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Explaining the Impossibility of Division by Zero: Approaches of Chinese and Korean Middle School Mathematics Teachers

Kim, Young-Ok
Department of Mathematics Education, Kyungnam University, 449 Woryeong-dong, Masan, Gyeongnam 631-701, South Korea; E-mail: youokim@kyungnam.ac.kr

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The present study explores mathematics teachers' understanding of division by zero and their approaches to explaining the impossibility of division by zero. This study analyzes Chinese and Korean middle school mathematics teachers' responses to the teaching task of explaining the impossibility of dividing 7 by zero, and examples of teachers' reasoned explanations for their answers are presented. The findings from this study suggest that most Korean teachers offer multiple types of mathematical explanations for justifying the impossibility of division by zero, while Chinese teachers' explanations were more uniform and based less on mathematical ideas than those of their Korean counterparts. Another finding from this study is that teachers' particular conceptions of zero were strongly associated with their justifications for the impossibility of division by zero, and the influence of the teachers' conceptions of zero was revealed as a barrier in composing a well-reasoned explanation for the impossibility of division by zero. One of the practical implications of this study is those teachers' basic attitudes toward always attempting to give explanations for mathematical facts or mathematical concepts do not seem to be derived solely from their sufficient knowledge of the facts or concepts of mathematics.

Keywords: division by zero, impossibility, lower secondary
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MSC2000 Classification: 97D40

TYPES OF APPROACHES SHOWING THE IMPOSSIBILITY OF DIVISION BY ZERO

Although division by zero has been widely documented in the literature (Henry 1969; Knifong & Burton 1980; Ball 1988; Watson 1991; Puritz 2005; Crespo & Nicol 2006), the issue is still vague and confusing to both students and teachers. Many teachers are uncomfortable when they must explain the impossibility of division by zero, even though they have many years of teaching experience (Henry 1969). Although it is worthwhile to guide students to experience for themselves the impossibility of division by zero (Watson 1991), teachers tend to believe that the issue can only be understood by gifted students (Knifong & Burton 1980) or to encourage students simply to memorize the answer without providing reasoned explanations.

Henry (1969), a mathematician who observed elementary school teachers' teaching of division by zero, pointed out the teachers' inadequate teaching of division by zero in the classroom as follows:

> All too often the teacher tells the students that division by zero is not permitted and ends the discussion there. Of course the statement is correct. Division by zero is not permitted. But why make it sound like a teacher-made rule? (The beleaguered student may believe that he must add division by zero to the list of things that he is not permitted to do-such as run in the halls, chew gum in class, or study spelling during arithmetic period.) How much better it would be to let him discover for himself that an attempt to divide by zero leads to no result (p. 366).

Henry (1969) proclaimed that whatever means teachers choose to explain the impossibility of division by zero; the important point is that they should guide their pupils to understand for themselves that the operation is without meaning, rather than simply proclaiming it to be meaningless. Teachers' rule-bound explanations of the impossibility of division by zero have not changed since Henry's report.

Ball (1988) explored elementary and secondary preservice teachers' knowledge of division by zero using the same task used in this study, and she reported that the secondary preservice teachers were better prepared than the elementary preservice teachers to deal with the task of providing the correct answer to problems involving division by zero. However, most of the secondary preservice teachers who provided correct answers justified their answers by stating a rule and emphasizing the importance of remembering the rule, as shown below.

> I'd just say... "It's undefined," and I'd tell them that this is a rule that you should never forget that anytime you divide by 0 you can't. You just can't. It's undefined, so...you just can't (p. 130).

Ball explained that the teachers' rule-bound explanations are caused by the teachers' conceptions of division by zero as a particular case for which there is a rule, instead of connecting the issue to their general concept of division.

Preservice teachers' rule-bound explanations of the operation of dividing a non-zero number by zero were also reported by Crespo & Nicol (2006). They examined 48 elementary preservice teachers' initial understanding of the mathematical topic, which revealed that the teachers' explanations were of two different types:

1) rule-bound, and

2) reasoned to reach either a correct answer or an incorrect answer.

The rule-bound explanations included: "I learned that anything divided by 0 is 0"; "That's all I remember"; "You can't divide by nothing"; "My teacher said."

