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H^f-SPACES FOR MAPS AND THEIR DUALS

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ABSTRACT. We define and study a concept of H^f-space for a map, which is a generalized concept of an H-space, in terms of the Gottlieb set for a map. For a principal fibration $E_k \to X$ induced by $k : X \to X'$ from $\epsilon : PX' \to X'$, we can obtain a sufficient condition to having an $H^{\bar{f}}$-structure on E_k, which is a generalization of Stasheff's result [17]. Also, we define and study a concept of co-H^g-space for a map, which is a dual concept of H^f-space for a map. Also, we get a dual result which is a generalization of Hilton, Mislin and Roitberg's result [6].

1. INTRODUCTION

A map $f : A \to X$ is *cyclic* [18] if there is a map $F : X \times A \to X$ such that $F|_X \sim 1_X$ and $F|_A \sim f$. It is clear that a space X is an H-space if and only if the identity map of X is cyclic. We call a space X as an H^f*-space* if there is a cyclic map $f : A \to X$. We show that if a space X is an H-space, then for any space A and any map $f : A \to X$, X is an H^f-space for a map $f : A \to X$, but the converse does not hold. Let $p_k : E_k \to X$ be a principal fibration induced by $k : X \to X'$ from $\epsilon : PX' \to X'$. Stasheff [17] showed that if X and X' are H-spaces, and $k : X \to X'$ is an H-map, then there is an H-structure on E_k such that $p_k : E_k \to X$ is an H-map. We can generalize the above result as follows. If X is an H^f-space with H^f-structure $F : X \times A \to X$ and X' is an $H^{f'}$-space with $H^{f'}$-structure $F' : X' \times A' \to X'$ such that $kF \sim F'(k \times l) : X \times A \to X'$, then there exists an $H^{\bar{f}}$-structure $\bar{F} : E_k \times E_l \to E_k$ on E_k such that $p_k\bar{F} \sim F(p_k \times p_l) : E_k \times E_l \to X$. Also, we can show that if X is an H^f-space with H^f-structure $F : X \times A \to X$, then there exists an H^{f_n}-structure $F_n : X_n \times A_n \to X_n$ for each Postnikov stage X_n of X such that $p_nF_n \sim F_{n-1}(p_n \times p'_n) : X_n \times A_n \to X_{n-1}$, where f_n is an induced map from

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f, and all the pair of k-invariants $(k_X^{n+2}, k_A^{n+2}) : f_n \to \tilde{f}_\#$ are H^{f_n}-primitive with respect to F_n, where $\tilde{f}_\# : K(\pi_{n+1}(A), n+2) \to K(\pi_{n+1}(X), n+2)$ is the induced map by $f : A \to X$. On the other hand, we define and study a concept of co-H^g-space of a map, which is a dual concept of H^f-space for a map. In 1978, Hilton, Mislin and Roitberg [6] showed that if X and X' are co-H-spaces, and $r : X' \to X$ is a co-H-map, then there is a co-H-structure on C_r such that $i_r : X \to C_r$ is a co-H-map. Then we can obtain a generalization of the above result to showing that if X is a co-H^g-space with co-H^g-structure $\theta : X \to X \vee A$ and X' is a co-$H^{g'}$-space with co-$H^{g'}$-structure $\theta' : X' \to X' \vee A'$ such that $(r \vee s)\theta' \sim \theta r : X' \to X \vee A$, then there exists a co-$H^{\bar{g}}$-structure $\bar{\theta} : C_r \to C_r \vee C_s$ on C_r satisfying $(i_r \vee i_s)\theta \sim \bar{\theta} i_r : X \to C_r \vee C_s$. In 1998, Golansinski and Klein [5] obtained some sufficient conditions for a map $r : X' \to X$ is a co-H-map as follows. Let $r : X' \to X$ be a map with X' and X 1-connected co-H-spaces. If the mapping cone C_r is a co-H-space, $i_r : X \to C_r$ a co-H-map and $\dim X' < conn\ X' + \min\{conn\ X, conn\ C_r\}$, then $r : X' \to X$ is a co-H-map. We can generalize the above result as follows; Let X be a simply connected co-H^g-space with co-H^g-structure $\theta : X \to X \vee A$. Let X' be a simply connected co-H-space and $(s, r) : g' \to g$ a map, where $g' : X' \to A'$ is a map. If there is a co-$H^{\bar{g}}$-structure $\bar{\theta} : C_r \to C_r \vee C_s$ on C_r satisfying $(i_r \vee i_s)\theta \sim \bar{\theta} i_r : X \to C_r \vee C_s$ and $\dim X' < \min\{conn\ (X') + conn\ (C_s), conn\ (A') + conn\ (X)\}$, then $(s, r) : g' \to g$ is a co-H^g-primitive with respect to θ.

2. H^f-SPACES FOR MAPS

Let $f : A \to X$ be a map. A based map $g : B \to X$ is called *f-cyclic* [15] if there is a map $\phi : B \times A \to X$ such that the diagram

$$\begin{array}{ccc} A \times B & \xrightarrow{\phi} & X \\ {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle \nabla} \\ A \vee B & \xrightarrow{(f \vee g)} & X \vee X \end{array}$$

is homotopy commute, where $j : A \vee B \to A \times B$ is the inclusion and $\nabla : X \vee X \to X$ is the folding map. We call such a map ϕ an *associated map* of a f-cyclic map g. Clearly, g is f-cyclic iff f is g-cyclic. In the case, $f = 1_X : X \to X$, $g : B \to X$ is called cyclic [18]. We denote the set of all homotopy classes of f-cyclic maps from B to X by $G(B; A, f, X)$ which is called the *Gottlieb set for a map* $f : A \to X$. In the case $f = 1_X : X \to X$, we called such a set $G(B; X, 1, X)$

the *Gottlieb set* denoted $G(B;X)$. In particular, $G(S^n;A,f,X)$ will be denoted by $G_n(A,f,X)$. Gottlieb [3, 4] introduced and studied the *evaluation subgroups* $G_n(X) = G_n(X,1,X)$ of $\pi_n(X)$.

In general, $G(B;X) \subset G(B;A,f,X) \subset [B,X]$ for any map $f : A \to X$ and any space B. However, there is an example [23] such that $G(B,X) \neq G(B;A,f,X) \neq [B,X]$.

The next proposition is an immediate consequence from the definition.

