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NULL CURVES IN A SEMI-RIEMANNIAN MANIFOLD OF INDEX 2

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ABSTRACT. The purpose of this paper is to study the geometry of null curves in a semi-Riemannian manifold (M, g) of index 2. We show that it is possible to construct new Frenet equations of two types of null curves in M.

1. INTRODUCTION

Many authors have believed that the null curves in an n-dimensional semi-Riemannian manifolds of index 2 has only one type of Frenet equations which are made up of 1-timelike and $(n-1)$-spacelike vector fields. But Duggal and I showed that it is possible to construct two types of Frenet frames suitable for the null curves in a semi-Riemannian manifolds (M, g) of index 2 such that each invariant under any causal change. This is then followed by constructing general Frenet equations (called *compound general Frenet equations*) which include all the possible forms of the two types ([4]).

The objective of the present paper is also to study null curves in a semi-Riemannian manifolds (M, g) of index 2. We show that it is possible to construct two types of new Frenet frames suitable for (M, g) (called *Natural Frenet frames*) such that each invariant under any causal change, which is more simple types than the Frenet frames in [4]. This Frenet frames and its Frenet equations are not unique as they depend on the parameters on the null curve and the screen vector bundles. To deal with this non-uniqueness problem, we construct a unique Frenet frame (called *Cartan frame*) along null curves in M with the minimum number of the curvature functions.

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Much of the work will be immediately generalized in a formal way to arbitrary semi-Riemannian manifolds of arbitrary index q.

2. General Frenet Frames

Let $(M, g) \equiv M_2$ be a real $(m+2)$-dimensional semi-Riemannian manifold of index 2 and C be a smooth null curve in M_2 locally given by

$$x^i = x^i(t), \quad t \in I \subset \mathbf{R}, \quad i \in \{0, 1, \ldots, (m+1)\}$$

for a coordinate neighborhood $\mathcal{U}$ on C. Then the tangent vector field $\frac{d}{dt} = C' \equiv \lambda$ of C on $\mathcal{U}$ satisfies

$$g(\lambda, \lambda) = 0.$$

Denote by TC the tangent bundle of C and $TC^\perp$ the TC-*perp* ([9]). Clearly, $TC^\perp$ is a vector bundle over C of rank $(m+1)$. Since λ is null, the tangent bundle TC is a vector subbundle of $TC^\perp$, of rank 1. Suppose $S(C)$ denotes the complementary vector subbundle to TC in $TC^\perp$, then we have

$$TC^\perp = TC \oplus_{orth} S(C),$$

where $\oplus_{orth}$ means the orthogonal direct sum. We call $S(C)$ a *screen vector bundle* of C. It follows that $S(C)$ is a non-degenerate vector subbundle of TM_2, of rank m ([3]). Thus we have

$$TM_2|_C = S(C) \oplus_{orth} S(C)^\perp. \tag{1}$$

Theorem 1 ([3]). *Let C be a null curve of M_2 and let $S(C)$ be a screen vector bundle of C. Then there exists a unique vector bundle $ntr(C)$ over C of rank 1 such that, on each coordinate neighborhood $\mathcal{U} \subset C$, there is a unique section $N \in ntr(C)|_{\mathcal{U}}$ satisfying*

$$g(\lambda, N) = 1, \quad g(N, N) = g(N, X) = 0, \quad {}^\forall X \in S(C)|_{\mathcal{U}}. \tag{2}$$

We call $ntr(C)$ the *null transversal vector bundle* of C with respect to the given screen vector bundle $S(C)$. Next consider the vector bundle

$$tr(C) = ntr(C) \oplus_{orth} S(C),$$

which, according to (1) and (2), is complementary but not orthogonal to TC in $TM_2|_C$. More precisely, we have

$$TM_2|_C = TC \oplus tr(C) = (TC \ominus ntr(C)) \oplus_{orth} S(C). \tag{3}$$

We call $tr(C)$ the *transversal vector bundle* of C with respect to $S(C)$. The vector field N in Theorem 1 is called the *null transversal vector field* of C with respect to λ. As $\{\lambda, N\}$ is a null basis of $(TC \oplus ntr(C))|_{\mathcal{U}}$, satisfying (2), we obtain

Proposition 2 ([3]). *Let C be a null curve of M_2. Then any screen vector bundle of C is Lorentzian.*

Now, we show that it is possible to construct two types of Frenet frames suitable for M_2 ([4]). First, we study a class of null curves whose Frenet frame is made up of two null vectors λ and N, one timelike and $(m-1)$ spacelike vector fields. Let us denote by *Type* 1 the Frenet frame and Frenet equations of this class of the null curves. Let ∇ be the Levi-Civita connection on M_2. Since $g(\nabla_\lambda \lambda, \lambda) = 0$ and $g(\nabla_\lambda \lambda, N) = h$, if the vector field $H_1 = \nabla_\lambda \lambda - h\lambda \in S(C)$ is non-null, we have

$$\nabla_\lambda \lambda = h\lambda + \kappa_1 \epsilon_1 \bar{W}_1,$$

where we let $\kappa_1 \epsilon_1 = \| H_1 \|$, $\bar{W}_1 = \frac{H_1}{\| H_1 \|}$ and ϵ_1 is the *sign* of H_1, i.e., $\epsilon_1 = +1$ or -1 according as H_1 is spacelike or timelike respectively. Also, from $g(\nabla_\lambda N, \lambda) = -h$, $g(\nabla_\lambda N, N) = 0$ and $g(\nabla_\lambda N, \bar{W}_1) = \kappa_2$, where κ_2 denotes the second curvature function, if the vector field $H_2 = \nabla_\lambda N + hN - \kappa_2 \epsilon_1 \bar{W}_1$ is also non-null, we have

$$\nabla_\lambda N = -hN + \kappa_2 \epsilon_1 \bar{W}_1 + \kappa_3 \epsilon_2 \bar{W}_2,$$

where we let $\kappa_3 \epsilon_2 = \| H_2 \|$, $\bar{W}_2 = \frac{H_2}{\| H_2 \|}$ and ϵ_2 is the sign of H_2. Repeating above process for the orthonormal basis $\{\bar{W}_1, \ldots, \bar{W}_m\}$ of $S(C)$ and then, setting $W_i = \epsilon_i \bar{W}_i$ for all $i \in \{1, \ldots, m\}$, we obtain the following equations

$$\begin{aligned}
\nabla_\lambda \lambda &= h\lambda + \kappa_1 W_1, \\
\nabla_\lambda N &= -hN + \kappa_2 W_1 + \kappa_3 W_2, \\
\epsilon_1 \nabla_\lambda W_1 &= -\kappa_2 \lambda - \kappa_1 N + \kappa_4 W_2 + \kappa_5 W_3, \\
\epsilon_2 \nabla_\lambda W_2 &= -\kappa_3 \lambda - \kappa_4 W_1 + \kappa_6 W_3 + \kappa_7 W_4, \\
\epsilon_3 \nabla_\lambda W_3 &= -\kappa_5 W_1 - \kappa_6 W_2 + \kappa_8 W_4 + \kappa_9 W_5, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\epsilon_{m-1} \nabla_\lambda W_{m-1} &= -\kappa_{2m-3} W_{m-3} - \kappa_{2m-2} W_{m-2} + \kappa_{2m} W_m, \\
\epsilon_m \nabla_\lambda W_m &= -\kappa_{2m-1} W_{m-2} - \kappa_{2m} W_{m-1},
\end{aligned} \tag{4}$$

where h and $\{\kappa_1, \cdots, \kappa_{2m}\}$ are smooth functions on $\mathcal{U}$, $\{W_1, \cdots, W_m\}$ is a certain orthonormal basis of $S(C)$ and $\epsilon_i \,(1 \le i \le m)$ is the sign of W_i. We call $F_1 = \{\lambda, N, W_1, \ldots, W_m\}$ a *general Frenet frame of Type* 1 on M_2 along C, with respect to the given screen vector bundle $S(C)$ and the equations (4) its *general Frenet equations of Type* 1. The functions $\{\kappa_1, \ldots, \kappa_{2m}\}$ are called *curvatures* of C with respect to the frame F_1. Since the screen vector bundle is Lorentzian, this implies that only one of the vector fields W_i is timelike and all others are spacelikes.

Next, we study a class of null curves whose Frenet frame is generated by a quasi-orthonormal basis consisting of the two null vector fields λ and N and another two null vector fields L_i and L_{i+1} such that $g(L_i, L_{i+1}) = 1$ and $(m-2)$ spacelike vector fields $\{W_\alpha\}$. We denote the Frenet frame and Frenet equations of this particular class of the null curves by *Type* 2. There are $(m-1)$ choices for L_i. In this case, if we set

$$W_i = \frac{L_i - L_{i+1}}{\sqrt{2}}, \qquad W_{i+1} = \frac{L_i + L_{i+1}}{\sqrt{2}}, \tag{5}$$

then W_i and W_{i+1} are timelike and spacelike vector fields respectively. Then we can exchange $\{L_i, L_{i+1}\}$ for $\{W_i, W_{i+1}\}$. We also denote the Frenet frame and Frenet equations of this class of C by *Type* 2.