Crespo & Nicol (2006, p. 89) pointed out that rule-bound explanations are the most problematic because they discourage pupils' attempts at mathematical sense making and understanding. Weston (2000), who defined the meaning of "to give an argument" in terms of "to offer a set of reasons or evidence in support of a conclusion" (p. xi), pointed out that a good argument does not merely repeat the conclusion. Instead it offers reasons and evidence so that other people can make up their minds for themselves (p. xii). In the same manner, if teachers understand the impossibility of division by zero in a way that is well-supported by reasons, then they can explain the topic using reasons and evidence.

Table 1. Types of Approaches Showing the Impossibility of Division by Zero

Types of approaches		7 ÷ 0 (Undefined)	0 ÷ 0 (Undefined)
Conceptual models of division approach	Repeated subtraction model	**Q:** How many times should 0 be subtracted from 7 to reach 0? **A:** There does not exist a number that will tell how many times 0 should be subtracted from 7 to reach 0 because no matter how many times one subtracts 0 from 7, the remainder will always be 7. Thus, the process of subtracting 0 from 7 is unending. => There is no number that represents the length of this unending process (infinity).	**Q:** How many times should 0 be subtracted from 0 to reach 0? **A:** Any number can be the answer because no matter how many times you subtract 0 from 0, the answer is always 0. => There is no unique number that serves as an answer = there are too many answers.
	Partitive model	**Q:** Given 7 candies to be divided evenly among zero children, how many candies will each child have? **A1:** There is no limit to the number that one can give to each of 0 children from the box with 7 candies. There is no number that represents the situation of infinity. **A2:** It is clear that there are no sets into which partitioning can take place. Hence the operation is impossible to perform, which means the operation undefined.	**Q:** If you have 0 candies to divide equally among 0 children, how many candies can you give to each child? **A:** It is clear that there are no candies and children into which partitioning can take place. Hence the operation is impossible to perform, which means the operation is undefined.

*Q: Real-world questions representing the meaning of division by zero
*A: Possible answers to the real-world questions (Table to becontinued)

Table 1 (continued)

Types of approaches		7 ÷ 0 (Undefined)	0 ÷ 0 (Undefined)
Conceptual models of division approach (cont.)	Measurement model	**Q:** You want to share 7 apples among a group of people. If you give 0 apples to each person, how many people can you give 0 apples to? **A:** You can give apples to a very large number of people, but there is no real number that can represent the infinitely large number of people. => There is no number that represents the situation of infinity.	**Q:** Suppose you have no rice to begin with, and we do not use any rice per day. After how many days will the amount of rice remaining reach 0? **A:** After any number of days. No matter how many days you use no rice, the remaining rice will always be 0 because you had no rice to begin with. => There is no unique number that serves as an answer. (= There are too many answers.)
	Rate model	**Q:** You have a book with 7 pages. If you read zero pages per minute, how long before you finish the book? **A:** You have to read the book forever because you will never finish reading the book. No matter how long you read, you will have 7 pages left to read. => There is no number that represents the situation of infinity.	**Q:** You have a book with 0 pages. If you read zero pages per minute, how long before you finish the book? **A:** After any minute. No matter how long you read the book, you will finish reading the book because you have a book with no pages. => There is no unique number that serves as an answer. (=There are too many answers.)
	Part-and-whole model	**Q:** Given a 7-inch pizza, can we cut it into zero (no) pieces? If so, how do we represent the size of each piece? **A:** It is impossible to perform this process, the result is declared undefined.	**Q:** Given no pizza, can we cut it into zero (no) pieces? If so, how do we represent the size of 0 pieces? **A:** It is impossible to perform this process, the result is declared undefined.

*Q: Real-world questions representing the meaning of division by zero
*A: Possible answers to the real-world questions (Table to becontinued)

Table 1 (continued)

The types of approaches		$7 \div 0$ (Undefined)	$0 \div 0$ (Undefined)
Intuition-based approach	Explaining pattern of divisors approaching zero	$\frac{7}{1} = 7$, $\frac{7}{0.1} = 70$, $\frac{7}{0.01} = 700$, $\frac{7}{0.001} = 7000, \ldots$ The answer for $\frac{7}{0}$ must be a very large quantity, but there is no real number that can represent this infinitely large quantity. =>There is no number that represents the situation of infinity.	
Deductive reasoning approach	The conception of division as the inverse operation of multiplication (Definition of division)	Consider the equation $7 \div 0 = \square$. The number that serves as an answer, if it existed, would satisfy the equation $\square \times 0 = 7$. Clearly there is no number that can be placed in the box to make the equation true, because the product of zero and any number is zero, never seven. => There is no number that serves as an answer.	Consider the equation $0 \div 0 = \square$. The number that serves as an answer, if it existed, would satisfy the equation $\square \times 0 = 0$. Clearly any number can be placed in the box to satisfy the equation, because the product of zero and any number is always zero. => There is no unique number that serves as an answer. (= there are too many answers.)