Proposition 2.1.

(1) *For any maps* $f : A \to X$, $\theta : C \to A$ *and any space* B, $G(B;A,f,X) \subset G(B;C,f\theta,X)$.

(2) $G(B,X) = G(B;X,1_X,X) \subset G(B;A,f,X) \subset G(B;A,*,X) = [B,X]$ *for any spaces* X, A *and* B.

(3) $G(B,X) = \cap\{G(B;A,f,X) | f : A \to X$ *is a map and* A *is a space*$\}$.

(4) *If* $h : C \to A$ *is a homotopy equivalence, then* $G(B;A,f,X) = G(B;C,fh,X)$.

(5) *For any map* $k : X \to Y$, $k_\#(G(B;A,f,X)) \subset G(B;A,kf,Y)$.

(6) *For any map* $k : X \to Y$, $k_\#(G(B,X)) \subset G(B;X,k,Y)$.

(7) *For any map* $s : C \to B, s^\#(G(B;A,f,X)) \subset G(C;A,f,X)$.

From Proposition 2.1(1), we have the following corollary.

Corollary 2.2. *If* $f : A \to X$ *has a right homotopy inverse, then* $G(B;A,f,X) = G(B,X)$ *for any space* B. *In that case,* $G_n(A,f,X) = G_n(X)$.

Let $Map\,(A,X)$ be the space of continuous maps from A to X with compact open topology. For a based map $f : A \to X$, let $Map\,(A,X;f)$ be the path component of $Map\,(A,X)$ containing f. Let $Map_*(A,X)$ and $Map_*(A,X;f)$ be the spaces of base point preserving maps in $Map\,(A,X)$ and $Map\,(A,X;f)$ respectively. Clearly, the evaluation map $\omega : Map\,(A,X) \to X$ is a fibration. Moreover, the restriction to path component $\omega_f = \omega_{|Map\,(A,X;f)} : Map\,(A,X;f) \to X$ is a fibration with fiber $Map_*(A,X;f)$.

Proposition 2.3 ([24]). *For the evaluation map* $\omega : Map\,(A,X;f) \to X$, $w_\#([B, Map\,(A,X;f)]) = G(B;A,f,X)$.

Thus we have the following corollary.

Corollary 2.4. *Let B be a co-H-group. Then $G(B; A, f, X)$ is a subgroup of $[B, X]$ for any $f : A \to X$.*

Clearly, a space X is an H-space means the identity map $1_X : X \to X$ is cyclic. Then the following proposition says that H-spaces are completely characterized by the Gottlieb sets.

Proposition 2.5 ([11]). *X is an H-space if and only if $G(B, X) = [B, X]$ for any space B.*

Now, for a map $f : A \to X$, we would like to introduce new spaces which can be characterized by the Gottlieb sets for a map $f : A \to X$.

Definition 2.6. A space X is called an *H^f-space for a map $f : A \to X$* if there is a map, H^f-structure on X, $F : X \times A \to X$ such that $Fj \sim \nabla(1 \vee f)$, where $j : X \vee A \to X \times A$ is the inclusion.

For the dual Puppe sequence of a fibration $\cdots \to \Omega E \to \Omega B \xrightarrow{\partial} F \xrightarrow{i} E \xrightarrow{p} B$, we have an operation [8, p. 97] $\rho : F \times \Omega B \to F$ of ΩB on F such that $\partial \sim \rho|_{\Omega B}$, that is, $\partial : \Omega B \to F$ is cyclic. Thus the space F is an H^∂-space for $\partial : \Omega B \to F$. Cyclicity can be used to characterize properties of the co-domain of a map.

Theorem 2.7. *X is an H^f-space for a map $f : A \to X$ if and only if $G(B; A, f, X) = [B, X]$ for any space B.*

Proof. Suppose that X is an H^f-space for a map $f : A \to X$. Then $f : A \to X$ is a cyclic map, and there is a map $F : X \times A \to X$ such that $Fj \sim \nabla(1 \vee f)$, where $j : X \vee A \to X \times A$ is the inclusion. Let $g \in [B, X]$. Consider the map $G = F(g \times 1) : B \times A \to X$. Then $Gj \sim \nabla(g \vee f)$ and $g \in G(B; A, f, X)$. On the other hand, suppose that $G(B; A, f, X) = [B, X]$ for any space B. Take $B = X$ and consider the identity map $1_X : X \to X$. Since $1_X \in G(X; A, f, X)$, we know that the identity map 1_X is f-cyclic and X is an H^f-space for a map $f : A \to X$. $\square$

It is well known fact that if X dominates A and X is an H-space, then A is an H-space. This fact can be generalized as the following corollary.

Corollary 2.8. *Let X be an H^i-space for a map $i : A \to X$.*

(1) If $i : A \to X$ has a left homotopy inverse $r : X \to A$, then A is an H-space.

(2) *If $i : A \to X$ has a right homotopy inverse $r : X \to A$, then X is an H-space.*

Proof. (1) Let B be any space. It is sufficient to show that $[B, A] \subset G(B, A)$ for any space B. Since X is an H^i-space for $i : A \to X$, we know, from Theorem 2.7, that $G(B; A, i, X) = [B, X]$. Thus we have, from Proposition 2.1(5), that $[B, A] = r_*[B, X] = r_*(G(B; A, i, X)) \subset G(B; A, ri, A) = G(B, A, 1, A) = G(B, A)$. Thus A is an H-space. (2) We show that $[B, X] \subset G(B, X)$ for any space B. By Theorem 2.7 and Proposition 2.1(1), we can obtain that $[B, X] = G(B; A, i, X) \subset G(B; X, ir, X) = G(B; X, 1, X) = G(B, X)$. Thus we know, from Proposition 2.5, that X is an H-space. □

Clearly, any H-space is an H^f-space for any map $f : A \to X$, but the converse does not hold.

Example 2.9. Consider the natural pairing $\mu : S^3/S^1 \times S^3 \to S^3/S^1$. Then we know that the Hopf map $\eta : S^3 \to S^2$ is cyclic. Thus S^2 is an H^η-space for $\eta : S^3 \to S^2$, but S^2 is not an H-space.

From Proposition 2.1(2) and (3), Proposition 2.5 and Theorem 2.7, we have the following corollary.

Corollary 2.10. *X is an H-space if and only if for any space A and any map $f : A \to X$, X is an H^f-space for a map $f : A \to X$.*

From some properties of cyclic maps [11], we have the following proposition.