To choose $\{L_1, L_2\}$, we let the vector field $\nabla_\lambda \lambda - h\lambda \in S(C)$ be null and define the curvature function σ_1 (in generally, $\sigma_1 = 1$) by

$$\nabla_\lambda \lambda - h\,\lambda = \sigma_1 L_1.$$

Since $S(C)$ is Lorentzian vector bundle, we can take another null vector field L_2 in $S(C)$ along C such that $g(L_1, L_2) = 1$. Set this case so that the equation (5) holds for $i = 1$. Therefore W_1 and W_2 are perpendicular to λ and N, then we have

$$\nabla_\lambda \lambda = h\,\lambda + \kappa_1 W_1 + \tau_1 W_2,$$

where $\kappa_1 = \tau_1 = \frac{\sigma_1}{\sqrt{2}}$. From $g(\nabla_\lambda N, \lambda) = -h$ and $g(\nabla_\lambda N, N) = 0$ and define the second and third curvatures by $g(\nabla_\lambda N, W_1) = -\kappa_2$ and $g(\nabla_\lambda N, W_2) = \kappa_3$ respectively, since the vector field $H_3 = \nabla_\lambda N + h\,N - \kappa_2 W_1 - \kappa_3 W_2$ is spacelike, we obtain

$$\nabla_\lambda N = -h\,N + \kappa_2 W_1 + \kappa_3 W_2 + \tau_2 W_3$$

if we let $\tau_2 = \| H_3 \|$ and $W_3 = \frac{H_3}{\| H_3 \|}$. Also, from the following results $g(\nabla_\lambda W_1, \lambda) = \kappa_1$, $g(\nabla_\lambda W_1, N) = \kappa_2$, $g(\nabla_\lambda W_1, W_1) = 0$ and define the fourth and fifth curvatures by $g(\nabla_\lambda W_1, W_2) = -\kappa_4$ and $g(\nabla_\lambda W_1, W_3) = -\kappa_5$ respectively, we obtain

$$\nabla_\lambda W_1 = \kappa_2\,\lambda + \kappa_1 N - \kappa_4 W_2 - \kappa_5 W_3 - \tau_3 W_4$$

if we let $\tau_3 = \| H_4 \|$ and $W_4 = \frac{H_4}{\| H_4 \|}$, where $H_4 = \nabla_\lambda W_1 - \kappa_2\,\lambda - \kappa_1 N + \kappa_4 W_2 + \kappa_5 W_3$ is a spacelike vector field on $S(C)$. Repeating above process for all the orthonormal basis $\{W_1, \cdots, W_m\}$ of $S(C)$, we also obtains the following equations

$$
(6)\qquad
\begin{aligned}
\nabla_\lambda \lambda &= h\,\lambda + \kappa_1 W_1 + \tau_1 W_2,\\
\nabla_\lambda N &= -\,h\,N + \kappa_2 W_1 + \kappa_3 W_2 + \tau_2 W_3,\\
\nabla_\lambda W_1 &= \kappa_2 \lambda + \kappa_1 N - \kappa_4 W_2 - \kappa_5 W_3 - \tau_3 W_4,\\
\nabla_\lambda W_2 &= -\,\kappa_3 \lambda - \tau_1 N - \kappa_4 W_1 + \kappa_6 W_3 + \kappa_7 W_4 + \tau_4 W_5,\\
\nabla_\lambda W_3 &= -\,\tau_2 \lambda - \kappa_5 W_1 - \kappa_6 W_2 + \kappa_8 W_4 + \kappa_9 W_5 + \tau_5 W_6,\\
\nabla_\lambda W_4 &= -\,\tau_3 W_1 - \kappa_7 W_2 - \kappa_8 W_3 + \kappa_{10} W_5 + \kappa_{11} W_6 + \tau_6 W_7\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\nabla_\lambda W_{m-1} &= -\tau_{m-2} W_{m-4} - \kappa_{2m-3} W_{m-3} - \kappa_{2m-2} W_{m-2} + \kappa_{2m} W_m,\\
\nabla_\lambda W_m &= -\tau_{m-1} W_{m-3} - \kappa_{2m-1} W_{m-2} - \kappa_{2m} W_{m-1}.
\end{aligned}
$$

We call $F_2 = \{\lambda, N, W_1, \cdots, W_m\}$ a *general Frenet frame of Type* 2 on M along C with respect to $S(C)$ and the equations (6) its *general Frenet equations of Type* 2.

Also, using the relations $g(\nabla_\lambda N, \lambda) = -h$, $g(\nabla_\lambda N, L_1) = \sigma_3$, $g(\nabla_\lambda N, L_2) = \sigma_2$, we can write

$$\nabla_\lambda N = -h\,\lambda + \sigma_2 L_1 + \sigma_3 L_2 + R$$

where $R \in S(C)$ is perpendicular to λ, N, L_1 and L_2 along C. From the relations $-\kappa_2 = (\sigma_3 - \sigma_2)/\sqrt{2}$, $\kappa_3 = (\sigma_3 + \sigma_2)/\sqrt{2}$ and

$$\nabla_\lambda N + h\,\lambda - \sigma_2 L_1 - \sigma_3 L_2 \;=\; \nabla_\lambda N + h\,\lambda - \kappa_2 W_1 - \kappa_3 W_2,$$

we conclude $R = \tau_2 W_3$. Therefore,

$$\nabla_\lambda N = -\,h\,\lambda + \sigma_2 L_1 + \sigma_3 L_2 + \tau_2 W_3.$$

Next, by the transformations (5) for $i = 1$, we have

$$\nabla_\lambda L_1 = \frac{1}{\sqrt{2}}\,(\nabla_\lambda W_2 + \nabla_\lambda W_1), \quad \nabla_\lambda L_2 = \frac{1}{\sqrt{2}}\,(\nabla_\lambda W_2 - \nabla_\lambda W_1).$$

Using the third and fourth relations of (6) in above equation and using the results $\sigma_4 = -\,\kappa_4$, $\sigma_5 = \frac{1}{\sqrt{2}}(\kappa_6 - \kappa_5)$, $\sigma_7 = \frac{1}{\sqrt{2}}(\kappa_6 + \kappa_5)$, we obtain

$$
\begin{aligned}
\nabla_\lambda L_1 &= -\,\sigma_3 \lambda + \sigma_4 L_1 + \sigma_5 W_3 + \sigma_6 W_4 + \mu_4 W_5,\\
\nabla_\lambda L_2 &= -\,\sigma_2 \lambda - \sigma_1 N - \sigma_4 L_2 + \sigma_7 W_3 + \sigma_8 W_4 + \mu_4 W_5,
\end{aligned}
$$

where $\mu_4 = \frac{\tau_4}{\sqrt{2}}$, $\sigma_6 = \frac{1}{\sqrt{2}}(\kappa_7 - \tau_3)$ and $\sigma_8 = \frac{1}{\sqrt{2}}(\kappa_7 + \tau_3)$. Repeating above process, we get the following equations

$$
\begin{aligned}
&\nabla_\lambda \lambda = h\lambda + \sigma_1 L_1,\\
&\nabla_\lambda N = -hN + \sigma_2 L_1 + \sigma_3 L_2 + \tau_2 W_3,\\
&\nabla_\lambda L_1 = -\sigma_3 \lambda + \sigma_4 L_1 + \sigma_5 W_3 + \sigma_6 W_4 + \mu_4 W_5,\\
&\nabla_\lambda L_2 = -\sigma_2 \lambda - \sigma_1 N - \sigma_4 L_2 + \sigma_7 W_3 + \sigma_8 W_4 + \mu_4 W_5,\\
(7)\qquad &\nabla_\lambda W_3 = -\tau_2 \lambda - \sigma_7 L_1 - \sigma_5 L_2 + \kappa_8 W_4 + \kappa_9 W_5 + \tau_5 W_6\\
&\nabla_\lambda W_4 = -\sigma_8 L_1 - \sigma_6 L_2 - \kappa_8 W_3 + \kappa_{10} W_5 + \kappa_{11} W_6 + \tau_6 W_7,\\
&\qquad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\nabla_\lambda W_{m-1} = -\tau_{m-2} W_{m-4} - \kappa_{2m-3} W_{m-3} - \kappa_{2m-2} W_{m-2} + \kappa_{2m} W_m,\\
&\nabla_\lambda W_m = -\tau_{m-1} W_{m-3} - \kappa_{2m-1} W_{m-2} - \kappa_{2m} W_{m-1}.
\end{aligned}
$$

We also call $F_2 = \{\lambda, N, L_1, L_2, W_3, \ldots, W_m\}$ a *general Frenet frame of Type* 2 on M along C, with respect to the screen vector bundle $S(C)$ and the equations (7) its *general Frenet equations of Type* 2.

To choose $\{L_2, L_3\}$, we let the vector field $\nabla_\lambda\lambda - h\lambda$ be non-null and $\nabla_\lambda N + hN - \kappa_2 W_1$ be null. To choose $\{L_3, L_4\}$, we let the vector fields and $\nabla_\lambda\lambda - h\lambda$ and $\nabla_\lambda N + hN - \kappa_2 W_1$ be non-nulls and $\nabla_\lambda W_1 + \kappa_2 \lambda + \kappa_1 N - \kappa_4 W_2$ be null. And so on, we finally obtain the following general equations

$$
\begin{aligned}
&\nabla_\lambda \lambda = h\lambda + \kappa_1 W_1,\\
&\nabla_\lambda N = -hN + \kappa_2 W_1 + \kappa_3 W_2,\\
(8)\qquad &\nabla_\lambda W_1 = -\kappa_2 \lambda - \kappa_1 N + \kappa_4 W_2 + \kappa_5 W_3,\\
&\qquad\cdots\cdots\cdots\cdots\\
&\nabla_\lambda L_i = -\sigma_{2i+1} W_{i-1} + \sigma_{2i+3} L_i + \sigma_{2i+4} W_{i+2} + \sigma_{2i+5} W_{i+3} + \mu_{i+2} W_{i+2},\\
&\nabla_\lambda L_{i+1} = -\sigma_{2i-1} W_{i-2} - \sigma_{2i} W_{i-1} - \sigma_{2i+3} L_{i+1} + \sigma_{2i+6} W_{i+2}\\
&\qquad + \sigma_{2i+7} W_{i+3} + \mu_{i+2} W_{i+2}, \qquad 2 \le i \le m-1,\\
&\qquad\cdots\cdots\cdots\cdots\\
&\nabla_\lambda W_{m-1} = -\tau_{m-2} W_{m-4} - \kappa_{2m-3} W_{m-3} - \kappa_{2m-2} W_{m-2} + \kappa_{2m} W_m,\\
&\nabla_\lambda W_m = -\tau_{m-1} W_{m-3} - \kappa_{2m-1} W_{m-2} - \kappa_{2m} W_{m-1}.
\end{aligned}
$$

