^k\mathbb{C}^n) \rtimes (*^k\mathbb{Z}_n).$$

Since $\mathbb{C}^n$ has the trivial $*$-homomorphism to $\mathbb{C}$, it induces a $*$-homomorphism from $*^k\mathbb{C}^n$ to $\mathbb{C}$. Since the unit of $*^k\mathbb{C}^n$ is invariant under the action of $*^k\mathbb{Z}_n$, there exists an onto $*$-homomorphism:

$$(*^k\mathbb{C}^n) \rtimes (*^k\mathbb{Z}_n) \to \mathbb{C} \rtimes (*^k\mathbb{Z}_n) \to 0,$$

and $\mathbb{C} \rtimes (*^k\mathbb{Z}_n) \cong C^*(*^k\mathbb{Z}_n)$ is the full group C^*-algebra of the free product $*^k\mathbb{Z}_n$. Furthermore, a canonical quotient from $*^k\mathbb{Z}_n$ to $\mathbb{Z}_n * \mathbb{Z}_n$ induces that there exists an onto $*$-homomorphism from $C^*(*^k\mathbb{Z}_n)$ to $C^*(\mathbb{Z}_n * \mathbb{Z}_n)$. It is shown in Nagisa [7] that $C^*(\mathbb{Z}_n * \mathbb{Z}_n)$ has stable rank ∞. This implies that $C^*(*^k\mathbb{Z}_n)$ also has stable rank ∞. Hence, $*^k M_n(\mathbb{C})$ has stable rank ∞. Since $\mathbb{K}$ is the c_0-direct limit of $M_n(\mathbb{C})$, $*^k\mathbb{K}$ is also the c_0-direct limit of $*^k M_n(\mathbb{C})$. Hence

$$\mathrm{sr}(*^k\mathbb{K}) = \sup_n \mathrm{sr}(*^k M_n(\mathbb{C})) = \infty.$$

Thus, $\mathrm{sr}((*^k\mathbb{K}) \otimes C_0(\mathbb{R})) = \infty$ by Rieffel [10, Theorem 4.3]. Therefore, by [10, Theorem 4.4] we obtain the conclusion. □

Corollary 2.6. *We obtain* $\mathrm{sr}(C^*(*^k\mathbb{N}) \rtimes_\alpha \mathbb{R}) = \infty$.

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