Proposition 2.11.

(1) *If X is an H^f-space for a map $f : A \to X$, then for any map $\theta : B \to A$, X is an $H^{f\theta}$-space for a map $v\theta : B \to X$.*

(2) *If $r : X \to Y$ has a right homotopy inverse and X is an H^f-space for a map $f : A \to X$, then Y is an H^{rf}-space for a map $rf : A \to Y$.*

(3) *If X is an H^f-space for a map $f : A \to X$ and Y is an H^g-space for a map $g : B \to Y$, then $X \times Y$ is an $H^{f \times g}$-space for a map $f \times g : A \times B \to X \times Y$.*

(4) *If X is an H^f-space for a map $f : A \to X$ and X is an H^g-space for a map $g : B \to X$, then X is an $H^{\nabla(f \vee g)}$-space for a map $\nabla(f \vee g) : A \vee B \to X$.*

Let $f : A \to X$, $f' : A' \to X'$, $l : A \to A'$, $k : X \to X'$ be maps. Then a pair of maps $(k, l) : (X, A) \to (X', A')$ is called *a map from f to f'* if the following diagram is commutative;

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle l}\downarrow & & {\scriptstyle k}\downarrow \\ A' & \xrightarrow{f'} & X'. \end{array}$$

It will be denoted by $(k,l) : f \to f'$.

Given maps $f : A \to X,\ f' : A' \to X'$, let $(k,l) : f \to f'$ be a map from f to f'. Let PX' and PA' be the spaces of paths in X' and A' which begin at $*$ respectively. Let $\epsilon_{X'} : PX' \to X'$ and $\epsilon_{A'} : PA' \to A'$ be the fibrations given by evaluating a path at its end point. Let $p_k : E_k \to X$ be the fibration induced by $k : X \to X'$ from $\epsilon_{X'}$. Let $p_l : E_l \to A$ induced by $l : A \to A'$ from $\epsilon_{A'}$. Then there is a map $\bar{f} : E_l \to E_k$ such that the following diagram is commutative

$$\begin{array}{ccc} E_l & \xrightarrow{\bar{f}} & E_k \\ {\scriptstyle p_l}\downarrow & & {\scriptstyle p_k}\downarrow \\ A & \xrightarrow{f} & X, \end{array}$$

where $E_l = \{(a,\xi) \in A \times PA' | l(a) = \epsilon(\xi)\}$, $E_k = \{(x,\eta) \in X \times PX' | k(x) = \epsilon(\eta)\}$, $\bar{f}(a,\xi) = (f(a), f' \circ \xi)$, $p_k(x,\eta) = x$, $p_l(a,\xi) = a$.

Definition 2.12. Let X be an H^f-space with H^f-structure $F : X \times A \to X$. A map $(k,l) : f \to f'$ is called an *H^f-primitive with respect to F* if there is an associate map $F' : X' \times A' \to X'$ of f' such that the following diagram is homotopy commutative;

$$\begin{array}{ccc} X \times A & \xrightarrow{F} & X \\ {\scriptstyle k\times l}\downarrow & & {\scriptstyle k}\downarrow \\ X' \times A' & \xrightarrow{F'} & X'. \end{array}$$

The following lemmas are standard.

Lemma 2.13. *A map $l : C \to X$ can be lifted to a map $C \to E_k$ if and only if $kl \sim *$.*

Lemma 2.14 ([7]). *Given maps $g_i : A_i \to E_k,\ i = 1,\ 2$ and $g : A_1 \times A_2 \to E_k$ satisfying $p_k g|_{A_i} \sim p_k g_i,\ i = 1,\ 2$, then there is a map $h : A_1 \times A_2 \to E_k$ such that $p_k h = p_k g$ and $h|_{A_i} \sim g_i, i = 1,\ 2$.*

Theorem 2.15. *If X is an H^f-space with H^f-structure $F : X \times A \to X$ and $(k,l) : f \to f'$ is an H^f-primitive with respective to F, then there exists an $H^{\bar{f}}$-structure $\bar{F} : E_k \times E_l \to E_k$ on E_k such that the following diagram is homotopy commutative;*

$$\begin{array}{ccc} E_k \times E_l & \xrightarrow{\bar{F}} & E_k \\ {\scriptstyle p_k \times p_l}\downarrow & & {\scriptstyle p_k}\downarrow \\ X \times A & \xrightarrow{F} & X. \end{array}$$

Proof. Since $(k,l) : f \to f'$ is an H^f-primitive with respect to F, there is a map $F' : X' \times A' \to X'$ such that the following diagram is homotopy commutative;

$$\begin{array}{ccc} X \times A & \xrightarrow{F} & X \\ {\scriptstyle k \times l}\downarrow & & {\scriptstyle k}\downarrow \\ X' \times A' & \xrightarrow{F'} & X'. \end{array}$$

Then $kF(p_k \times p_l) \sim F'(k \times l)(p_k \times p_l) = F'(k \circ p_k \times l \circ p_l) \sim F'(* \times *) \sim * : E_k \times E_l \to X'$. From Lemma 2.13, there is a lifting $\tilde{F} : E_k \times E_l \to E_k$ of $F(p_k \times p_l) : E_k \times E_l \to X$, that is, $p_k\tilde{F} = F(p_k \times p_l)$. Then $p_k \circ \tilde{F}|_{E_k} \sim F|_X \circ p_k \sim p_k \circ 1$ and $p_k \circ \tilde{F}|_{E_l} \sim F|_A \circ p_l \sim f \circ p_l = p_k \circ \bar{f}$. Thus we have, from Lemma 2.14, that there is a map $\bar{F} : E_k \times E_l \to E_k$ such that $p_k\bar{F} = p_k\tilde{F} = F(p_k \times p_l)$ and $\bar{F}|_{E_k} \sim 1$, $\bar{F}|_{E_l} \sim \bar{f}$. This proves the theorem. □

Taking $f = 1_X$ and $l = k$, we can obtain the following corollary.