In the above case also the equations (8) are the general Frenet equations of *Type* 2 with the general Frenet frame $F_2 = \{\lambda, N, W_1, \ldots, L_i, L_{i+1}, \ldots, W_m\}$. Also, by replacing L_i and L_{i+1} in (8) with its values in terms of W_i and W_{i+1} (see relations (5)), we can get another set of general Frenet equations in terms of one timelike and all others spacelike basis of its general Frenet frame. Finally, proceeding as before, one can show that the general Frenet equations of Type 2 are equivalent to the follow set of general equations

$$(9)\qquad \begin{aligned} \nabla_\lambda \lambda &= h\,\lambda + \kappa_1\, W_1 + \tau_1\, W_2, \\ \nabla_\lambda N &= -\,h\,N + \kappa_2\, W_1 + \kappa_3\, W_2 + \tau_2\, W_3, \\ \epsilon_1\, \nabla_\lambda W_1 &= -\,\kappa_2\,\lambda - \kappa_1\, N + \kappa_4\, W_2 + \kappa_5\, W_3 + \tau_3\, W_4, \\ \epsilon_2\, \nabla_\lambda W_2 &= -\,\kappa_3\,\lambda - \tau_1\, N - \kappa_4\, W_1 + \kappa_6\, W_3 + \kappa_7\, W_4 + \tau_4\, W_5, \\ \epsilon_3\, \nabla_\lambda W_3 &= -\,\tau_2\,\lambda\, - \,\kappa_5\, W_1 - \kappa_6\, W_2 + \kappa_8\, W_4 + \kappa_9\, W_5 + \tau_5\, W_6, \\ \epsilon_4\, \nabla_\lambda W_4 &= -\,\tau_3\, W_1 - \kappa_7\, W_2 - \kappa_8\, W_3 + \kappa_{10}\, W_5 + \kappa_{11}\, W_6 + \tau_6\, W_7 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \epsilon_{m-1} \nabla_\lambda W_{m-1} &= -\tau_{m-2}\, W_{m-4}\, - \,\kappa_{2m-3} W_{m-3} - \kappa_{2m-2} W_{m-2} + \kappa_{2m} W_m, \\ \epsilon_m\, \nabla_\lambda\, W_m &= -\tau_{m-1}\, W_{m-3}\, - \,\kappa_{2m-1}\, W_{m-2} - \kappa_{2m}\, W_{m-1}. \end{aligned}$$

The equations (9) include all m-th different general Frenet equations of Type 1 and all $(m-1)$-th different general Frenet equations of Type 2. Hence we call the equations (9) the *compound general Frenet equations* of the null curve C, $F_2 = \{\lambda, N, W_1, \cdots, W_m\}$ the *compound general Frenet frame* on M_2 along C and the functions $\{\kappa_1, \ldots, \kappa_{2m}\}$ and $\{\tau_1, \ldots, \tau_{m-1}\}$ the *curvatures* and *torsions* of C for the compound general Frenet equations (9) respectively.

3. Invariance of Types

In this section, we show that each type of the Frenet frames always transform to the same type by the transformations of the coordinate neighborhood and the screen vector bundle of C.

In the first case, with respect to a given screen vector bundle $S(C)$, we consider two Frenet frames F and F^* along two neighborhoods $\mathcal{U}$ and $\mathcal{U}^*$ respectively with non-null intersection. Then we have

$$(10)\qquad \lambda^* = \frac{dt}{dt^*}\,\lambda, \quad N^* = \frac{dt^*}{dt}\,N, \quad W^*_\alpha = \sum_{\beta=1}^{m} A^\beta_\alpha\, W_\beta \;\; (1 \le \alpha \le m),$$

where A_α^β are smooth functions on $\mathcal{U} \cap \mathcal{U}^*$ and the $m \times m$ matrix $[A_\alpha^\beta(x)]$ is an element of the Lorentzian group $O(1, m-1)$ for any $x \in \mathcal{U} \cap \mathcal{U}^*$. We call (10) the *transformation of coordinate neighborhood* (or *parameter transformation*) of C with respect to the given screen vector bundle $S(C)$.

Let F and $\bar{F}$ be two Frenet frames with respect to $(t, S(C), \mathcal{U})$ and $(\bar{t}, \bar{S}(C), \bar{\mathcal{U}})$ respectively. Then the general transformation that relate elements of F and $\bar{F}$ on $\mathcal{U} \cap \bar{\mathcal{U}} \neq \emptyset$ are given by

$$
\begin{aligned}
\bar{\lambda} &= \frac{dt}{d\bar{t}}\lambda, \\
\bar{N} &= -\frac{1}{2}\frac{dt}{d\bar{t}}\sum_{\alpha=1}^{m}\epsilon_\alpha(c_\alpha)^2\lambda + \frac{d\bar{t}}{dt}N + \sum_{\alpha=1}^{m}c_\alpha W_\alpha, \qquad (11)\\
\bar{W}_\alpha &= \sum_{\beta=1}^{m}B_\alpha^\beta\left(W_\beta - \epsilon_\beta\frac{dt}{d\bar{t}}c_\beta\lambda\right) \quad (1 \le \alpha \le m),
\end{aligned}
$$

where c_α and B_α^β are smooth funtions on $\mathcal{U} \cap \bar{\mathcal{U}}$ and the $m \times m$ matrix $[B_\alpha^\beta(x)]$ is also an element of the Lorentzian group $O(1, m-1)$ for each $x \in \mathcal{U} \cap \bar{\mathcal{U}}$. We call (11) the *transformation of screen vector bundle* (or *screen transformation*) of C.

Denote $\widetilde{F} \equiv \{F^*; \bar{F}\}$ and $P_\alpha^\beta \equiv \{A_\alpha^\beta; B_\alpha^\beta\}$. Using the parameter and screen transformation (10) and (11) that relate elements of F and $\widetilde{F}$ on $\mathcal{U} \cap \widetilde{\mathcal{U}}$ and the first equation of (9) for both F and $\widetilde{F}$, we obtain

$$(\widetilde{\kappa}_1, \widetilde{\tau}_1, 0, \cdots, 0)\left[P_\alpha^\beta(x)\right] = (\kappa_1, \tau_1, 0, \cdots, 0)\left(\frac{dt}{d\widetilde{t}}\right)^2. \qquad (12)$$

Theorem 3. *Let C be a null curve of $\mathbf{M}_2$. Then the type of general Frenet equations is invariant to the parameter and screen transformations of C.*

Proof. In the first case suppose $\widetilde{F} = \widetilde{F}_2$ and $F = F_1$. Then we have $\widetilde{\tau}_1 = \widetilde{\kappa}_1$ and $\tau_1 = 0$. This means from equation (12) that $P_1^2 = P_2^2$. Since $\widetilde{W}_1$ and $\widetilde{W}_2$ are timlike and spacelike vector fields respectively, the first row $(P_1^1, P_1^2, 0, \cdots, 0)$ of $[P_\alpha^\beta(x)]$ is timelike and the second row $(P_2^1, P_2^2, 0, \cdots, 0)$ is spacelike vector field of $\mathbf{R}_1^m$ and these vectors are perpendicular to each other. Thus, we have

$$(P_1^1)^2 - 1 = (P_2^1)^2 + 1 = P_1^1 P_2^1.$$

From the last relations, we have the contradictory relation $P_1^1 = P_2^1$. Hence this case is not possible.

Conversely, if $\widetilde{F} = \widetilde{F}_1$ and $F = F_2$, then $\widetilde{\tau}_1 = 0$ and $\tau_1 = \kappa_1$. From the equation

(12), we have $P_1^1 = -P_1^2$. This means that the first row $(P_1^1, P_1^2, 0, \cdots, 0)$ of the matrix $[P_\alpha^{'j}(x)]$ is null vector field, hence the vector field $\widetilde{W}_1$ is a null vector field. Thus we conclude that this case is also not possible, which complete the proof. □

Now if we write the first equations of the general Frenet equations (4) and (9) for both F and $\widetilde{F}$ and use (10) and (11), we obtain

$$\frac{d^2 t}{d\widetilde{t}^2} - \widetilde{h}\frac{dt}{d\widetilde{t}} + O_3(\widetilde{c}_i) = -h\left(\frac{dt}{d\widetilde{t}}\right)^2,$$

where the remainder $O_3(c_i^*) = 0$ and $O_3(\bar{c}_i) = \kappa_1\epsilon_1 c_1 \left(\frac{dt}{dt}\right)^3$ for Type 1 and $O_3(\bar{c}_i) = \kappa_1(\epsilon_1 c_1 + \epsilon_2 c_2)\left(\frac{dt}{dt}\right)^3$ for Type 2. We consider the following differential equation

$$\frac{d^2 t}{d\widetilde{t}^2} - \widetilde{h}\frac{dt}{d\widetilde{t}} = 0$$

whose general solution comes from

$$t = a\int_{\widetilde{t}_0}^{\widetilde{t}} \exp\left(\int_{s_0}^{s} \widetilde{h}(\widetilde{t})d\widetilde{t}\right) ds + b, \quad a,\, b \in R.$$

It follows that any of these solutions, with $a \neq 0$, might be taken as special parameter on C, such that $h = 0$. Denote one such solution by $p = \frac{t-b}{a}$, where t is the general parameter as defined in above equation. We call p a *distinguished parameter* of C, in terms for which $h = 0$. When t is replaced by p, we let $C \equiv C(p)$ and $\xi \equiv \frac{d}{dp}$.

Theorem 4. *Let C be a null curve of M_2. Then C is a null geodesic of M_2 if and only if the first curvature κ_1 vanishes identically on C.*

Proof. In case $\kappa_1 = 0$, then, since $\tau_1 = 0$ or $\tau_1 = \kappa_1$, the first equations of (4) and (9) such that $h = 0$ takes the following familiar form

$$\frac{d^2 x^i}{dp^2} + \Gamma^i_{jk}\frac{dx^j}{dp}\frac{dx^k}{dp} = 0, \quad i,\, j,\, k \in \{0, \ldots, m\}$$

where Γ^i_{jk} are the Christoffel symbols of the second type induced by ∇. Hence C is a null geodesic of M. The converse follows easily. □

4. Natural Frenet Frames of Type 1

Let F and $\widetilde{F}$ be two Frenet frames along neighborhoods $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ respectively with non-null intersection. From Theorem 3, F and $\widetilde{F}$ are both Type 1 or both Type 2. In this section we assume that F and $\widetilde{F}$ are both Type 1.

