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THE PROCESS OF THE DEVELOPMENT OF HYPOXIA IN AN ABNORMAL BLOOD FLOW II

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ABSTRACT. The oxygen distribution at steady state is analyzed mathematically in a hexagonal cylinder. The domain is penetrated by parallel cylindrical capillaries of different oxygen squirt. Asymptotic solution is used to determine the effect of axial diffusion. Oxygen concentration profiles are displayed at some positions of capillary-beds. At the venous end some tissue areas suffer from a shortage of oxygen.

Keywords : Hypoxia; Oxygen diffusion; Oxyhemoglobin;

1. INTRODUCTION

Measurements of oxygen concentration on a tissue domain is a fundamental problem in the study of oxygen transport. The basic single capillary model was introduced by Krogh[1]. The Krogh-Erlang equation formulated based on Krogh's model was used to express oxygen tension in the highly regular capillary beds of skeletal muscle, and has been the foundation of most physiological estimates for the last 70 years. The major developments in quantitative and qualitative comprehension of oxygen transport to tissue lie in the studies of hemoglobin-oxygen kinetics, the role of hemoglobin and myoglobin in promoting oxygen diffusion, and the role of morphological hemodynamic heterogeneities.

Salathe and coworker[4] investigated oxygen distribution in steady state in a Krogh cylinder by considering axial diffusion in the tissue and capillary, and an arbitrary oxyhemoglobin dissociation relationship. The effect of heterogeneity in skeletal muscle was studied by Popel and Charny[2], and Ronald and Chang[7]. Hoofd[6] extended Krogh circle to arbitrary shapes with nonhomogeneous multi- source including myoglobin role to facilitate oxygen diffusion. The geometrical influences such as capillary network anastomoses and tortuosity were presented in the paper[3]. But, in spite of previous remarkable achievements, their works were limited to constant solute out.

A mathematical analysis of steady state oxygen distribution in a regular hexagonal cylinder is presented in this paper(see Fig. 1). In our mathematical equations the oxyhemoglobin dissociation relationship and axial diffusion in capillary are included. The exact solution for the oxygen concentration in the capillary is obtained using perturbation analysis.

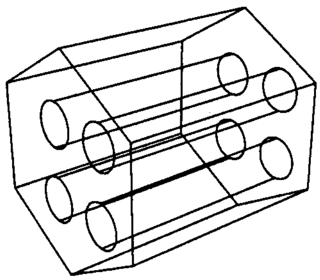


Fig 1. A multi-capillary hexagonal cylinder domain.

2. MATHEMATICAL MODELLING AND PERTURBATION ANALYSIS

2.1. Mathematical Modelling

We consider a regular hexagonal cylinder of length L for tissue domain. The cylinder is penetrated by parallel cylindrical N capillaries arranged in uneven location and diffusion strength (Fig. 1.). Let $(\overline{\rho}_j, \varphi_j)$ be local cylindrical coordinates centered at each capillary and ρ_{jr} be the radius of the jth capillary. The oxygen flux $\overline{q}_j$ per unit volume of the jth capillary is defined by

$$\overline{q}_j = -D_{\overline{T}}\rho_{jr} \int_0^{2\pi} \frac{\partial \overline{C_t}}{\partial \overline{\rho}_j}|_{\overline{\rho}_j \to \rho_{jr}} d\varphi_j, \tag{1}$$

where $\overline{C_t}$ is the oxygen concentration in the tissue and $D_{\overline{r}}$ is the radial oxygen diffusivity. The oxygen concentration in the blood of the jth capillary, C_{bj}, varies with both radial and axial location within the capillary. However, the convective mixing within the bolus of fluid between each red blood cell results in a fairy uniform distribution of oxygen. We may thus neglect radial variation of concentration in the capillary. If $\overline{F}_j$ denotes the volume blood flow rate of the jth capillary and $\overline{z}$ denotes distance measured along the capillary, then oxygen concentration in the blood is

$$\overline{F}_j\frac{\partial}{\partial\overline{z}}\{C_{bj}(\overline{z})+\overline{MS}[C_{bj}(\overline{z})]\}=-\overline{q}_j+\epsilon D_b\frac{\partial^2 C_{bj}}{\partial\overline{z}^2}. \tag{2}$$

The $\overline{M}$ denotes the oxygen capacity of blood, $\overline{S}[C_{bj}]$ denotes the oxyhemoglobin dissociation relationship, and the D_b denotes the diffusivity of blood. The ϵ is related to the diffusivity in the direction $\overline{z}$[4], but we regard it as a small positive number, that is, $0<\epsilon\ll 1$. The function $\overline{S}[C_{bj}]$ may be approximated by the empirical formula

$$\overline{S}[C_{bj}]=\frac{\alpha C_{bj}^n}{1+\alpha C_{bj}^n} \tag{3}$$

for suitable choice of the constants α and n. At the arterial end, the oxygen concentration of the jth capillary is set:

$$C_{bj}(0)=\overline{q}_j(0). \tag{4}$$

Let κ be the constant consumption per volume of tissue. The longitudinal diffusion of solute may be neglected in the body because the length of the blood vessel cylinder is about 100 times cylinder diameter. At a given $\overline{z}=\overline{z}_0$ the oxygen distribution chart of each cross section of the hexagonal cylinder is investigated. The oxygen concentration in the tissue, $\overline{C_t}(\overline{x},\overline{y},\overline{z}_0)$, at a fixed $\overline{z}=\overline{z}_0$ $(0\leq\overline{z}_0\leq L)$, satisfies the equation and boundary condition

$$\frac{\partial}{\partial\overline{x}}D_{\overline{x}}\frac{\partial\overline{C_t}}{\partial\overline{x}}+\frac{\partial}{\partial\overline{y}}D_{\overline{y}}\frac{\partial\overline{C_t}}{\partial\overline{y}}=\kappa, \tag{5}$$

$$\frac{\partial \overline{C_t}}{\partial n} = 0, \qquad \text{at} \quad \overline{z} = \overline{z}_0. \tag{6}$$

The $D_{\overline{x}}$ and $D_{\overline{y}}$ are the oxygen diffusivities of directions x and y, respectively, and we set that $D = D_{\overline{x}} = D_{\overline{y}}$ is a constant.

Let us suppose that, at the arterial end (at $\overline{z} = 0$), everywhere the consumption is κ per volume. Then the balance of mass flux on $\overline{q}_j$ is

$$\sum_{j=1}^{N} \overline{q}_j(0) = \kappa R, \tag{7}$$

where R is the area of the regular hexagon (the cross section of the hexagonal cylinder). In terms of non-dimension variables: $C_{Bj} = C_{bj}/\kappa R$, $x = \overline{x}/K$, $y = \overline{y}/K$, $z = \overline{z}/L$, $q_j = \overline{q}_j/\kappa R$, $C = \overline{C_t}D/\kappa K^2$, $M = \overline{M}/\kappa R$, $S[C_{Bj}] = \overline{S}[C_{Bj}\kappa R]$, we obtain equations;

$$\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} = 1 \tag{8}$$

$$\beta_j \frac{\partial}{\partial z}\{C_{Bj}(z) + MS[C_{Bj}(z)]\} = -q_j + \epsilon\tau \frac{\partial^2 C_{Bj}}{\partial z^2} \tag{9}$$

$$\frac{\partial C}{\partial n} = 0, \qquad \text{at } z = z_0, \quad 0 \le z_0 \le 1 \tag{10}$$

$$\sum_{j=1}^{N} q_j = 1, \qquad \text{at } z = 0 \tag{11}$$

$$C_{Bj} = q_j(0), \qquad \text{at } z = 0. \tag{12}$$

Here $\beta_j = \overline{F_j}/L$, and $\tau = D_b/L^2$.

2.2. Perturbation Analysis

We assume that q_j, $1 \le j \le N$, are independent of z and let us expand C_{Bj} in the form of an asymptotic series

$$C_{Bj} \sim C_{j0} + \epsilon C_{j1} + \cdots. \tag{13}$$

The expansion $S[C_{Bj}] = S[C_{j0}] + \epsilon S'[C_{j0}]C_{j1}$ then yields the consecutive differential equations

$$\beta_j \frac{\partial}{\partial z}\{C_{j0} + MS[C_{j0}]\} = -q_j \tag{14}$$

$$\beta_j \frac{\partial}{\partial z}\{C_{j1} + MS'[C_{j0}]C_{j1}\} = \tau \frac{\partial^2 C_{j0}}{\partial z^2}. \tag{15}$$

The solution of Eq (14) satisfying (12) is

$$C_{j0}(z) + MS[C_{j0}(z)] = \frac{-q_j}{\beta_j} z + q_j + MS[q_j] \tag{16}$$

Using the Eq (16) the solution of Eq (15) is

$$C_{j1}(z)\{1 + MS'[C_{j0}(z)]\} = -\frac{\tau q_j}{\beta_j^2(1 + MS'[C_{j0}(z)])} + \frac{\tau q_j}{\beta_j^2(1 + MS'[q_j])}. \tag{17}$$

We assume that the value of each parameter used in the asymptotic solution is independent of capillary, and the data are shown in table 1. When we combine the equations (16) and (17) for the oxygen concentration in the blood, $\varepsilon = 10^{-2}$ are employed for 2-capillary. The changes of the oxygen concentration in the blood along the capillaries are depicted in figure 2.

Table 1. Values of Parameters

F	L	D_b	α	n	M
1.13×10^{-3}	6×10^{-4}	1.7×10^{-1}	8.55×10^{5}	2.0	0.204

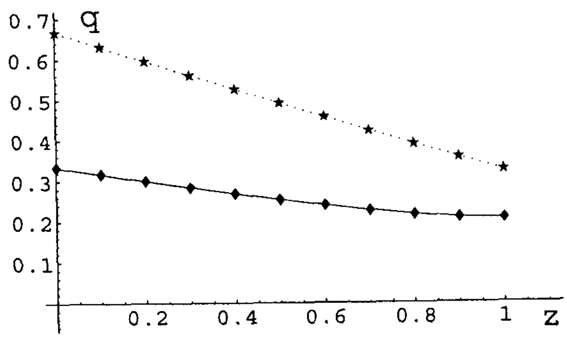


Fig 2. The variation of oxygen concentration in blood.