Corollary 2.16 ([17]). *If X and X' are H-spaces and $k : X \to X'$ is an H-map, then there is an H-structure on E_k such that $p_k : E_k \to X$ is an H-map.*

In 1951, Postnikov [16] introduced the notion of the Postnikov system as follows; *A Postnikov system for X (or homotopy decomposition of X)* $\{X_n, i_n, p_n\}$ consists of a sequence of spaces and maps satisfying (1) $i_n : X \to X_n$ induces an isomorphism $(i_n)_\# : \pi_i(X) \to \pi_i(X_n)$ for $i \leq n$. (2) $p_n : X_n \to X_{n-1}$ is a fibration with fiber $K(\pi_n(X), n)$. (3) $p_n i_n \sim i_{n+1}$. It is well known fact [13] that if X is a 1-connected space having a homotopy type of CW-complex, then there is a Postnikov system $\{X_n, i_n, p_n\}$ for X such that $p_{n+1} : X_{n+1} \to X_n$ is the fibration induced from the path space fibration over $K(\pi_{n+1}(X), n+2)$ by a map $k^{n+2} : X_n \to K(\pi_{n+1}(X), n+2)$.

Theorem 2.17. *Let A and X be spaces having the homotopy type of 1-connected countable CW-complexes, and $\{A_n, i'_n, p'_n\}$ and $\{X_n, i_n, p_n\}$ be Postnikov systems* *for A and X respectively. If X is an H^f-space with H^f-structure $F : X \times A \to X$, then there exists an H^{f_n}-structure $F_n : X_n \times A_n \to X_n$ for each stage X_n such that*

$$\begin{array}{ccc} X_n \times A_n & \xrightarrow{F_n} & X_n \\ {\scriptstyle p_n \times p'_n}\downarrow & & {\scriptstyle p_n}\downarrow \\ X_{n-1} \times A_{n-1} & \xrightarrow{F_{n-1}} & X_{n-1}, \end{array}$$

where f_n is an induced map from f, and all the pair of k-invariants (k_X^{n+2}, k_A^{n+2}): $f_n \to \tilde{f}_\#$ are H^{f_n}-primitive with respect to F_n, where $\tilde{f}_\# : K(\pi_{n+1}(A), n+2) \to K(\pi_{n+1}(X), n+2)$ is the induced map by $f : A \to X$.

Proof. Clearly $\{X_n \times A_n, i_n \times i'_n, p_n \times p'_n\}$ is a Postnikov system for $X \times A$. Then we have, by Kahn's result [9, Theorem 2.2], that there are families of maps $f_n : A_n \to X_n$ and $F_n : X_n \times A_n \to X_n$ such that $p_n f_n = f_{n-1} p'_n$ and $i_n f \sim f_n i'_n$, and $p_n F_n = F_{n-1}(p_n \times p'_n)$ and $i_n F \sim F_n(i_n \times i'_n)$ for $n = 2, 3, \cdots$ respectively, and there are homotopy commutative diagrams

$$\begin{array}{ccc} A_n & \xrightarrow{f_n} & X_n \\ {\scriptstyle k_A^{n+2}}\downarrow & & {\scriptstyle k_X^{n+2}}\downarrow \\ K(\pi_{n+1}(A), n+2) & \xrightarrow{\tilde{f}_\#} & K(\pi_{n+1}(X), n+2), \\ X_n \times A_n & \xrightarrow{F_n} & X_n \\ {\scriptstyle k_X^{n+2} \times k_A^{n+2}}\downarrow & & {\scriptstyle k_X^{n+2}}\downarrow \\ K(\pi_{n+1}(X), n+2) \times K(\pi_{n+1}(A), n+2) & \xrightarrow{\tilde{F}_\#} & K(\pi_{n+1}(X), n+2), \end{array}$$

where $k_A^{n+2} : A_n \to K(\pi_{n+1}(A), n+2)$ and $k_X^{n+2} : X_n \to K(\pi_{n+1}(X), n+2)$ are k-invariants of A and X respectively, $\tilde{f}_\# : K(\pi_{n+1}(A), n+2) \to K(\pi_{n+1}(X), n+2)$ and $\tilde{F}_\# : K(\pi_{n+1}(X), n+2) \times K(\pi_{n+1}(A), n+2) \approx K(\pi_{n+1}(X \times A), n+2) \to K(\pi_{n+1}(X), n+2)$ are the induced maps by $f : A \to X$ and $F : X \times A \to X$ respectively. Since $F|_X \sim 1$ and $F_n|_{A_n} \sim f_n$, we know, from Kahn's another result [10, Theorem 1.2], that $F_{n|X_n} = (F|_X)_n \sim 1$ and $F_{n|A_n} = (F|_A)_n \sim f_n$. Thus there exists an H^{f_n}-structure $F_n : X_n \times A_n \to X_n$ for each stage X_n such that

$$\begin{array}{ccc} X_n \times A_n & \xrightarrow{F_n} & X_n \\ {\scriptstyle p_n \times p'_n}\downarrow & & {\scriptstyle p_n}\downarrow \\ X_{n-1} \times A_{n-1} & \xrightarrow{F_{n-1}} & X_{n-1}, \end{array}$$

where f_n is an induced map from f, and all the pair of k-invariants $(k_X^{n+2}, k_A^{n+2}) : f_n \to \tilde{f}_\#$ are H^{f_n}-primitive with respect to F_n, where $\tilde{f}_\# : K(\pi_{n+1}(A), n+2) \to K(\pi_{n+1}(X), n+2)$ is the induced map by $f : A \to X$. □

In fact, the above theorem follows from Theorem 2.15 if we can show that all the pair of k-invariants $(k_X^{n+2}, k_A^{n+2}) : f_n \to \tilde{f}_\#$ are H^{f_n}-primitive with respect to F_n.

We can obtain an equivalent condition for E_k is an $H^{\bar{f}}$-space for $\bar{f}$.