Theorem 5. *Let C be a non-geodesic Type 1 null curve of M_2. Then there exists a screen vector bundle $\bar{S}(C)$ which induces a Frenet frame $\bar{F}$ on $\bar{\mathcal{U}}$ such that $\bar{\kappa}_4 = \bar{\kappa}_5 = 0$ on $\bar{\mathcal{U}}$.*

Proof. From the first equation of the general Frenet equations (4), we have

$$\bar{\kappa}_1 B_1^1 = \kappa_1 \left(\frac{dt}{d\bar{t}}\right)^2; \qquad \bar{\kappa}_1 B_1^\alpha = 0, \quad 2 \le \alpha \le m.$$

Since $\kappa_1 \neq 0$ on $\mathcal{U} \cap \bar{\mathcal{U}}$, we get $\bar{\kappa}_1 \neq 0$ and $B_1^\alpha = 0$ $(2 \le \alpha \le m)$. From the fact that $[B_\alpha^\beta(x)]$ is a Lorentz matrix, we have $B_1^1 = B_1 = \pm 1$ and $B_\alpha^1 = 0$ $(2 \le \alpha \le m)$. Also, from the third equation of the general Frenet equations (4), we have

$$\begin{aligned}
\bar{\kappa}_4 B_2^2 + \bar{\kappa}_5 B_3^2 &= B_1^1 \left(\kappa_4 + \kappa_1 c_2 \frac{dt}{d\bar{t}}\right) \frac{dt}{d\bar{t}}, \\
\bar{\kappa}_4 B_2^3 + \bar{\kappa}_5 B_3^3 &= B_1^1 \left(\kappa_5 + \kappa_1 c_3 \frac{dt}{d\bar{t}}\right) \frac{dt}{d\bar{t}}, \\
\bar{\kappa}_4 B_2^\alpha + \bar{\kappa}_5 B_3^\alpha &= B_1^1 \kappa_1 c_\alpha \left(\frac{dt}{d\bar{t}}\right)^2, \quad 4 \le \alpha \le m.
\end{aligned}$$

Taking into account that

$$c_2 = -\frac{\kappa_4}{\kappa_1}\frac{d\bar{t}}{dt}; \quad c_3 = -\frac{\kappa_5}{\kappa_1}\frac{d\bar{t}}{dt}; \quad c_\alpha = 0, \quad 4 \le \alpha \le m$$

in the last equations, we have $\bar{\kappa}_4 = \bar{\kappa}_5 = 0$. □

Relabeling $N = \bar{N}$, $W_1 = \bar{W}_1, W_2 = \bar{W}_2$, $\kappa_i = \bar{\kappa}_i$, $i \in \{1, 2, 3\}$ and $S(C) = \bar{S}(C)$ in the process of the above theorem and we take only the first four equations in (4)

$$\begin{aligned}
\nabla_\lambda \lambda &= h\,\lambda + \kappa_1 W_1, \\
\nabla_\lambda N &= -\,h\,N + \kappa_2 W_1 + \kappa_3 W_2, \\
\epsilon_1 \nabla_\lambda W_1 &= -\,\kappa_2 \lambda - \kappa_1 N, \\
\epsilon_2 \nabla_\lambda W_2 &= -\,\kappa_3 \lambda + R_1,
\end{aligned}$$

where $R_1 \in S(C)$. Since C is Type 1, R_1 is a non-null vector field perpendicular to λ, N, W_1 and W_2. Denote $\rho_3 = \| R_1 \|$ and define the new fourth curvature function κ_4 by $\kappa_4 = \epsilon_3 \rho_3$, where ϵ_3 is the sign of R_1. If ρ_3 is also non-zero for any t, we set new vector field $\bar{W}_3 = \frac{R_1}{\rho_3}$, then $\bar{W}_3$ is a non-null unit vector field with the same causality as R_1 along C. Thus we have

$$\nabla_\lambda W_2 = -\,\kappa_3 \lambda + \kappa_4 \epsilon_3 \bar{W}_3.$$

Repeating above process for all the orthonormal basis $\{W_1, W_2, \bar{W}_3, \dots, \bar{W}_m\}$ of $S(C)$ and then, setting $W_i = \epsilon_i \bar{W}_i$; $3 \le i \le m$, we obtain the following;

$$
(13)\qquad
\begin{aligned}
\nabla_\lambda \lambda &= h\lambda + \kappa_1 W_1,\\
\nabla_\lambda N &= -hN + \kappa_2 W_1 + \kappa_3 W_2,\\
\epsilon_1 \nabla_\lambda W_1 &= -\kappa_2 \lambda - \kappa_1 N,\\
\epsilon_2 \nabla_\lambda W_2 &= -\kappa_3 \lambda + \kappa_4 W_3,\\
\epsilon_2 \nabla_\lambda W_3 &= -\kappa_4 W_2 + \kappa_5 W_4,\\
&\cdots\cdots\cdots\cdots\\
\epsilon_i \nabla_\lambda W_i &= -\kappa_{i+1} W_{i-1} + \kappa_{i+2} W_{i+1}, \quad 3 \le i \le m-1,\\
\epsilon_m \nabla_\lambda W_m &= -\kappa_{m+1} U_{m-1}.
\end{aligned}
$$

We call the frame $F_1 = \{\lambda, N, W_1, \dots, W_m\}$ a *Natural Frenet frame of Type* 1 on M_2 along C with respect to the given screen vector bundle $S(C)$ and the equations (13) is called its *Natural Frenet equations of Type* 1. Finally, the functions $\{\kappa_1, \dots, \kappa_{m+1}\}$ are called *curvature functions* of C with respect to the frame F_1.

Let $C(p)$ be a Type 1 null curve of M_2, parameterized by the distinguished parameter p. Then, from the equations (13), there exists a Natural Frenet frame $F_1 = \{\xi, N, W_1, \dots, W_m\}$ satisfying the Natural Frenet equations

$$
\begin{aligned}
\nabla_\xi \xi &= \kappa_1 W_1,\\
\nabla_\xi N &= \kappa_2 W_1 + \kappa_3 W_2,\\
\epsilon_1 \nabla_\xi W_1 &= -\kappa_2 \xi - \kappa_1 N,\\
\epsilon_2 \nabla_\xi W_2 &= -\kappa_3 \xi + \kappa_4 W_3,\\
\epsilon_2 \nabla_\xi W_3 &= -\kappa_4 W_2 + \kappa_5 W_4,\\
&\cdots\cdots\cdots\cdots\\
\epsilon_i \nabla_\xi W_i &= -\kappa_{i+1} W_{i-1} + \kappa_{i+2} W_{i+1}, \quad 3 \le i \le m-1,\\
\epsilon_m \nabla_\xi W_m &= -\kappa_{m+1} U_{m-1}.
\end{aligned}
$$

Theorem 6. *Let $C(p)$ be a non-geodesic Type* 1 *null curve of M_2, parameterized by the distinguished parameter p. Then there exists a Frenet frame $\mathcal{F}_1 = \{\xi, N, W_1, \dots, W_m\}$ satisfying the Frenet equations*

$$
\begin{aligned}
\nabla_\xi \xi &= W_1,\\
\nabla_\xi N &= \kappa_1 W_1 + \kappa_2 W_2,
\end{aligned}
$$

$$
(14)\qquad
\begin{aligned}
\epsilon_1 \nabla_\xi W_1 &= -\kappa_1 \xi - N,\\
\epsilon_2 \nabla_\xi W_2 &= -\kappa_2 \xi + \kappa_3 W_3,\\
\epsilon_2 \nabla_\xi W_3 &= -\kappa_3 W_2 + \kappa_4 W_4,\\
&\cdots\cdots\cdots\cdots\\
\epsilon_i \nabla_\xi W_i &= -\kappa_i W_{i-1} + \kappa_{i+1} W_{i+1}, \quad 3 \le i \le m-1,\\
\epsilon_m \nabla_\xi W_m &= -\kappa_m W_{m-1}.
\end{aligned}
$$

Proof. For a Type 1 non-geodesic null curve, since the vector field C'' is non-null, without any loss of generality, we assume that C is parameterized by a special parameter p such that $h = 0$ and $g(C'', C'') = \epsilon_1$, where ϵ_1 is the sign of C''. Choose $\xi = C'$ and $W_1 = C''$ so that $\nabla_\xi \xi = W_1$. Since $g(C^{(3)}, C') = -\epsilon_1$, the null transversal vector field N is given by

$$
N = -\epsilon_1 C^{(3)} - \kappa_1 C', \qquad \text{where} \qquad \kappa_1 = \frac{1}{2} g(C^{(3)}, C^{(3)}).
$$

Thus we have $\epsilon_1 \nabla_\xi W_1 = -\kappa_1 \xi - N$. From $g(\nabla_\xi N, \xi) = 0$ and $g(\nabla_\xi N, W_1) = \epsilon_1 \kappa_1$, we have $\nabla_\xi N = \kappa_1 W_1 + R_2$. If R_2 is non-null, we let $\rho_1 = \| R_2 \|$ and $\bar W_2 = \frac{R_2}{\rho_1}$. Define the second curvature function κ_2 by $\kappa_2 \epsilon_2 = \rho_1$, where ϵ_2 is the sign of R_2. Then we have $\nabla_\xi N = \kappa_1 W_1 + \kappa_2 \epsilon_2 \bar W_2$, Also, from $g(\nabla_\xi \bar W_2, \xi) = g(\nabla_\xi \bar W_2, W_1) = 0$, $g(\nabla_\xi \bar W_2, N) = -\kappa_2$, we have $\nabla_\xi \bar W_2 = -\kappa_2 \xi + R_3$. If R_3 is also non-null, we let $\rho_2 = \| R_3 \|$ and $\bar W_3 = \frac{R_3}{\rho_2}$, Define the third curvature function κ_3 by $\kappa_3 \epsilon_3 = \rho_2$, where ϵ_3 is the sign of R_3. Then we have $\nabla_\xi \bar W_2 = -\kappa_2 \xi - \kappa_3 \bar W_3$. Repeating above process for the orthonormal basis $\{W_1, \bar W_2, \ldots, \bar W_m\}$ of $S(C)$ and then setting $W_i = \epsilon_i \bar W_i$, $2 \le i \le m$, we obtain the equations (14). □