*Q: Real-world questions representing the meaning of division by zero
*A: Possible answers to the real-world questions

Although rule-bound explanations of division by zero prevail in the classroom, mathematicians and mathematics educators (Henry 1969; Knifong & Burton 1980; Watson 1991; Crespo & Nicol 2006) have provided teachers with pedagogically useful approaches to explain two cases of division by zero: $a \div 0$ and $0 \div 0$ (where a = any non-zero real number). The approaches are summarized in Table 1.

METHODS AND PROCEDURES

Subjects

The participants in this study are 19 middle school mathematics teachers in China and South Korea. In China, 9 middle school mathematics teachers were interviewed from three urban middle schools in Changsha, Hunan. The other 10 teachers came from nine Korean middle schools: four located in Seoul, and six in southern Korea. Table 2 summarizes demographic information about the participating teachers.

Table 2. Demographic Summary of Participating Teachers

	Chinese Teachers ($n = 9$)	Korean Teachers ($n = 10$)
Gender	Female (3), Male (6)	Female (7), Male (3)
Educational Background	High school graduate (1) Bachelor's degree in mathematics education (8)	Bachelor's degree in mathematics education (4) Master's degree in mathematics education (5) Bachelor's degree in mathematics and Master's degree in mathematics education (1)
Average teaching experience	12 years	8 years

Data Collection

Participating teachers were presented with a division by zero teaching task to investigate their approaches to explaining the impossibility of division by zero.

Scenario

Suppose you have a pupil who asks you what 7 divided by 0 is. How would you respond? (Choose an age you might teach, and think about the question for that age.) The teaching task presented above is one of the Teacher Education and Learning to Teach Study (TELT) mathematics interview questions developed by the National Center for Research on Teacher Education (NCRTE) at Michigan State University (Kennedy *et al.* 1993).

The teaching task used in this study was used previously by Ball (1988) and Ma (1999) to investigate preservice and inservice teachers' subject matter knowledge of division by zero. Shulman (1986, p. 9) explained that teachers' subject matter knowledge for teaching goes beyond knowledge of the facts or concepts of the subject matter; it requires "the capability of explaining why an accepted truth in a domain is deemed warranted, why it is worth knowing, and how it is relates to other truths in the domain" as a display of "syntactic" knowledge (Schwab 1978). Thus, the teaching task above is appropriate for investigating the teachers' understanding of division by zero.

Procedure

Teacher interviews were conducted outside of normal class time. Data collection in China was conducted with a professional Korean-Chinese translator. The interviews consisted of two sessions, which together lasted about thirty minutes. The first session of the interview included a brief questionnaire and general questions. The brief questionnaire was designed to elicit respondents' demographic and background information. The general questions concerned participants' personal and academic histories and their views on some general issues about teaching and learning mathematics. The purpose of the general questions was to establish rapport between the interviewer and the respondent by demonstrating the researcher's interest in the respondents and removing tension.

The scenario problem that comprised the teaching task was conducted in the second session, after participants had completed the questionnaire and general questions. Participants were not allowed to use any resources while completing the scenario problem. Participants were asked follow-up questions that were specific to particular situations that arose in the interviews. The interviews were audio taped and transcribed.

RESULTS

Types of teachers' answers to dividing 7 by zero

The question "what is 7 divided by zero?" was no problem for the Chinese and Korean middle school mathematics teachers. All of the teachers knew that zero cannot be a divisor and no one provided a number as the answer to division by zero. However, one Chinese teacher and one Korean teacher believed that division by zero is possible and suggested "infinity" as the answer to 7 ÷ 0.

Table 3. Types of Chinese and Korean Teachers' Answers to "What is 7 divided by 0?"