Theorem 2.18. *Let $(k,l) : f \to f'$ be a map. Then E_k is an $H^{\bar{f}}$-space for $\bar{f} : E_l \to E_k$ if and only if there is a map $G : E_k \times E_l \to X$ such that $Gj \sim \nabla(p_k \vee p_k \circ \bar{f})$ and $kG \sim *$, where $j : E_k \vee E_l \to E_k \times E_l$ is the inclusion.*

Proof. Suppose that E_k is an $H^{\bar{f}}$-space for $\bar{f} : E_l \to E_k$. Then there is a map $\bar{F} : E_k \times E_l \to E_k$ such that $\bar{F}j' \sim \nabla(1 \vee \bar{f})$. Let $G = p_k\bar{F} : E_k \times E_l \to X$. Then $Gj \sim \nabla(p_k \vee p_k \circ \bar{f})$, where $j : E_k \vee E_l \to E_k \times E_l$ is the inclusion. Since G has a lifting $\bar{F}$, by Lemma 2.13, we know that $kG \sim *$. On the other hand, suppose there is a map $G : E_k \times E_l \to X$ such that $Gj \sim \nabla(p_k \vee p_k \circ \bar{f})$ and $kG \sim *$, where $j : E_k \vee E_l \to E_k \times E_l$ is the inclusion. Since $kG \sim *$, there is a map $H : E_k \times E_l \to E_k$ such that $p_kH \sim G$. For maps $1 : E_k \to E_k$ and $\bar{f} : E_l \to E_k$, we can easily know that $p_kH_{|E_k} \sim p_k \circ 1_{E_k}$ and $p_kH_{|E_l} \sim p_k \circ \bar{f}$. Thus we have, from Lemma 2.14, that there is a map $\bar{F} : E_k \times E_l \to E_k$ such that $p_k\bar{F} = p_kH$ and $\bar{F}_{|E_k} \sim 1$ and $\bar{F}_{E_l} \sim \bar{f}$. Thus we know that E_k is an $H^{\bar{f}}$-space for $\bar{f} : E_l \to E_k$. □

Now we can obtain the converse of Theorem 2.15 under some conditions as follows;

Theorem 2.19. *Suppose that there are maps $s_k : X \to E_k$ and $s_l : A \to E_l$ such that $p_ks_k \sim 1_X$ and $p_ls_l \sim 1_A$. If there exists an $H^{\bar{f}}$-structure $\bar{F} : E_k \times E_l \to E_k$ on E_k such that the following diagram is homotopy commutative;*

$$\begin{array}{ccc} E_k \times E_l & \xrightarrow{\bar{F}} & E_k \\ {\scriptstyle p_k \times p_l}\downarrow & & \downarrow{\scriptstyle p_k} \\ X \times A & \xrightarrow{F} & X, \end{array}$$

then X is an H^f-space with H^f-structure $F : X \times A \to X$.

Proof. Since E_k is an $H^{\bar{f}}$-space for $\bar{f} : E_l \to E_k$, there is a map $G : E_k \times E_l \to X$ such that $Gj \sim \nabla(p_k \vee p_k \circ \bar{f})$ and $kG \sim *$, where $j : E_k \vee E_l \to E_k \times E_l$ is the inclusion. Consider the map $F = G(s_k \times s_l) : X \times A \to X$. Then $Fj' \sim \nabla(1 \vee f)$

and $kF(p_k \times p_l) \sim *$, where $j' : X \vee A \to X \times A$ is the inclusion. Thus we know that X is an H^f-space with H^f-structure $F : X \times A \to X$. □

3. Co-H^g-space for Maps

Let $g : X \to A$ be a map. A based map $f : X \to B$ is called *g-coclic* [15] if there is a map $\theta : X \to A \vee B$ such that the following diagram is homotopy commutative;

$$\begin{array}{ccc} X & \xrightarrow{\theta} & A \vee B \\ \Delta\downarrow & & j\downarrow \\ X \times X & \xrightarrow{(g\times f)} & A \times B, \end{array}$$

where $j : A \vee B \to A \times B$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal map. We call such a map θ a *coassociated map* of a g-cocyclic map f.

In the case $g = 1_X : X \to X$, $f : X \to B$ is called *cocyclic* [18]. Clearly any cocyclic map is a g-cocyclic map and also $f : X \to B$ is g-cocyclic iff $g : X \to A$ is f-cocyclic. The *dual Gottlieb set* $DG(X, g, A; B)$ *for a map* $g : X \to A$ is the set of all homotopy classes of g-cocyclic maps from X to B. In the case $g = 1_X : X \to X$, we called such a set $DG(X, 1, X; B)$ the *dual Gottlieb set* denoted $DG(X; B)$, that is, the dual Gottlieb set is exactly same with the dual Gottlieb set for the identity map. In particular, $DG(X, g, A; K(\pi, n))$ will be denoted by $G^n(X, g, A; \pi)$. Haslam [8] introduced and studied the *coevaluation subgroups* $G^n(X; \pi)$ of $H^n(X; \pi)$. $G^n(X; \pi)$ is defined to be the set of all homotopy classes of cocyclic maps from X to $K(\pi, n)$.

In general, $DG(X; B) \subset DG(X, g, A; B) \subset [X, B]$ for any map $g : X \to B$ and any space B. However, there is an example in [22] such that $DG(X, B) \neq DG(X, g, A; B) \neq [X, B]$.

The next proposition is an immediate consequence from the definition.

Proposition 3.1.

(1) *For any maps* $g : X \to A$, $h : A \to B$ *and any space* C, $DG(X, g, A; C) \subset DG(X, hg, B; C)$.

(2) $DG(X, B) = DG(X, 1_X, X; B) \subset DG(X, g, A; B) \subset DG(X, *, A; B) = [X, B]$ *for any spaces* X, A *and* B.

(3) $DG(X, B) = \cap\{DG(X, g, A; B) | g : X \to A$ *is a map and* A *is a space*$\}$.

(4) *If* $h : A \to B$ *is a homotopy equivalence, then* $DG(X, g, A; C) = DG(X, hg, B; c)$.

(5) *For any map* $k : Y \to X$, $k^{\#}(DG(X, g, A; B)) \subset DG(Y, gk, A; B)$.

(6) *For any map* $k : Y \to X$, $k^{\#}(DG(X; B)) \subset DG(Y, k, X; B)$.

(7) *For any map* $s : B \to C$, $s_{\#}(DG(X, g, A; B)) \subset DG(X, g, A; C)$.

It is well known [7] that $G^n(X; \pi)$ is a subgroup of $H^n(X; \pi)$. Moreover, it is also shown [12] that if B is an H-group, then $DG(X, B)$ is a subgroup of $[X, B]$.

But we do not know whether $DG(X, g, A; B)$ is a group. So we would like to investigate the relationship among $DG(X; B)$, $DG(X, g, A; B)$ and $[X, B]$.

Corollary 3.2.