We call the frame $\mathcal{F}_1 = \{\xi, N, W_1, \ldots, W_m\}$ the *Cartan frame of Type 1* on M_2 along C, the equations (14) is called its *Cartan equations of Type 1* and the functions $\{\kappa_1, \ldots, \kappa_m\}$ are called *Cartan curvatures* of C with respect to the frame $\mathcal{F}_1$. Finally, we call the curve C with the Cartan frame of Type 1 the *Cartan null curve of Type 1.*

5. NATURAL FRENET FRAMES OF TYPE 2

In this section we assume that F and $\widetilde{F}$ are both Type 2. Now we introduce another form of Franet equations of Type 2. First, we assume that $\tau_1 \neq 0$, i.e., C''

is a null vector field, then, from the equation (12), we have

$$P_1^1 + P_2^1 = P_1^2 + P_2^2, \qquad P_1^\alpha = -P_2^\alpha \quad (3 \le \alpha \le m), \tag{15}$$

because $\widetilde{\tau}_1 = \widetilde{\kappa}_1$ and $\tau_1 = \kappa_1$. Using (10), (11), (15) and $L_1 = \frac{1}{\sqrt{2}}(W_1 + W_2)$, we obtain

$$L_1^* = \mathcal{A}_1 L_1, \qquad \bar{L}_1 = \mathcal{B}_1 \left\{ L_1 - \mathcal{C}_2 \frac{dt}{d\bar{t}} \lambda \right\}, \tag{16}$$

where $\mathcal{A}_1 = A_1^1 + A_2^1 \neq 0$ and $\mathcal{B}_1 = B_1^1 + B_2^1 \neq 0$, otherwise the matrix $[P_\alpha^\beta(x)]$ is singular and $\mathcal{C}_2 = \frac{1}{\sqrt{2}}(c_2 - c_1)$. Since the vector fields $\widetilde{W}_1$ and $\widetilde{W}_2$ are timelike and spacelike respectively, the first row $(P_1^1, \ldots, P_1^m)$ and the second row $(P_2^1, \ldots, P_2^m)$ of $[P_\alpha^\beta(x)]$ are mutually orthogonal timelike and spacelike vector fields in $\mathbf{R}_1^m$ respectively. Thus, we have

$$(P_1^1)^2 - (P_1^2)^2 - 1 = (P_2^1)^2 - (P_2^2)^2 + 1 = -P_1^1 P_2^1 + P_1^2 P_2^2.$$

From this relations we have the following two relations

$$P_1^1 P_2^2 - P_1^2 P_2^1 = 1, \qquad (P_1^1 + P_2^1)(P_1^1 - P_1^2) = 1. \tag{17}$$

Using (6), (10), (11) and (15) for $\alpha \in \{3, \ldots, m\}$, we obtain

$$\widetilde{W}_\alpha = \frac{1}{\sqrt{2}}(P_\alpha^2 + P_\alpha^1)\, L_1 + \frac{1}{\sqrt{2}}(P_\alpha^2 - P_\alpha^1)\, L_2 + \sum_{\beta \ge 3}^{m} P_\alpha^\beta W_\beta - O_1(\widetilde{\lambda}),$$

where $O_1(\lambda^*) = 0$ and $O_1(\bar{\lambda}) = \sum_{i \ge 1}^m B_\alpha^i \epsilon_i c_i \bar{\lambda}$. From $g(\widetilde{L}_1, \widetilde{W}_\alpha) = 0$, we have

$$P_\alpha^1 = P_\alpha^2 \quad (3 \le \alpha \le m). \tag{18}$$

Also, using (6), (10), (11) and (15) for $\alpha \in \{3, \ldots, m\}$, we obtain

$$\widetilde{L}_2 = \frac{1}{2}(P_2^1 - P_1^1 + P_2^2 - P_1^2)\, L_1 + (P_1^1 - P_1^2)\, L_2 - \sqrt{2} \sum_{\alpha \ge 3}^{m} P_1^\alpha W_\alpha - O_2(\widetilde{\lambda}),$$

where $O_2(\lambda^*) = 0$ and $O_2(\bar{\lambda}) = \frac{1}{\sqrt{2}} \sum_{i \ge 1}^m (B_2^i - B_1^i)\, \epsilon_i c_i \bar{\lambda}$. Since $\widetilde{L}_2$ is null, we have

$$(P_1^1 - P_1^2)(P_2^1 - P_1^1 + P_2^2 - P_1^2) + 2 \sum_{\alpha \ge 3}^{m} \epsilon_\alpha (P_1^\alpha)^2 = 0.$$

Using this equation and the first equation of (15), we obtain

$$P_1^1 = P_2^2;\ P_1^2 = P_2^1 \iff P_2^1 - P_1^1 + P_2^2 - P_1^2 = 0 \iff \sum_{\alpha \ge 3}^{m} \epsilon_\alpha (P_1^\alpha)^2 = 0.$$

As a consequence of the fundamental transformation of matrices with respect to the rows and columns for the matrix $[P^{\alpha}_{\beta}(x)]$ and using the equations (15), (17) and (18), the matrix $[P^{\alpha}_{\beta}(x)]$ is transformed as

$$\mathcal{T}_1(C) \equiv \begin{pmatrix} 0 & P^1_1 + P^1_2 & 0 & \cdots & 0 \\ -P^1_1 + P^2_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & P^3_3 & \cdots & P^m_3 \\ 0 & 0 & P^3_4 & \cdots & P^m_4 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & P^3_m & \cdots & P^m_m \end{pmatrix}. \tag{19}$$

Thus the screen vector bundle $S(C)$ is a orthogonal direct sum of two invariant subspaces $Span\{L_1, L_2\} = Span\{W_1, W_2\}$ and $Span\{W_3, \ldots, W_m\}$ by the parameter and screen transformation of C.

Proposition 7. *Let C be a non-geodesic Type 2 null curve of M_2. If C'' is a null vector field, then the screen vector bundle $S(C)$ is a orthogonal direct sum of two invariant subspaces $Span\{L_1, L_2\} = Span\{W_1, W_2\}$ and $Span\{W_3, \ldots, W_m\}$ by the parameter and screen transformations of C.*

Using the transformations (19) that relate elements of F and $\widetilde{F}$ on $\mathcal{U} \cap \widetilde{\mathcal{U}}$ such that $\tau_1 \neq 0$, we have (16) and

$$\begin{aligned} L^*_2 &= \mathcal{A}_2 L_2, \qquad \bar{L}_2 = \mathcal{B}_2 \left\{ L_2 - \mathcal{C}_1 \frac{dt}{d\bar{t}} \lambda \right\}, \\ \bar{N} &= -\frac{dt}{d\bar{t}} \sum_{i\geq 1}^{m} \epsilon_i (c_i)^2 \lambda + \frac{d\bar{t}}{dt} N + \mathcal{C}_1 L_1 + \mathcal{C}_2 L_2 + \sum_{\alpha \geq 3}^{m} c_\alpha W_\alpha, \end{aligned} \tag{20}$$

where $\mathcal{A}_2 = A^1_1 - A^2_1$, $\mathcal{B}_2 = B^1_1 - B^2_1$, $\mathcal{C}_1 = \frac{1}{\sqrt{2}}(c_1 + c_2)$ and $\mathcal{A}_1\mathcal{A}_2 = \mathcal{B}_1\mathcal{B}_2 = 1$.

Theorem 8. *Let C be a non-geodesic Type 2 null curve of M_2. If C'' is a null vector field, then there exists a screen vector bundle $\bar{S}(C)$ which induces another Frenet frame $\bar{F}$ on $\bar{\mathcal{U}}$ such that $\bar{\sigma}_4 = \bar{\sigma}_7 = \bar{\sigma}_8 = \bar{\mu}_4 = \bar{\kappa}_4 = \bar{\tau}_4 = 0$ on $\bar{\mathcal{U}}$.*

Proof. From the fourth equation of the general Frenet equations (7), we have

$$\bar{\sigma}_4 = \left\{ \sigma_4 + \sigma_1 \mathcal{C}_2 \frac{dt}{d\bar{t}} - \mathcal{B}_2 \frac{d\mathcal{B}_1}{dt} \right\} \frac{dt}{d\bar{t}},$$

$$\bar{\sigma}_7 B_3^3 + \bar{\sigma}_8 B_4^3 + \bar{\mu}_4 B_5^3 = \mathcal{B}_2\left(\sigma_7 + \sigma_1 c_3 \frac{dt}{d\bar{t}}\right)\frac{dt}{d\bar{t}},$$
$$\bar{\sigma}_7 B_3^4 + \bar{\sigma}_8 B_4^4 + \bar{\mu}_4 B_5^4 = \mathcal{B}_2\left(\sigma_8 + \sigma_1 c_4 \frac{dt}{d\bar{t}}\right)\frac{dt}{d\bar{t}},$$
$$\bar{\sigma}_7 B_3^5 + \bar{\sigma}_8 B_4^5 + \bar{\mu}_4 B_5^5 = \mathcal{B}_2\left(\mu_4 + \sigma_1 c_5 \frac{dt}{d\bar{t}}\right)\frac{dt}{d\bar{t}},$$
$$\bar{\sigma}_7 B_3^\alpha + \bar{\sigma}_8 B_4^\alpha + \bar{\mu}_4 B_5^\alpha = \sigma_1 c_\alpha \mathcal{B}_2\left(\frac{dt}{d\bar{t}}\right)^2.$$