Types of Answers		Chinese Teachers ($n = 9$)	Korean Teachers ($n = 10$)	Total
Incorrect	It is possible to divide 7 by 0. (It is **infinity**.)	1	1	2
Correct	Division by zero **is not allowed**. (You cannot divide by zero.)	3	5	8
	Division by zero is **meaningless**.	5	–	5
	It is a kind of promise.	–	2	2
	It is impossible to divide by zero.	–	1	1
	Division by zero is **undefined**.	–	1	1

These incorrect answers resulted from the teachers' misconceptions about the concept of infinity. Table 3 presents the distribution of teachers' answers to 7 ÷ 0.

Teachers' reactions to: "What is 7 divided by zero?"

When the teaching task was given to the Chinese teachers, their responses differed considerably from those of the Korean teachers in two ways. First, most Chinese teachers did not attempt to give explanations for their answers until the researcher asked them to explain their reasons, while all but one of the Korean teachers justified their answers with explanations when the task was originally presented. Thus, the follow-up interview question asking teachers to react to a pupil's erroneous idea, 7 ÷ 0 = 7, served as a tool to extract reasons from most Chinese teachers, while it was used to collect additional responses from most Korean teachers. Second, most Korean teachers provided multiple types of mathematical explanations for their own answers, while Chinese teachers' explanations were uniform and based less on mathematical ideas than those of their Korean counterparts. The next section further examines these differences between the two teacher groups' explanations.

Chinese teachers' explanations of the impossibility of dividing 7 by zero

Regardless of the correctness of answers, of the 9 Chinese teachers, 4 (44%) provided reasoned explanations for their own answers.

Table 4. Distributions of Types of Chinese Teachers' Explanations of the Impossibility of Dividing 7 by Zero

Types of Explanations		Incorrect	Correct		Total
		Infinity (n=1)	It is not allowed (n=3)	Meaningless (n=5)	
Reasoned explanations	Partitive model of division			3	3
	Pattern of divisors approaching zero	1			1
Rule-bound explanations	It is a rule of division			1	1
	My teacher said so		1		1
	Students may not be able to understand my explanation	1			1
	All students know that we cannot divide by zero		2		2

The remaining 5 Chinese teachers (55%) did not provide reasoned explanations for their answers by emphasizing the absoluteness of the impossibility of division, all students' awareness of that, the lack of capability, or outside authority such as "my teacher said." Table 4 presents the distribution of types of Chinese teachers' explanations, and examples of the Chinese teachers' reasoned explanations are presented in the following discussion.

Examples of Chinese teachers' reasoned explanations

Three of the four Chinese teachers who provided reasoned explanations justified their answers using the partitive model of division as equal sharing. They represented the meaning of $7 \div 0$ in a real-world situation of partitioning something evenly among people. In the real world representations, the meaning of zero was interpreted as "no one," such that there are no sets into which partitioning can take place. Hence the operation $7 \div 0$ is impossible to perform. An example of this type of approach is presented below.

> Division represents the act of sharing something. Thus, there must exist people into which partitioning can take place. For example, the act of sharing seven of something equally among a few people has its meaning as a division problem. But, when the divisor is zero it means that there is no one into which partitioning can take place because zero means nothing. Therefore, in that case, the act of partitioning something is impossible to perform. Thus, dividing $7 \div 0$ is meaning-less.

The remaining teacher who provided a reasoned explanation justified why $7 \div 0$ is meaningless with an explanation based on a pattern of divisors approaching zero. His explanation is presented below.

> It is possible to divide 7 by zero. It is infinity. For example, 7/1 is 7, 7/0.1 is 70, and 7/0.01 is 700. In the same pattern, as the divisor tends to approaching zero, its computational value is getting larger and larger. Once the divisor is almost closer to zero, the computational value will be infinity larger than larger. But, we don't know the exact value of $7 \div 0$ except for knowing that it is an infinitely large number, so $7 \div 0$ is meaningless.

This teacher actually provided two answers to 7 ÷ 0; it is infinity and it is meaningless. According to the response above, however, he seemed to provide the first statement "7 ÷ 0 is infinity" as a shorthand for the statement that 7 ÷ 0 tends to infinity (*i.e.*, increases beyond all bounds) as x tends to 0, although he did not give clear notice that this statement is intended as shorthand for the full statement. But, this teacher's answer was categorized as an incorrect answer because his statements that "7 ÷ 0 is possible and it is infinity" can encourage pupils' to have a certain misconception about infinity regardless of the teachers' intention: infinity is a number so that the equation "7 ÷ 0 = infinity" is possible.