(1) *If* $g : X \to A$ *has a left homotopy inverse, then* $DG(X, g, A; B) = DG(X; B)$ *for any space* B. *In that case,* $G^n(X, g, A; \pi) = G^n(X; \pi)$ *is a group.*

(2) *If* $g : X \to A$ *is a map such that* $G^n(X, g, A; \pi) \subset g^{\#}(G^n(A; \pi))$, *then* $G^n(X, g, A; \pi)$ *is a subgroup of* $H^n(X; \pi)$.

Theorem 3.3. *Let* $g : X \to A$ *be a map and* B *an* H*-group. Then*

(1) *For any* $[\gamma] \in g^{\#}(DG(A; B))$ *and any* $[\alpha] \in DG(X, g, A; B)$, $[\gamma] + [\alpha] \in DG(X, g, A; B)$.

(2) *For any* $[\alpha] \in DG(X, g, A; B)$, $-[\alpha] \in DG(X, g, A; B)$.

Proof. Let $m : B \times B \to B$ and $\mu : B \to B$ be the H-structure and the inverse on B respectively. We can easily know, from Proposition 3.1(7), that $-[\alpha] = [\mu\alpha] = \mu_*([\alpha]) \in DG(X, g, A; B)$ for any $[\alpha] \in DG(X, g, A; B)$. Thus $DG(X, g, A; B)$ is closed under inversion. To show the property (1), let $[\gamma] \in g^{\#}(DG(A; B))$ and $[\alpha] \in DG(X, g, A; B)$. Since $[\gamma] \in g^{\#}(DG(A; B))$, there is $[\beta] \in DG(A; B)$ such that $\beta g \sim \gamma : X \to B$. Thus there are maps $\theta_1 : A \to A \vee B$ and $\theta_2 : X \to A \vee B$ such that $j\theta_1 \sim (1 \times \beta)\Delta$ and $j\theta_2 \sim (g \times \alpha)\Delta$, where $j : A \vee B \to A \times B$ is the inclusion. Let $\lambda = (1 \vee m)i(\theta_1 \vee 1)\theta_2 : X \to A \vee B$, where $i : A \vee (B \vee B) \to A \vee (B \times B)$ is the inclusion. Then we have $j\lambda \sim (1 \times m)((1 \times \beta)\Delta \times 1)(g \times \alpha)\Delta = (1 \times m)(g \times \beta g)\Delta \times \alpha)\Delta \sim (g \times m(\gamma \times \alpha)\Delta)\Delta = (g \times (\gamma + \alpha))\Delta$, where $j : A \vee B \to A \times B$ is the inclusion. Thus we know that $[\gamma] + [\alpha] \in DG(X, g, A; B)$. □

Corollary 3.4 ([12]).

(1) $DG(X; B)$ *is a subgroup of* $[X, B]$ *for an* H*-group* B.

(2) *For any map $g : X \to A$, the group $g^{\#}(G^n(A;\pi))$ acts on $G^n(X, g, A;\pi)$.*

The following proposition says that co-H-spaces are completely characterized by the dual Gottlieb sets.

Proposition 3.5 ([12]). *X is a co-H-space if and only if $DG(X, B) = [X, B]$ for any space B.*

Now, for a map $g : X \to A$, we would like to introduce new spaces which can be characterized by the dual Gottlieb sets for a map $g : X \to A$.

Definition 3.6. A space X is called a *co-H^g-space for a map $g : X \to A$* if there is a map, a co-H^g-structure, $\theta : X \to X \vee A$ such that $j\theta \sim (1 \times g)\Delta$, where $j : X \vee A \to X \times A$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal map.

Proposition 3.7 ([22]). *X is a co-H^g-space for a map $g : X \to A$ if and only if $DG(X, g, A; B) = [X, B]$ for any space B.*

It is well known fact that if X dominates A and X is a co-H-space, then A is a co-H-space. This fact can be generalized as follows;

Corollary 3.8. *Let X be a co-H^r-space for a map $r : X \to A$.*

(1) *If $r : X \to A$ has a right homotopy inverse $i : A \to X$, then A is a co-H-space.*

(2) *If $r : X \to A$ has a left homotopy inverse $i : A \to X$, then X is a co-H-space.*

Proof. (1) Let B be any space. It is sufficient to show that $[A, B] \subset DG(A, B)$. Since X is a co-H-space for a map $r : X \to A$, we have that $DG(X, r, A; B) = [X; B]$. Thus we know, from Proposition 3.1(5), that $[A, B] = i^{\#}[X, B] = i^{\#}DG(X, r, A; B) \subset DG(A, ri, A; B) = DG(A, 1, A; B) = DG(A, B)$. (2) For any space B, we can obtain, from Proposition 3.7 and Proposition 3.1(1), that $[X, B] = DG(X, r, A; B) \subset DG(X, ir, X; B) = DG(X, 1, X) = DG(X, B)$. □

Given maps $g : X \to A$, $g' : X' \to A'$, let $(s, r) : g' \to g$ be a map from g' to g, that is, the following diagram is commutative;

$$\begin{array}{ccc} X' & \xrightarrow{g'} & A' \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{g} & A. \end{array}$$

It is a well known fact that $Y \xrightarrow{\iota} cY \to \Sigma Y$ is a cofibration, where $\iota(y) = [y, 1]$. Let

$i_r : X \to C_r$ be the cofibration induced by $r : X' \to X$ from $\iota_{X'} : X' \to cX'$. Let $i_s : A \to C_s$ be the cofibration induced by $s : A' \to A$ from $\iota_{A'} : A' \to cA'$. Then there is a map $\bar{g} : C_t \to C_s$ such that the following diagram is commutative

$$\begin{array}{ccc} X & \xrightarrow{g} & A \\ {\scriptstyle i_r}\downarrow & & {\scriptstyle i_s}\downarrow \\ C_r & \xrightarrow{\bar{g}} & C_s, \end{array}$$

where $C_t = cX' \amalg X/[x',1] \sim t(x')$, and $C_s = cA' \amalg A/[a',1] \sim s(a')$. $\bar{g} : C_t \to C_s$ is given by $\bar{g}([x',t]) = [g'(x'),t]$ if $[x',t] \in cX'$ and $\bar{g}(x) = g(x)$ if $x \in X$, $i_r(x) = x$, $i_s(a) = a$.