Taking into account that $\mathcal{B}_1$ is a constant and

$$\mathcal{C}_2 = -\frac{\sigma_4}{\sigma_1}\frac{d\bar{t}}{dt}; \quad c_3 = -\frac{\sigma_7}{\sigma_1}\frac{d\bar{t}}{dt}; \quad c_4 = -\frac{\sigma_8}{\sigma_1}\frac{d\bar{t}}{dt}; \quad c_5 = -\frac{\mu_4}{\sigma_1}\frac{d\bar{t}}{dt};$$
$$c_\alpha = 0, \ \alpha \in \{6, \cdots, m\}$$

in the last equations, we have $\bar{\sigma}_4 = \bar{\sigma}_7 = \bar{\sigma}_8 = \bar{\mu}_4 = 0$. Since (5), (6) and (7) imply $\bar{\kappa}_4 = -\bar{\sigma}_4$ and $\bar{\tau}_4 = \sqrt{2}\bar{\mu}_4$, we have $\bar{\kappa}_4 = \bar{\tau}_4 = 0$. □

Relabeling $N = \bar{N}$; $L_\alpha = \bar{L}_\alpha$, $\alpha \in \{1, 2\}$; $W_3 = \bar{W}_3$; $\sigma_4 = \bar{\sigma}_5$, $\sigma_5 = \bar{\sigma}_6$; $\kappa_5 = \bar{\kappa}_8$ and $S(C) = \bar{S}(TC^\perp)$ and we take only the first five equations in (7) as follows:

$$\nabla_\lambda \lambda = h\,\lambda + \sigma_1 L_1,$$
$$\nabla_\lambda N = -h\,N + \sigma_2 L_1 + \sigma_3 L_2 + \tau_2 W_3,$$
$$\nabla_\lambda L_1 = -\sigma_3 \lambda + \sigma_4 W_3 + \sigma_5 W_4,$$
$$\nabla_\lambda L_2 = -\sigma_2 \lambda - \sigma_1 N,$$
$$\nabla_\lambda W_3 = -\tau_2 \lambda - \sigma_4 L_2 + \kappa_5 W_4 + R_4,$$

where $R_4 \in S(C)$ is perpendicular to L_1, L_2, W_3 and W_4. Since the vector field R_4 is a spacelike. Now, we define the new third torsion function τ_3 by $\tau_3 = \| R_4 \|$ and $W_5 = \frac{R_4}{\tau_3}$, then W_5 is also a unit spacelike vector field along C. We have

$$\nabla_\lambda W_3 = -\tau_2 \lambda + \sigma_4 L_2 + \kappa_5 W_4 + \tau_3 W_5.$$

Repeating above process for the quasi-orthonormal basis

$$F_2 = \{\lambda, N, L_1, L_2, W_3, \cdots, W_m\}$$

of $S(C)$, we obtain the following;

$$\nabla_\lambda \lambda = h\,\lambda + \sigma_1 L_1,$$
$$\nabla_\lambda N = -h\,N + \sigma_2 L_1 + \sigma_3 L_2 + \tau_2 W_3,$$
$$\nabla_\lambda L_1 = -\sigma_3 \lambda + \sigma_4 W_3 + \sigma_5 W_4,$$

$$\begin{aligned}
&\nabla_\lambda L_2 = -\sigma_2\,\lambda - \sigma_1\,N,\\
(21)\quad &\nabla_\lambda\, W_3 = -\tau_2\,\lambda - \sigma_4\,L_2 + \kappa_5\,W_4 + \tau_4\,W_5,\\
&\nabla_\lambda\, W_4 = -\sigma_5\,L_2 - \kappa_5\,W_3 + \kappa_6\,W_5 + \tau_5\,W_6,\\
&\nabla_\lambda W_5 = -\tau_4 W_3 - \kappa_6\,W_4 + \kappa_7\,W_6 + \tau_6\,W_7,\\
&\qquad\qquad \cdots\cdots\cdots\cdots\cdots\cdots\\
&\nabla_\lambda W_i = -\tau_{i-1}W_{i-2} - \kappa_{i+1}W_{i-1} + \kappa_{i+2}W_{i+1} + \tau_{i+1}W_{i+2},\ 5 \le i \le m-1,\\
&\nabla_\lambda W_m = -\tau_{m-1}W_{m-2} - \kappa_{m+1}W_{m-1}.
\end{aligned}$$

We call the frame F_2 a *Natural Frenet frame of Type* 2 on M along C with respect to the given screen vector bundle $S(C)$ and the equations (21) are called its *Natural Frenet equations of Type* 2. Finally, the functions $\{\kappa_1, \ldots, \kappa_{m+1}\}$ and $\{\tau_1, \ldots, \tau_{m-1}\}$ are called the *curvature* and *torsion functions* of C with respect to the frame F_2.

Theorem 9. *Let $C(p)$ be a non-geodesic Type 2 null curve of M_2, parameterized by the distinguished parameter p such that C'' is a null. Then there exists a Frenet frame $\mathcal{F}_2 = \{\xi, N, L_1, L_2, W_3, \ldots, W_m\}$ satisfying the equations*

$$\begin{aligned}
&\nabla_\xi \xi = L_1,\\
&\nabla_\xi N = \sigma_1\,L_1 + \sigma_2\,L_2 + \tau_1\,W_3,\\
&\nabla_\xi L_1 = -\sigma_2\,\xi + \sigma_3\,W_3 + \sigma_4\,W_4,\\
&\nabla_\xi L_2 = -\sigma_1\,\xi - N,\\
(22)\quad &\nabla_\xi\, W_3 = -\tau_1\,\xi - \sigma_3\,L_2 + \kappa_4\,W_4 + \tau_2\,W_5,\\
&\nabla_\xi\, W_4 = -\sigma_4\,L_2 - \kappa_4\,W_3 + \kappa_5\,W_5 + \tau_3\,W_6,\\
&\nabla_\xi W_5 = -\tau_2 W_3 - \kappa_5\,W_4 + \kappa_6\,W_6 + \tau_4\,W_7,\\
&\qquad\qquad \cdots\cdots\cdots\cdots\cdots\cdots\\
&\nabla_\xi W_i = -\tau_{i-3}W_{i-2} - \kappa_i W_{i-1} + \kappa_{i+1}W_{i+1} + \tau_{i-1}W_{i+2},\ 5 \le i \le m-1,\\
&\nabla_\xi W_m = -\tau_{m-3}W_{m-2} - \kappa_m W_{m-1}.
\end{aligned}$$

Proof. Since C'' is a null vector field, choose $\xi = C'$ and $L_1 = C''$ so that $\sigma_1 = 1$. A direct computation and renaming the curvature functions $\{\sigma_i = \sigma_{i+1}\,(1 \le i \le 4)\,;\ \kappa_i = \kappa_{i+1}\,(4 \le i \le m)\,;\ \tau_1 = \tau_2$ and $\tau_i = \tau_{i+2}\,(2 \le i \le m-3)\}$, we obtain the equations (22). □

We call the frame $\mathcal{F}_2 = \{\xi, N, L_1, L_2, W_3, \ldots, W_m\}$ the *Cartan frame of Type 2* on M_2 along C, the equations (22) its *Cartan equations of Type 2* and the functions $\{\kappa_1, \ldots, \kappa_m\}$ and $\{\tau_1, \cdots, \tau_{m-2}\}$ the *Cartan curvature* and *Cartan torsion* of C with respect to the frame $\mathcal{F}_2$. Finally, we call the curve C with the Cartan frame of Type 2 the *Cartan null curve of Type* 2.

Next, using the transformations (19) such that $\tau_1 = 0$, i.e., C'' is a non-null vector field, we have

Theorem 10. *Let C be a non-geodesic Type 2 null curve of M_2. If C'' is a non-null vector field, then there exists a screen vector bundle $\bar{S}(C)$ which induces another Frenet frame $\bar{F}_2$ on $\bar{\mathcal{U}}$ such that $\bar{\kappa}_4 = \bar{\kappa}_5 = \bar{\tau}_3 = 0$ on $\bar{\mathcal{U}}$.*

Proof. Since $\kappa_1 \neq 0$, we have $B_1^\alpha = B_\alpha^1 = 0\,(\alpha \neq 1)$ and $B_1^1 = B_1 = \pm\, 1$. Also, using (11) and the third equation of the general Frenet equations (9), we have

$$\begin{aligned}
\bar{\kappa}_4\, B_2^2 + \bar{\kappa}_5\, B_3^2 + \bar{\tau}_3\, B_4^2 &= B_1\left(\kappa_4 + \kappa_1 c_2 \frac{dt}{d\bar{t}}\right)\frac{dt}{d\bar{t}}, \\
\bar{\kappa}_4\, B_2^3 + \bar{\kappa}_5\, B_3^3 + \bar{\tau}_3\, B_4^3 &= B_1\left(\kappa_5 + \kappa_1 c_3 \frac{dt}{d\bar{t}}\right)\frac{dt}{d\bar{t}}, \\
\bar{\kappa}_4\, B_2^4 + \bar{\kappa}_5\, B_3^4 + \bar{\tau}_3\, B_4^4 &= B_1\left(\tau_3 + \kappa_1 c_4 \frac{dt}{d\bar{t}}\right)\frac{dt}{d\bar{t}}, \\
\bar{\kappa}_4\, B_2^\alpha + \bar{\kappa}_5\, B_3^\alpha + \bar{\tau}_3\, B_4^\alpha &= B_1\, \kappa_1 c_\alpha \left(\frac{dt}{d\bar{t}}\right)^2, \quad 5 \leq \alpha \leq m.
\end{aligned}$$

Taking into account that

$$c_2 = -\frac{\kappa_4}{\kappa_1}\frac{d\bar{t}}{dt}; \quad c_3 = -\frac{\kappa_5}{\kappa_1}\frac{d\bar{t}}{dt}; \quad c_4 = -\frac{\tau_3}{\kappa_1}\frac{d\bar{t}}{dt}; \quad c_\alpha = 0,\ 5 \leq \alpha \leq m$$

in the last equations, we have $\bar{\kappa}_4 = \bar{\kappa}_5 = \bar{\tau}_3 = 0$. □