Korean teachers' explanations of the impossibility of dividing 7 by zero

Of the 10 Korean teachers, one teacher explained the impossibility of dividing 7 by zero based on outside authority, such as mathematical convention, and 9 teachers attempted to justify their answers with reasoned explanations using five types of approaches: the pattern of divisors approaching zero, fractional function graphs, the standard partitive model of division, the measurement model of division, and the concept of division as the inverse of multiplication.[1] The distribution of types of Korean teachers' explanations is presented in Table 5. As shown in Table 5, the five types of reasoned explanations are classified under two types of approaches: intuitive approach and formal mathematical approach.

According to Table 5, eight Korean teachers (80%) preferred the intuitive approaches, and only one Korean teacher used the formal mathematical approach. Examples of Korean teachers' reasoned explanations are presented in the following section.

Examples of Korean teachers' reasoned explanations

1. Approach using a pattern of divisors approaching zero

The intuitive approach using a pattern of divisors approaching zero was the approach most commonly used by the Korean teachers. Three Korean teachers considered the operation 7 ÷ 0 as an equation ($\frac{7}{x}$), and then they demonstrated the value of $\frac{7}{x}$ as the divisor, x, approaches zero. The demonstration aimed to encourage pupils intuitively to realize that division by zero does not result in a special number as an answer; instead, the result of the equation approaches infinity as the divisor, x, approaches zero. The teachers preferred to use this approach as an alternative explanation of the formal definition:

$$\lim_{x \to +0} \frac{7}{x} = \infty .$$

[1] The approach combining the partitive model of division and the fractional function graphs is not considered as a different type of explanation showing the impossibility of division by zero.

Table 5. Distribution of Types of Korean Teachers' Explanations of the Impossibility of Dividing 7 by Zero

Explanations	Incorrect	Correct				Total
	Infinity	It is not allowed	It is a promise	It is impossible	Undefined	
Reasoned Explanations						
Intuitive approach The pattern of divisors approaching zero	1	2				3
Fractional function graphs		2				2
The partitive model		1				1
The measurement model					1	1
The partitive model + fractional function graphs			1			1
Formal mathematical Approach The conception of division as inverse of multiplication				1		1
Rule-bound Explanation						
It is a kind of mathematical convention			1			1

An example of this type of approach to explaining the impossibility of dividing 7 by zero is provided in below:

If the students are middle school students, I would use an intuitive approach to explain why dividing by zero is not allowed.

If I use 7… let us think about

$$\frac{7}{100}, \frac{7}{10}, \frac{7}{1}, \frac{7}{0.1}, \frac{7}{0.01}, \dots$$

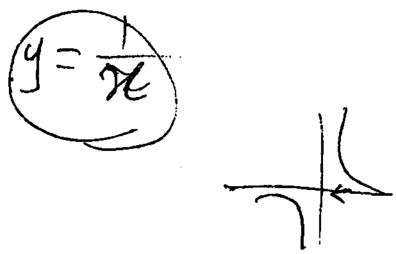
These fractions are the same with 0.07, 0.7, 7, 70, 700, …. what if this pattern is going

> on…as the divisors tend to 0, the fractions tend to a tremendously large number. However, in mathematics, the result of getting larger and larger numbers is not treated as a number. We call it infinity. It is not a number. Hence, dividing by zero does not have a quotient, so dividing by zero is not allowed in mathematics. However, middle school students may be able to intuitively understand it, but still my explanation might be hard for them. Most of all, most students seemed to be not interested in knowing why we cannot divide by zero. So, I usually skip this explanation in class.

2) Fractional function graphs approach

Three Korean teachers used fractional function graphs, such as $y = \frac{7}{x}$ and $y = \frac{1}{x}$, to show the impossibility of division by zero. After drawing the function graphs, the teachers encouraged pupils to observe visually the y value approaching infinity as the x value tends to 0. This approach is conceptually identical with the approach using a pattern of divisors approaching zero as an alternative explanation of the formal definition of $\lim_{x \to +0} \frac{7}{x} = \infty$. However, the teachers who presented the fractional function graphs noticed that the explanation using fractional function graphs would be difficult for middle school students to understand. An example of this approach is given below.