Definition 3.9. *Let X be a co-H^g-space with co-H^g-structure $\theta : X \to X \vee A$. Then a map $(s,r) : g' \to g$ is called a co-H^g-primitive with respect to $\theta : X \to X \vee A$ if there is a coassociate map $\theta' : X' \to X' \vee A'$ of g' such that the following diagram is homotopy commutative;*

$$\begin{array}{ccc} X' & \xrightarrow{\theta'} & X' \vee A' \\ {\scriptstyle r}\downarrow & & {\scriptstyle r\vee s}\downarrow \\ X & \xrightarrow{\theta} & X \vee A. \end{array}$$

The following lemmas are standard.

Lemma 3.10. *Let $f : X \to B$ be a map. Then there is a map $h : C_r \to B$ such that $hi_r = f$ if and only if $fr \sim *$.*

Lemma 3.11 ([21]). *Let $g_t : C_r \to B_t (t = 1,2)$ and $g : C_r \to B_1 \vee B_2$ a map such that $p_t j g i_k \sim g_t i_r (t = 1,2)$, where $j : B_1 \vee B_2 \to B_1 \times B_2$ is the inclusion and $p_t : B_1 \times B_2 \to B_t, t = 1,2$ are projections. Then there is a map $h : C_r \to B_1 \vee B_2$ such that $gi_r = hi_r$ and $p_t j' h \sim g_t (t = 1,2)$.*

Theorem 3.12. *If X is a co-H^g-space with co-H^g-structure $\theta : X \to X \vee A$ and $(s,r) : g' \to g$ is a co-H^g-primitive with respect to θ, then there exists a co-$H^{\bar{g}}$-structure $\bar{\theta} : C_r \to C_r \vee C_s$ on C_r satisfying commutative diagram*

$$\begin{array}{ccc} C_r & \xrightarrow{\bar{\theta}} & C_r \vee C_s \\ {\scriptstyle i_r}\uparrow & & {\scriptstyle i_r\vee i_s}\uparrow \\ X & \xrightarrow{\theta} & X \vee A. \end{array}$$

Proof. Since $(s,r) : g' \to g$ is a co-H^g-primitive with respect to θ, then there is a map $\theta' : X' \to X' \vee A'$ satisfying commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\theta'} & X' \vee A' \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle r\vee s} \\ X & \xrightarrow{\theta} & X \vee A. \end{array}$$

Then we have that $(i_r \vee i_s)\theta r \sim (i_r \vee i_s)(r \vee s)\theta' \sim (i_r \circ r \vee i_s \circ s)\theta \sim *$. Thus we know, from Lemma 3.10, that there is a map $\tilde{\theta} : C_r \to C_r \vee C_s$ such that $\tilde{\theta} i_r = (i_r \vee i_s)\theta$. Then $p_1 j \tilde{\theta} i_r = p_1 j (i_r \vee i_s)\theta \sim p_1(i_r \times i_s)(1 \times g)\Delta \sim i_r = 1 \circ i_r$ and $p_2 j \tilde{\theta} i_r \sim p_2(i_r \times i_s)(1 \times g)\Delta \sim i_s \circ g = \bar{g} \circ i_r$. Thus we have, from Lemma 3.11, that there is a map $\bar{\theta} : C_r \to C_r \vee C_s$ such that $\bar{\theta} i_r = \tilde{\theta} i_r = (i_r \vee i_s)\theta$. □

Taking $g = 1_X$ and $s = r$, we can get the following corollary.

Corollary 3.13 ([6]). *If X and X' are co-H-spaces, and $r : X' \to X$ is a co-H-map, then there is a co-H-structure on C_r such that $i_r : X \to C_r$ is a co-H-map.*

In 1959, Eckmann and Hilton [1] introduced a dual concept of Postnikov system as follows; *A homology decomposition of X* consists of a sequence of spaces and maps $\{X_n, q_n, i_n\}$ satisfying (1) $q_n : X_n \to X$ induces an isomorphism $(q_n)_* : H_i(X_n) \to H_i(X)$ for $i \leq n$. (2) $i_n : X_n \to X_{n+1}$ is a cofibration with cofiber $M(H_{n+1}(X), n)$(a Moore space of type $(H_{n+1}(X), n)$). (3) $q_n \sim q_{n+1} \circ i_n$. It is known by [8] that if X be a 1-connected space having the homotopy type of CW complex, then there is a homology decomposition $\{X_n, q_n, i_n\}$ of X such that $i_n : X_n \to X_{n+1}$ is the principal cofibration induced from $\iota : M(H_{n+1}(X), n) \to cM(H_{n+1}(X), n)$ by a map $r : M(H_{n+1}(X), n) \to X_n$ which is called the dual Postnikov invariants.

From Theorem 3.12, we have the following corollary.

Corollary 3.14. *Let X and A be spaces having the homotopy type of 1-connected countable CW-complexes, and $\{X_n, q_n, i_n\}$ and $\{A_n, q_n', i_n'\}$ be homology decompositions for X and A respectively. If X is a co-H^g-space with co-H^g-structure $\theta : X \to X \vee A$ and for each $n \geq 2$, the pair of r daul invariants $(r_A^n, r_X^n) : \tilde{g}_* \to g_n$ are co-H^{g_n}-primitive with respect to $\theta_n : X_n \to X_n \vee A_n$, where $\tilde{g}_* : M(H_{n+1}(X), n) \to M(H_{n+1}(A), n)$ and g_n are induced maps from $g : X \to A$, then there exists a co-$H^{g_{n+1}}$-structure on X_{n+1} such that $(i_{n+1}', i_{n+1}) : g_n \to g_{n+1}$ is a co-$H^{g_{n+1}}$-primitive with respect to $\theta_n : X_n \to X_n \vee A_n$.*

For a map $f : X \to Y$, we write *conn* $f = n$ if the induced map of homotopy groups $\pi_k(f) : \pi_k(X) \to \pi_k(Y)$ is an isomorphism for $k < n$ and an epimorphism for $k = n$. In particular, for the constant map $C_* : X \to *$ we put *conn* $X =$ *conn* C_*. Let $X \flat Y$ and $X \wedge Y$ be the flat product and the smash product of spaces X and Y, respectively. There is a homotopy equivalence $X\flat Y \simeq \Sigma\Omega X \wedge \Omega Y$(see e.g., [8, p. 216]). We now show that the converse of Theorem 3.12 also holds, provided some conditions are satisfied.