Relabeling $N = \bar{N}$, $W_1 = \bar{W}_1$, $W_2 = \bar{W}_2$, $\kappa_i = \bar{\kappa}_i$, $i \in \{1, 2, 3\}$, $\tau_2 = \bar{\tau}_2$, $\kappa_4 = \bar{\kappa}_6$ and $S(C) = \bar{S}(TC^\perp)$ in the process of the above theorem and we take only the first four equations in (9) with $\tau_1 = 0$ as follows:

$$\begin{aligned}
\nabla_\lambda \lambda &= h\,\lambda + \kappa_1\, W_1, \\
\nabla_\lambda N &= -\,h\,N + \kappa_2\, W_1 + \kappa_3\, W_2 + \tau_2\, W_3, \\
\epsilon_1\, \nabla_\lambda W_1 &= -\,\kappa_2\, \lambda - \kappa_1\, N, \\
\epsilon_2\, \nabla_\lambda W_2 &= -\,\kappa_3\, \lambda + \kappa_4\, W_3 + R_5,
\end{aligned}$$

where $R_5 \in S(C)$. Assume that $\tau_2 \neq 0$, then W_2 is a timelike and W_3 is a spacelike vector field. Thus R_5 is a spacelike. Define the new torsion function τ_3 by $\tau_3 = \| R_5 \|$ and $W_4 = \frac{R_5}{\tau_3}$. Then we have $\nabla_\lambda W_2 = \kappa_3 \lambda - \kappa_4 W_4 - \tau_3 W_4$. Assume that $\tau_2 = 0$, then we have

$$\begin{aligned}
\nabla_\lambda \lambda &= h\,\lambda + \kappa_1 W_1, \\
\nabla_\lambda N &= -\,h\,N + \kappa_2 W_1 + \kappa_3 W_2, \\
\epsilon_1 \nabla_\lambda W_1 &= -\,\kappa_2 \lambda - \kappa_1 N, \\
\epsilon_2 \nabla_\lambda W_2 &= -\,\kappa_3 \lambda + R_6.
\end{aligned}$$

We assume that R_6 is null. Let $R_6 = \kappa_4 W_3 + \tau_3 W_4$, where $\kappa_4 = \tau_3 = \frac{1}{\sqrt{2}}$ and W_3 is a timelike and W_4 is a spacelike. Then we have $\nabla_\lambda W_2 = -\,\kappa_3 \lambda + \kappa_4 W_4 + \tau_3 W_4$. Repeating above process for all the orthonormal basis $\{W_1, \ldots, W_m\}$ of $S(C)$, we obtain the following;

$$\begin{aligned}
\nabla_\lambda \lambda &= h\lambda + \kappa_1 W_1, \\
\nabla_\lambda N &= -\,hN + \kappa_2 W_1 + \kappa_3 W_2 + \tau_2 W_3, \\
\epsilon_1 \nabla_\lambda W_1 &= -\,\kappa_2 \lambda - \kappa_1 N, \\
\epsilon_2 \nabla_\lambda W_2 &= -\,\kappa_3 \lambda + \kappa_4 W_3 + \tau_3 W_4, \\
\epsilon_3 \nabla_\lambda W_3 &= -\tau_2 \lambda - \kappa_4 W_2 + \kappa_5 W_4 + \tau_4 W_5, \\
\epsilon_4 \nabla_\lambda W_4 &= -\,\tau_3 W_2 - \kappa_5 W_3 + \kappa_6 W_5 + \tau_5 W_6, \qquad (23)\\
\epsilon_5 \nabla_\lambda W_5 &= -\,\tau_4 W_3 - \kappa_6 W_4 + \kappa_7 W_6 + \tau_6 W_7, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\epsilon_i \nabla_\lambda W_i &= -\,\tau_{i-1} W_{i-2} - \kappa_{i+1} W_{i-1} + \kappa_{i+2} W_{i+1} + \tau_{i+1} W_{i+2},\ 3 \le i \le m-1, \\
\epsilon_m \nabla_\lambda W_m &= -\,\tau_{m-1} W_{m-2} - \kappa_{m+1} W_{m-1}.
\end{aligned}$$

We call the frame $F_2 = \{\lambda, N, W_1, \ldots, W_m\}$ a *Natural Frenet frame of Type* 2 on M along C with respect to the given screen vector bundle $S(C)$ and the equations (23) are called its *Natural Frenet equations of Type* 2. Finally, the functions $\{\kappa_1, \ldots, \kappa_{m+1}\}$ and $\{\tau_2, \cdots, \tau_{m-1}\}$ are called *curvature* and *torsion functions* of C with respect to the frame F_2.

Theorem 11. *Let $C(p)$ be a non-geodesic Type 2 null curve of M_2, parameterized by the distinguished parameter p such that C'' is non-null. Then there exists a Frenet frame $\mathcal{F}_2 = \{\xi, N, W_1, \ldots, W_m\}$ satisfying the equations*

$$\begin{aligned}
&\nabla_\xi \xi = W_1, \\
&\nabla_\xi N = \kappa_1 W_1 + \kappa_2 W_2 + \tau_1 W_3, \\
&\epsilon_1 \nabla_\xi W_1 = -\kappa_1 \xi - N, \\
&\epsilon_2 \nabla_\xi W_2 = -\kappa_2 \xi + \kappa_3 W_3 + \tau_2 W_4, \\
&\epsilon_3 \nabla_\xi W_3 = -\tau_1 \xi - \kappa_3 W_2 + \kappa_4 W_4 + \tau_3 W_5, \\
(24) \qquad &\epsilon_4 \nabla_\xi W_4 = -\tau_2 W_2 - \kappa_4 W_3 + \kappa_5 W_5 + \tau_4 W_6, \\
&\epsilon_5 \nabla_\xi W_5 = -\tau_3 W_3 - \kappa_5 W_4 + \kappa_6 W_6 + \tau_5 W_7, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&\epsilon_i \nabla_\xi W_i = -\tau_{i-2} W_{i-2} - \kappa_i W_{i-1} + \kappa_{i+1} W_{i+1} + \tau_i W_{i+2}, \; 3 \le i \le m-1, \\
&\epsilon_m \nabla_\xi W_m = -\tau_{m-2} W_{m-2} - \kappa_m W_{m-1}.
\end{aligned}$$

Proof. Since the vector field C'' is a non-null and one of the vector fields $\nabla_\lambda N - \kappa_1 W_1$, $\epsilon_2 \nabla_\lambda W_2 + \kappa_2 \lambda$ and $\epsilon_i \nabla_\lambda W_i + \kappa_i W_{i-1}$ $(3 \le i \le m-1)$ be null, without any loss of generality, we assume that C is parameterized by a special parameter p such that $h = 0$ and $g(C'', C'') = \epsilon_1$, where ϵ_1 is the sign of C''. Choose $\xi = C'$ and $W_1 = C''$ so that $\nabla_\xi \xi = W_1$. Also, we have

$$N = -\epsilon_1 C^{(3)} - \kappa_1 C', \quad \text{where } \kappa_1 = \frac{1}{2} g(C^{(3)}, C^{(3)}).$$

Thus we have $\epsilon_1 \nabla_\xi W_1 = -\kappa_1 \xi - N$. From $g(\nabla_\xi N, \xi) = 0$ and $g(\nabla_\xi N, W_1) = \epsilon_1 \kappa_1$, we have $\nabla_\xi N = \kappa_1 W_1 + R_7$. If $R_7 \equiv L_2$ is a null, by using (5), we have $L_2 = \kappa_2 W_2 + \tau_1 W_3$ and $\kappa_2 = \tau_1 = \frac{1}{\sqrt{2}}$. Thus, we have $\nabla_\xi N = \kappa_1 W_1 + \kappa_2 W_2 + \tau_1 W_3$. Also, from $g(\nabla_\xi W_2, \xi) = 0$, $g(\nabla_\xi W_2, N) = \kappa_2$, $g(\nabla_\xi W_2, W_1) = 0$, $g(\nabla_\xi W_2, W_3) \equiv -\kappa_3$, we have $\nabla_\xi W_2 = \kappa_2 \xi - \kappa_3 W_3 + R_8$. Since R_8 is a spacelike vector field, we let $-\tau_2 = \| R_8 \|$ and $W_4 = R_8 / -\tau_2$, we have $\nabla_\xi W_2 = \kappa_2 \xi - \kappa_3 W_3 - \tau_2 W_4$. Repeating above process, we obtain the equations (24). □

We call the frame $F_2 = \{\xi, N, W_1, \ldots, W_m\}$ the *Cartan frame of Type* 2 on M_2 along C, the equations (24) its *Cartan equations of Type* 2 and the functions $\{\kappa_1, \ldots, \kappa_m\}$ the *Cartan curvature* of C with respect to the frame $\mathcal{F}_2$. Finally, we call the curve C with the Cartan frame of Type 2 the *Cartan null curve of Type* 2.

Note. If C'' and $\nabla_\lambda N - \kappa_1 W_1$ are non-null vector fields and $\epsilon_2 \nabla_\lambda W_2 + \kappa_2 \lambda \equiv L_3$ is null, then we also have the equations (24) such that $\tau_1 = 0$, $\tau_2 = \kappa_3$ and all other torsion functions τ_i $(3 \le i \le m-1)$ are survive. Also, if C'', $\nabla_\lambda N - \kappa_1 W_1$ and

$\epsilon_2 \nabla_\lambda W_2 + \kappa_3 \lambda$ are non-null and $\epsilon_i \nabla_\lambda W_i + \kappa_{i+1} W_{i-1}$ $(3 \le i \le m-1)$ is null, we also have the equations (24) such that $\tau_1 = \cdots = \tau_{i-1} = 0$, $\tau_i = \kappa_{i+1}$ $(3 \le i \le m-1)$ and all other torsion functions τ_j $(i+1 \le j \le m-1)$ are survive.

6. Transformations of Curvature Functions

In this section, we examine the transformations of the curvature functions in the Natural Frenet equations of Type 1 and Type 2 on both the parameter and the screen transformations of C.