> If there is a student who asks about why we cannot divide by zero, I would consider that the student has a potential to be able to do mathematics well, but I will not fully explain the reason because they may not be able to understand it. I would say, "You will learn the reason later. If I explain the reason now briefly… When you learn function graphs later, you will learn the reciprocal function graph like this [drawing the graph of $y = \frac{1}{x}$]. Look at this. "Where does y approach as x tends to 0? [drawing an arrow on the right hand of the x-axis]" In the same manner, I will try to stimulate the student's curiosity. However, most of all, there are few students who desire to know the reason. When I was a novice teacher, I used to attempt to explain the reason using the graph of $\frac{1}{x}$, but most students could not understand the explanation [laugh].



At the high school level, I would explain it using the concept of limits because they might learn fractional function graphs, such as $y = \frac{7}{x}$. Does y have a value as x approaches 0? It does not have a value. Once the divisor is zero, we particularly say that the y is plus infinity or minus infinity. However, the case is not allowed in the equation $y = \frac{7}{x}$. In fact, this type of explanation is possible at the middle school level because they learn the reciprocal function graph. After drawing only this part [pointing to the first quadrant in the coordinate plane], I would say, "Does y have a matching value as x approaches 0?" Their answers would be "No" or "Infinity." I will react to the answers by saying, "So, we don't consider the case of division by zero."

3) Approach using the partitive model of division

Of the 9 Korean teachers who provided reasoned explanations, two teachers attempted to justify the impossibility of division by zero using the partitive model of division as equal sharing. The approach based on the partitive model of division justified the impossibility of $7 \div 0$ by reaching infinity in the equal sharing situation: there is no limit to the number that one gives to each of 0 people from the 7 bread rolls. The example of this approach is presented below.

> To elementary and middle school students, I would say "we don't think about dividing by zero. It is a kind of promise among people." If I recall the explanation that I learned from my teacher, this ($7 \div 0$) is a situation of sharing something evenly. Is it right? For example, $7 \div 2$ represents the situation: There are 7 bread rolls. If I share the rolls equally between two people, how many rolls can one person eat? ...3.5 rolls... Again, there are 7 rolls. If I give all of them to one person, how many rolls can the person eat? ...7 rolls... Then, there are 7 and 0. This division represents a situation sharing 7 rolls equally among no people, and the question is, "how many rolls can each of 0 people have?" This case leads a paradoxical situation. I can give an infinitely many number of rolls to each of 0 people because no matter how many rolls might be given to each of 0 people, there are still 7 rolls left over. I will give this explanation only to the students who seem to have a sense of being able to understand what I am saying. For other students, I will just say "we don't think about dividing by zero."

4) Approach using the measurement model of division

The teacher who stated that the answer to $7 \div 0$ is "undefined" represented the meaning of $7 \div 0$ in a real-world situation based on the measurement model of division, and her explanation is presented below.

This Korean teacher represented the quotient of $7 \div 0$ as a box, and then presented the relationship between the division form and its related multiplication form. As an example, she represented the meaning of $15 \div 3$ in a real-world situation based on the measurement model of division, and then, by analyzing the conceptual structure of the real-world situation, she conceptualized the box as the number of equal groups and the divisor as the number of members that will determine the size of the equal groups.

Division is the inverse operation of multiplication. Pupils often represent the computational value of a division problem as a box. If 7 ÷ 0 = □, then □ × 0 = 7. In this form, 15 ÷ 3 is represented in finding out how many groups having 3 members each can be formed from the 15 people. They will say, "five." That is, the box always represents the number of groups. Hence, 7 ÷ 0 would be interpreted as "how many groups having no members can be formed from a total of 7 people? Such groups do not exist because there are no members in these groups. That is, the value of 7 ÷ 0 does not exist. Hence dividing by zero is not allowed. In this case, the zero means "nonexistence."	

She expressed the divisor zero in terms of nonexistent members, so that accordingly it is impossible to form the equal groups. Hence there is no number that is represented by the box, so the operation $7 \div 0$ is impossible to perform, which means the operation is undefined. This justification is different than the justification that was derived by the same approach using the measurement model of division in Table 1. According to Table 1, the approach using the measurement model of division leads to infinity, while this Korean teacher's approach using the same model of division results in the impossibility of performing the division.