Lemma 3.15. *For any two maps* $f : X \to Y$, $g : X' \to Y'$, *conn* $(f\flat\ g) = \min\{conn\ (f) + conn\ (Y'),\ conn\ (X) + conn\ (g)\}$

Proof. Since $f\flat\ g = (f\flat\ 1_{Y'}) \circ (1_X\flat\ g) : X\flat\ X' \overset{1_X\flat\ g}{\to} X\flat\ Y' \overset{f\flat\ 1_Y}{\to} Y\flat\ Y'$ and $conn\ (f\flat\ 1_{Y'}) = conn\ (\Sigma\Omega(f) \wedge 1_{\Omega Y'}) = conn\ (f) + conn\ (Y')$ and $conn\ (1_X\flat\ g) = conn\ (X) + conn\ (g)$, we have that

$$conn\ (f\flat\ g) = \min\{conn\ (f) + conn\ (Y'),\ conn\ (X) + conn\ (g)\}.$$

□

Lemma 3.16. ([19, Theorem 7.16]) *For a map* $f : X \to Y$, *conn* $(f) = n$ *if and only if* (1) *for every CW complex* K *with dim* $K < n$, $f_\# : [K, X] \to [K, Y]$ *is one-to-one correspondence (2) for every CW complex* K *with* $\dim K = n$, $f_\# : [K, X] \to [K, Y]$ *is onto.*

Theorem 3.17. *Let* X *be a simply connected co-*H^g*-space with co-*H^g*-structure* $\theta : X \to X \vee A$. *Let* X' *be a simply connected co-*H*-space and* $(s, r) : g' \to g$ *a map, where* $g' : X' \to A'$ *is a map. If there is a co-*$H^{\bar{g}}$*-structure* $\bar{\theta} : C_r \to C_r \vee C_s$ *on* C_r *satisfying commutative diagram*

$$\begin{array}{ccc} C_r & \xrightarrow{\bar{\theta}} & C_r \vee C_s \\ {\scriptstyle i_r}\uparrow & & \uparrow{\scriptstyle i_r \vee i_s} \\ X & \xrightarrow{\theta} & X \vee A \end{array}$$

and $\dim X' < \min\{conn\ (X') + conn\ (C_s), conn\ (A') + conn\ (X)\}$, *then* $(s, r) : g' \to g$ *is a co-*H^g*-primitive with respect to* θ.

Proof. Let $\mu' : X' \to X' \vee X'$ be a co-H-structure on X'. It is known by [2] that a co-H-structure μ' on a 1-connected space X' admits an inversion. It is also known by [14] that for a co-H-space X' with a co-H-structure μ' and an inversion map, and any spaces X and A, there is a split short exact sequence

$$0 \to [X', X\flat A] \to [X', X \vee A] \underset{\gamma}{\overset{j_\#}{\rightleftharpoons}} [X', X \times A] \to 0,$$

where $j_\#\gamma = 1$ with $\gamma([\alpha_1, \alpha_2]) = [(\alpha_1 \vee \alpha_2)\mu']$. Denote the induced operation on $[X', X \vee A]$ additively and let $\beta = 1 - \gamma j_\# : [X', X \vee A] \to [X', X \vee A]$. Since $j_\#\beta = 0$, we know that $\beta : [X', X \vee A] \to [X', X\flat A]$. Consider the commutative diagram

$$\begin{array}{ccccc} [X', X] & \xrightarrow{\theta_\#} & [X', X \vee A] & \xrightarrow{\beta} & [X', X\flat A] \\ {\scriptstyle i_{r\#}}\downarrow & & {\scriptstyle (i_r \vee i_s)_\#}\downarrow & & {\scriptstyle (i_r\flat\, i_s)_\#}\downarrow \\ [X', C_r] & \xrightarrow{\bar\theta_\#} & [X', C_r \vee C_s] & \xrightarrow{\beta'} & [X', C_r\flat\, C_s]. \end{array}$$

Then $(i_r\flat\, i_s)_\#\beta\theta_\#([r]) = \beta'\bar\theta_\# i_\#([r]) = \beta'\bar\theta_\#([i_r \circ r]) = 0$. Clearly $conn\ i_r = conn\ X'$ and $conn\ i_s = conn\ A'$. By Lemma 3.15, we know that $conn\ (i_r\flat\, i_s) = \min\{conn\ i_r + conn\ C_s, conn\ i_s + conn\ X\}$ and $\dim X' < conn\ (i_r\flat\, i_s)$. Thus we have, from Lemma 3.16, that $(i_r\flat\, i_s)_\#$ is an isomorphism, so we get that $\beta\theta_\#([r]) = 0$. From the definition of β, we know that $0 = \beta\theta_\#([r]) = (1-\gamma j_\#)([\theta r]) = [\theta r] - \gamma[j\theta r]$. Thus we have that $\theta r \sim (p_1 j\theta r \vee p_2 j\theta r)\mu' \sim (r \vee g \circ r)\mu' \sim (r \vee s)(1 \vee g')\mu'$ and $(s, r) : g' \to g$ is a co-H^g-primitive with respect to θ. □

Taking $g = 1_X$, $g' = 1_{X'}$ and $s = r$, we have the following corollary.

Corollary 3.18 ([5]). *Let $r : X' \to X$ be a map with X' and X 1-connected co-H-spaces. If the mapping cone C_r is a co-H-space, $i_r : X \to C_r$ a co-H-map and $\dim X' < conn\ X' + \min\{conn\ X, conn\ C_r\}$, then $r : X' \to X$ is a co-H-map.*

In particular, let $M(A, n)$ be the Moore space of type (A, n) for $n \geq 2$. Then $\dim M(A, n) \leq n + 1$ and $conn\ M(A, n) = n - 1$. Thus we have the following corollary.

Corollary 3.19. *Let X be a 2-connected co-H^g-space with co-H^g-structure $\theta : X \to X \vee A$. Let $(s, r) : g' \to g$ be a map, where $g' : M(A, n) \to M(A', n)$ $(n \geq 2)$ is a map. If there is a co-$H^{\bar g}$-structure $\bar\theta : C_r \to C_r \vee C_s$ on C_r satisfying commutative diagram*

$$\begin{array}{ccc} C_r & \xrightarrow{\bar\theta} & C_r \vee C_s \\ {\scriptstyle i_r}\uparrow & & {\scriptstyle i_r \vee i_s}\uparrow \\ X & \xrightarrow{\theta} & X \vee A, \end{array}$$

then $(s, r) : g' \to g$ is a co-H^g-primitive with respect to θ.

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