Using the Natural Frenet equations (13) of Type 1 and the parameter transformation (10), we have

Proposition 12. *Let C be a Type 1 null curve of $\mathbf{M}_2$ and F and F^* be two Natural Frenet frames related by (10). Suppose $\prod_{i=1}^{m+1} \kappa_i \neq 0$, then we have*

$$
\begin{aligned}
&\kappa_1^* = \kappa_1 A_1 \left(\frac{dt}{dt^*}\right)^2, \\
(25)\qquad &\kappa_2^* = \kappa_2 A_1, \\
&\kappa_3^* = \kappa_3 A_2, \\
&\kappa_\alpha^* = \kappa_\alpha A_{\alpha-1} \frac{dt}{dt^*}, \ 4 \le \alpha \le m+1, \ \text{where } A_\alpha = \pm 1.
\end{aligned}
$$

Hence κ_2 and κ_3 are invariant functions up to a sign, with respect to the parameter transformations on C.

Proof. From the relations (12) we have $\tau_1 = \tau_1^* = 0$. Therefore, $k_1^* \neq 0$ on $\mathcal{U} \cap \mathcal{U}^*$ and $A_1^2 = \ldots = A_1^m = 0$. Since $[\,A_\alpha^\beta(x)\,]$ is a Lorentzian matrix, we infer that $A_1^1 = A_1 = \pm 1$ and $A_2^1 = \ldots = A_m^1 = 0$. Then from the second equation of the Natural Frenet equations (13) of Type 1 with respect to F and F^* and taking into account that $\kappa_3 \neq 0$, we obtain $\kappa_3^* \neq 0$ on $\mathcal{U} \cap \mathcal{U}^*$ which implies $A_2^3 = A_3^2 = \ldots = A_2^m = A_m^2 = 0$ and $A_2^2 = A_2 = \pm 1$. Repeating this process for all other equations, we obtain all the relations in (25), which completes the proof. □

Also, using the Natural Frenet equations (13) of Type 1 and the screen transformations (11), we have

Proposition 13. *Let C be a Type 1 null curve of $\mathbf{M}_2$ and F and $\bar{F}$ be two Natural Frenet frames related by (11). Suppose $\prod_{i=1}^{m+1} \kappa_i \neq 0$, then we have*

$$\bar{\kappa}_1 = \kappa_1 B_1 \left(\frac{dt}{d\bar{t}}\right)^2,$$
$$\bar{\kappa}_2 = \left\{\kappa_2 + \bar{h}\, c_1 + \frac{dc_1}{d\bar{t}} - \frac{1}{2}\kappa_1 c_1^2 \left(\frac{dt}{d\bar{t}}\right)^2\right\} B_1,$$
$$\bar{\kappa}_3 = \kappa_3 B_2,$$
$$\bar{\kappa}_\alpha = \kappa_\alpha B_{\alpha-1}\frac{dt}{d\bar{t}}, \quad 4 \le \alpha \le m+1, \ \text{where } B_\alpha = \pm 1,$$

and $c_2 = \cdots = c_m = 0$. *Hence* κ_3 *is invariant function up to a sign, with respect to the screen transformations of* C.

Proof. From (11) and the first equation of (13), we have the first equation of this proposition and $\bar{\kappa}_1 B_1^\alpha = 0\,(\alpha \neq 1)$ on $\mathcal{U} \cap \bar{\mathcal{U}}$. Thus we have $B_1^\alpha = 0\,(\alpha \neq 1)$. Since $[\,B_\alpha^\beta(x)\,]$ is a Lorentzian matrix, we infer that $B_1^1 = B_1 = \pm 1$ and $B_\alpha^1 = 0\,(\alpha \neq 1)$. While, from (11) and the third equation of (13), we have the first and second equations of this proposition and $\bar{\kappa}_1 c_\alpha = 0\,(\alpha \neq 1)$ on $\mathcal{U} \cap \bar{\mathcal{U}}$. Thus we have $c_\alpha = 0\,(\alpha \neq 1)$. Also, from (11) and the second equation of (13), we have the second equation of proposition and $\bar{\kappa}_3 B_2^2 = \kappa_3$; $\bar{\kappa}_3 B_2^\alpha = 0\,(\alpha \ge 3)$. Thus we have $B_2 = B_2^2 = \pm 1$ and $B_2^\alpha = B_\alpha^2 = 0\,(\alpha \ge 3)$. Repeating this process for all other equations of (13) and set $B_\alpha = B_{\alpha-1}^{\alpha-1} B_\alpha^\alpha\,(\alpha \ge 3)$, we obtain all the relations in proposition, which completes the proof. □

Using the parameter transformations (10) that relate elements of F_2 and F_2^* on $\mathcal{U} \cap \mathcal{U}^*$ such that $\tau_1 \neq 0$, we have

Proposition 14. *Let* C *be a Type 2 null curve of* $\mathbf{M}_2$ *and* F *and* F^* *be two Natural Frenet frames related by* (10). *Suppose* $\prod_{i=1}^{m-1} \tau_i \neq 0$, *then we have*

$$\begin{aligned}
&\sigma_1^* = \sigma_1 A_2 \left(\frac{dt}{dt^*}\right)^2,\\
&\sigma_2^* = \sigma_2 A_2, \quad \sigma_3^* = \sigma_3 A_1, \quad \tau_2^* = \tau_2 A_3,\\
&\sigma_4^* = \sigma_4 A_3 A_1 \frac{dt}{dt^*}, \quad \sigma_5^* = \sigma_5 A_4 A_1 \frac{dt}{dt^*},\\
&\kappa_\alpha^* = \kappa_\alpha A_{\alpha-1}\frac{dt}{dt^*}, \ 5 \le \alpha \le m+1, \ \text{where } A_\alpha = \pm 1.\\
&\tau_\alpha^* = \tau_\alpha A_{\alpha+1}\frac{dt}{dt^*}, \ 4 \le \alpha \le m+1, \ \text{where } A_\alpha = \pm 1.
\end{aligned} \tag{26}$$

Hence σ_2, σ_3 *and* τ_2 *are invariant functions up to a sign, with respect to the parameter transformations on* C.

Proof. From the equation (16) and the first equations of (10) and (21) respectively, we have $\sigma_1^* = \sigma_1 \mathcal{A}_2 (\frac{dt}{dt^*})^2$. Also, from (20) and the second equations of (10) and (21) respectively, we have $\sigma_2^* = \sigma_2 \mathcal{A}_2$, $\sigma_3^* = \sigma_3 \mathcal{A}_1$ and $\tau_2^* A_3^3 = \tau_2$; $\tau_2^* A_3^\alpha = 0$ for all $\alpha \in \{4, \cdots, m\}$. Since $\tau_2 \neq 0$, therefore, $\tau_2^* \neq 0$ on $\mathcal{U} \cap \mathcal{U}^*$ and $A_3^4 = \ldots = A_3^m = 0$. Since $[A_\alpha^\beta]$ is a Lorentzian matrix, we infer that $A_3^3 = A_3 = \pm 1$ and $A_4^3 = \ldots = A_m^3 = 0$. Then from the third equation of the Natural Frenet equations (21) of Type 2 with respect to F and F^*, we have $\mathcal{A}_1$ is a constant, $\sigma_4^* = \sigma_4 A_3 \mathcal{A}_1 \frac{dt}{dt^*}$, $\sigma_5^* A_4^4 = \sigma_5 \mathcal{A}_1 \frac{dt}{dt^*}$ and $\sigma_5^* A_4^\alpha = 0$ for all $\alpha \in \{5, \cdots, m\}$. Since $\tau_3 \neq 0$ and $\sigma_5 = -\sqrt{2}\tau_3$, we have $\sigma_5 \neq 0$, therefore $\sigma_5^* \neq 0$ on $\mathcal{U} \cap \mathcal{U}^*$ and $A_4^5 = \ldots = A_4^m = 0$. We infer that $A_4^4 = A_4 = \pm 1$ and $A_5^4 = \ldots = A_m^4 = 0$. Repeating this process for all other equations of (21), we obtain all the relations in (26), which completes the proof. □

Also, using the screen transformations (11) that relate elements of F_2 and $\bar{F}_2$ on $\mathcal{U} \cap \bar{\mathcal{U}}$ such that $\tau_1 \neq 0$, we have

Proposition 15. *Let C be a Type 2 null curve of $\mathbf{M}_2$ and F and $\bar{F}$ be two Natural Frenet frames related by* (11)*. Suppose $\tau_1 \neq 0$ on $\mathcal{U} \cap \bar{\mathcal{U}} \neq \emptyset$. Then, their curvature functions are related by*

$$\bar{\sigma}_1 = \sigma_1 \mathcal{B}_2 \left(\frac{dt}{d\bar{t}}\right)^2, \quad \bar{\sigma}_2 = \left\{\sigma_2 + \bar{h}\, \mathcal{C}_1 + \frac{d\mathcal{C}_1}{d\bar{t}}\right\} \mathcal{B}_2, \quad \bar{\sigma}_3 = \sigma_3 \mathcal{B}_1 \tag{27}$$

and $\mathcal{C}_2 = c_3 = \cdots = c_m = 0$. Hence σ_3 is invariant function up to a sign, with respect to the transformations of the screen vector bundle of C.

Proof. From (16), (20) and the fourth equation of (21), we have the first equation of (27) and $\bar{\sigma}_1 \mathcal{C}_2 = 0$; $\bar{\sigma}_1 c_\alpha = 0\, (\alpha \geq 3)$ on $\mathcal{U} \cap \bar{\mathcal{U}}$. Since $\sigma_1 = \sqrt{2}\tau_1 \neq 0$, we have $\mathcal{C}_2 = c_\alpha = 0\, (\alpha \geq 3)$. Thus (11) is reduced to

$$\begin{aligned} \bar{N} &= \frac{d\bar{t}}{dt} N + \mathcal{C}_1 L_1, \quad \bar{L}_1 = \mathcal{B}_1 L_1, \\ \bar{L}_2 &= \mathcal{B}_2 \left\{ L_2 - \mathcal{C}_1 \frac{dt}{d\bar{t}} \lambda \right\}, \quad \mathcal{C}_1 = \frac{1}{\sqrt{2}}(c_1 + c_2), \\ \bar{W}_\alpha &= \sum_{\beta \geq 3}^{m} B_\alpha^\beta W_\beta, \quad \alpha,\ \beta \in \{3, \cdots, m\}. \end{aligned} \tag{28}$$

Also, from the second equation of (21), we have the second and third equations of (27). □

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