5) Division as the inverse of multiplication approach

One Korean teacher explained the impossibility of two cases of division by zero using the concept of division as the inverse of multiplication. Of all 10 Korean teachers, this was the only teacher who expressed the answer to 7 ÷ 0 by stating, "It is impossible", even though the expression, "It is impossible", is a formal expression used in most Korean mathematics textbooks.

DISCUSSION AND IMPLICATIONS

The teaching task of responding to a pupil asking "what is 7 divided by 0?" revealed Chinese and Korean teachers' understanding of the impossibility of dividing a nonzero dividend by zero. The teachers' understandings of 7 ÷ 0 were embedded in their explanations to the impossibility of 7 ÷ 0, and the terms they used to express the answer to 7 ÷ 0 were characterized according to the type of approach they used. For example, almost all of the Chinese and Korean teachers who argued for the impossibility of 7 ÷ 0 using the alternative explanations of the formal concept of limits (the patterns of divisors approaching zero and the graphs of and $y = \frac{1}{x}$ and $y = \frac{7}{x}$) provided "infinity" and "it is not allowed" as answers to $7 \div 0$.

If the students are middle school students, I would say, "It is impossible to divide 7 by zero." When dividing 7 by 0, the quotient can be represented as an unknown number x. Based on the relationship of division and multiplica-tion, the equation can be changed to this form: $0 \times x = 7$. However, there is no number that represents the x satisfying the equation $0 \times x = 7$. Thus, it is impossible to divide 7 by zero. In addition to this explanation, I would explain why it is impossible to determine the answer to $0 \div 0$ (*Bu-jeong*: 부정, 不定) to show the difference between the two cases of division by zero:$0 \div 0$ and $\Delta \div 0$ (Δ = nonzero number). In the same manner, the quotient of $0 \div 0$ is represented as an unknown number x. The equation is changed to the new form: $0 \times x = 0$ based on the relationship between division and multiplication, and it is clear that arbitrarily any number can be the x satisfying the equation.	

The alternative explanations demonstrate that as the value of x (divisor) gets closer and closer to zero, the y-value (the quotient) gets larger and larger without bound. Through these observations, the teachers reached two types of conclusions:

It is impossible to tell the y-value, which means that we cannot divide 7 by zero (*i.e.*, dividing 7 by 0 is not allowed), or we call the situation infinity, which means that $7 \div 0$ is infinity.

It is interesting that the Chinese teachers' explanations of the impossibility of $7 \div 0$ were strongly associated with their particular conceptions of zero, and the influence of the teachers' conceptions of zero was revealed as a barrier in composing a well-reasoned explanation. Of the four Chinese teachers who provided reasoned explanations, three teachers generated real-world situations based on the partitive model of division to represent the meaning of $7 \div 0$. Using real-world models, they all argued that $7 \div 0$ is meaningless in a similar manner. Below I summarize their approaches to reveal the commonality among their answers.

> In equal sharing situations, the divisor number factors represent the sets into which partitioning can take place. When 7 ÷ 0 is represented in the partitioning situation, there are no sets because zero means nothing. How can we perform the partitioning when there are no sets into which partitioning can take place? Hence, the operation 7 ÷ 0 is meaningless.

As shown in the summarized interview response above, the conception of zero as nothing[2] was essential to the explanation of the impossibility of 7 ÷ 0 in the partitive model approach. However, although the approach is somewhat useful in showing pupils the strange situation that results when attempting to divide 7 by 0 (as opposed to division problems with whole number divisors), the approach has certain drawbacks. For instance, the approach involving referring to the lack of sets into which partitioning can take place may cause pupils to mistakenly assume that 0 ÷ 7 is also meaningless.

Consider the case of 0 ÷ 7. When the meaning of 0 ÷ 7 is considered in a real-world situation based on the partitive model of division, according to the Chinese teachers' arguments above, there are no objects to divide equally among 7 sets. Hence, it would seem that 0 ÷ 7 is impossible because there are no objects to partition into the seven sets, and thus 0 ÷ 7 is meaningless. However, 0 ÷ 7 is clear and well defined (0 ÷ 7 = 0) in division.