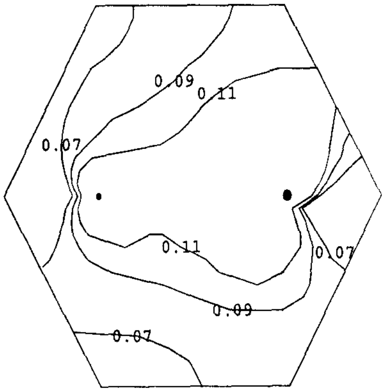
3. NUMERICAL RESULTS AND DISCUSSION

The level curves of oxygen concentration are found using MATHEMATICA 4.1. The term "HYPOXIA" is used in our figures to indicate the tissue area that the oxygen concentration C is less than 0. We consider a hexagonal cylinder which is penetrated by two parallel cylindrical capillaries. The oxygen concentrations in the blood obtained using the perturbation analysis are diffused numerically based on the finite difference method[5]. The locations and strengths of capillaries at $z = 0$ are shown below in table 2 and the amount decreases as z increases.

Table 2. Locations and strengths of 2-capillary

k	q_k	x_k	y_k
1	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{4}$
2	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{3}{4}$

The oxygen concentration profiles along the axis z are displayed in figures 3-(a) $\sim$ 3-(f). As shown in figures 3-(a) $\sim$ 3-(d) the low lever concentration curves show an inclination that more oxygen disperse to the right upper corner of each hexagon. Hypoxic area appears at $z = 0.8$ (see Fig 3-(e)) and develops at the end of capillary bed (see Fig-(f)). The tissue area near the side 1 and the left upper corner of each hexagon are most susceptible to hypoxia. Moreover, Fig. 3-(f) shows that the upper part of side 2 is susceptible to hypoxia even though the distance to capillary is not far away.



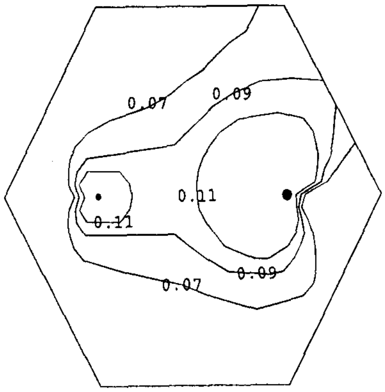
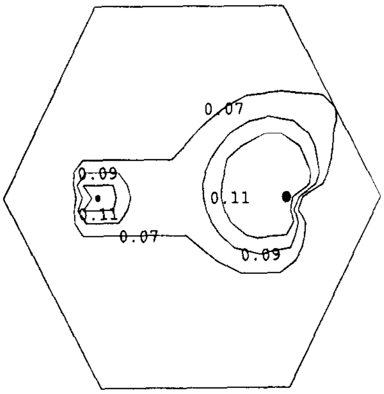


Fig 3-(a). 2-capillary domain at $z = 0.0$. Fig 3-(b). 2-capillary domain at $z = 0.2$.



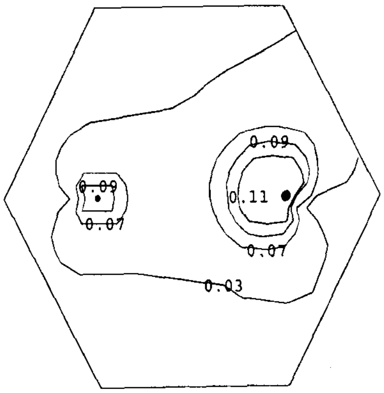
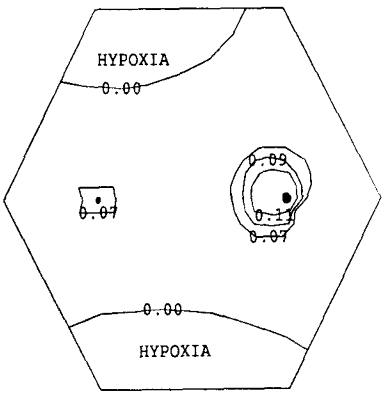


Fig 3-(c). 2-capillary domain at $z = 0.4$. Fig 3-(d). 2-capillary domain at $z = 0.6$.



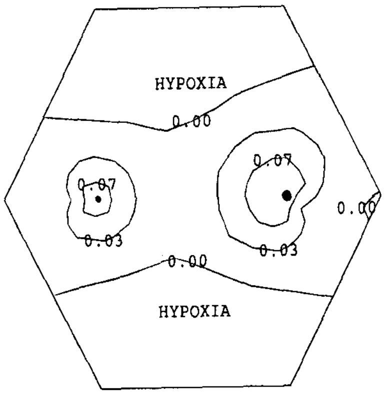


Fig 3-(e). 2-capillary domain at $z = 0.8$. Fig 3-(f). 2-capillary domain at $z = 1.0$.

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