One Chinese teacher's response took the conception of zero as nothing to be the entire reason for the impossibility of 7 ÷ 0. The teacher said, "Zero is meaningless. Accordingly, 7 ÷ 0 is meaningless." This rule-bound explanation overlooked implications for the case of 0 ÷ 7, just as the explanations of the previous Chinese teachers did. The findings from these Chinese teachers' responses suggest that teachers should consider using the second type of approach, generating an infinity situation using the standard partitive model of division, in order to demonstrate the impossibility of dividing 7 by 0. Otherwise, they risk leading the students to wrongly believe that it is impossible to divide 0 by 7.

The Korean teachers' explanations, by contrast, did not depend much on their conceptions of zero. While approaches using the conceptual models of division are strongly associated with teachers' conceptions of zero, many of the Korean teachers preferred approaches using the alternative explanations of the formal definition of limits, which are not strongly associated with teachers' conceptions of zero. For example, the Korean teacher who completed the reasoned explanation using the partitive model of

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[2] During the interviews with Chinese teachers, almost all teachers explained the meaning of zero in terms of "meaningless" or "nothing." These two terms were interchangeable in the manner that "nothing" indicates the condition having no discernible meaning, but Chinese teachers preferred to use the formal term, "meaningless."

division focused on reaching the infinity situation in the model, even though she had the concept of zero as nothing.

One Korean teacher used her conception of zero as self-evident proof of the impossibility of 7 ÷ 0. She generated a real-world situation representing the meaning of 7 ÷ 0 based on the measurement model of division, and then argued that because there is no size of the measurement unit used in measuring off from the given total quantity 7, the division, 7 ÷ 0, cannot be performed. Hence, the operation is not allowed in mathematics. This Korean teacher's approach has a weakness similar to the weakness of the Chinese teachers' approaches. The Korean teacher's approach similarly ignores the case of 0 ÷ 7. Thus, the approach deriving the infinity situation is pedagogically useful when using the measurement model of division as well.

In comparison to the results from the Chinese teachers' responses, Korean teachers more frequently approached explaining the impossibility of division using the concept of limits, even though this is a concept usually taught at the high school or college level. The Korean teachers were concerned about the difficulties of elementary or middle school students in understanding the formal definition and description of limits, so the Korean teachers attempted to explain the concept using alternative descriptions. The alternative explanations used the pattern of divisors approaching zero or the graphs of reciprocal or fractional functions, and the alternative explanations aimed to encourage the pupils' intuitive understanding of the concept of infinity. Although the Korean teachers were successful in visually representing the infinity situation, most of them failed in explaining how the infinity situation verifies the mathematically accepted fact: division by zero is undefined/not allowed.

The failure to properly explain the impossibility of division by zero by generating an infinity situation resulted from the teachers' ignorance of the fact that infinity is not a number, which means that infinity cannot serve as the quotient of division because the quotient can be represented only by a real number. Only one Korean teacher was clearly familiar with these two fundamental mathematical facts, and this teacher succeeded in declaring the impossibility of division by zero based on the infinity situation.

The teaching task of responding to a pupil's question: "What is 7 divided by 0?" revealed not only teachers' understanding of the impossibility of division by zero, but also their basic attitudes about explaining a mathematically accepted truth: division by zero is undefined. While most Korean teachers (90%) initially attempted to provide an explanation of the accepted fact, almost half of the Chinese teachers (about 44%) avoided giving an explanation.

The results from this study suggest that the teachers who consider the impossibility of division by zero as a particular rule in mathematics tend to provide a rule-bound explanation and merely repeat the rule. To justify their attitudes, these teachers emphasized outside authorities such as "my teacher said," or they over-generalized beliefs about students' learning, such as "all students already know the rule." By appealing to authority, these teachers may even discourage students' desire to know why a mathematically accepted fact is true. According to the analysis of the Chinese and Korean teachers' responses, even among the teachers who know why it is true, some teachers did not attempt to explain the reason because they believe that the young students cannot understand their explanations.

Therefore, the implication of this study is that the teachers' basic attitudes toward always attempting to give explanations for mathematical facts or mathematical concepts do not seem to be derived solely from their sufficient knowledge of the facts or concepts of mathematics. In providing well-reasoned explanations, the teachers' basic pedagogical attitudes play a strong role along with their sufficient understanding of the relevant mathematical facts and concepts. In particular, a basic pedagogical attitude is essential for prospective and inservice mathematics teachers. Since a number of definitions and theorems are taught in mathematics, mathematics teachers should be encouraged to give students well-reasoned explanations of those mathematical facts in a manner that considers the students' learning development.

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