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CODING THEOREMS ON A GENERALIZED INFORMATION MEASURES.

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Abstract: In this paper a generalized parametric mean length $L(P^{\nu},R)$ has been defined and bounds for $L(P^{\nu},R)$ are obtained in terms of generalized R-norm information measure.

Key words and phrases: Parametric mean length, Entropy, Holder's inequality.

1. Introduction

Consider the model A given below for a finite scheme random experiment having $(A_1, A_2, ..., A_n)$ as the complete system of events with respective probabilities $P=(p_1, p_2, ..., p_n), p_i \geq 0$ and $\sum_{i=1}^{n} p_i = 1$. Denote

$$A = \begin{bmatrix} A_1 & A_2 & & A_n \\ p_1 & p_2 & & p_n \end{bmatrix} \tag{1.1}$$

We call the scheme (1.1) as a finite information scheme. Every finite scheme describes a state of uncertainty. Shannon [5] introduced a quantity which in a reasonable way, measures the amount of uncertianity (entropy) associated with a given finite scheme. This measure is given by

$$H(P) = -\sum_{i=1}^{n} p_i \log p_i \tag{1.2}$$

can serve as a very suitable measure of entropy of the finite scheme (1.1). Through out the paper, logarithms are taken to base $D(D \geq 2)$.

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Let $X=(x_1,x_2,...,x_n)$ be the finite set of input symbols which are to be encoded using alphabet of D symbols. It has been shown Feinstein [2] that there is a unique decipherable code with lengths $l_1,l_2,...,l_n$ and satisfying

$$\sum_{i=1}^{n} D^{-l_i} \le 1 \tag{1.3}$$

where D is the size of the code alphabet.
Noiseless coding theorem for Shannon's entropy with ordinary code mean length

$$\sum_{i=1}^{n} l_i p_i \tag{1.4}$$

under the condition (1.3),has played an important role in ordinary communication theory, (see Shannon [5]).

Khan and Haseen [3], Khan, Autar and Haseen [4], Boekee et al [1] have studied generalized coding theorems by considering different generalized measures of (1.2) and (1.4) under the condition (1.3) of unique decipherability.

In this paper, we study coding theorems by considering a new function depending on the parameters α and ν. Our motivation for studying this new function is that it generalizes some entropy functions already existing in the literature.

2. Coding theorems

Consider a function

$$H(P^{\nu},R)=\frac{R}{R-1}\left[1-\left(\frac{\sum_{i=1}^{n} p_i^{R+\nu-1}}{\sum_{i=1}^{n} p_i^{\nu}}\right)^{\frac{1}{R}}\right] \tag{2.1}$$

for all $R\in\Re_{+}\,(\neq 1),\nu\neq 1\ ,\sum_{i=1}^{n} p_i=1\ ,\ i=1,2,...,n$

(i) When $\nu=1$, (2.1) reduces to the R-norm information due to Boekee et al [1].
(ii) When $R\to 1,\nu=1$, (2.1) reduces to the measure due to Shannon [5].
Further consider

$$L(P^{\nu},R)=\frac{R}{R-1}\left[1-\frac{\sum_{i=1}^{n} p_i^{\nu} D^{-l_i\left(\frac{R-1}{R}\right)}}{\sum_{i=1}^{n} p_i^{\nu}}\right] \tag{2.2}$$

where $R\in\Re_{+}$, $R\neq 1$.
(i) For $\nu=1$, (2.2) reduces to the mean length due to Boekee et al [1].
(ii) For $R\to 1,\nu=1$, (2.2) reduces to the optimal code length defined by Shannon [5].

We now establish a result, that in a sense, gives a characterization of $H(P^{\nu},R)$ under the condition

(2.3) $$\sum_{i=1}^{n} p_i^{\nu-1} D^{-l_i} \le \sum_{i=1}^{n} p_i^{\nu}$$

Remark: For $\nu=1$ and $\sum_{i=1}^{n} p_i = 1$, (2.3) is a generalization of (1.3).

Theorem 1: For every code whose lengths $l_1, l_2, \ldots, l_n$ satisfies (2.3), the average length satisfies

(2.4) $$L(P^{\nu},R) \ge H(P^{\nu},R)$$

equality holds if and only if

(2.5) $$l_i = -\log \frac{p_i^{R}}{\left(\frac{\sum_{i=1}^{n} p_i^{R+\nu-1}}{\sum_{i=1}^{n} p_i^{\nu}}\right)}$$

Proof: we use Holders inequality [6]

(2.6) $$\sum_{i=1}^{n} x_i y_i \ge \left[\sum_{i=1}^{n} x_i^{p}\right]^{\frac{1}{p}} \left[\sum_{i=1}^{n} y_i^{q}\right]^{\frac{1}{q}}$$

for all $x_i > 0$, $y_i > 0$, $i = 1,2,\ldots,n$, $p < 1 (\ne 0)$ and $p^{-1} + q^{-1} = 1$

with equality if and only if there exists a positive number c such that

(2.7) $$x_i^{p} = c\, y_i^{q}$$

Setting

$$x_i = p_i^{\frac{\nu R}{1-R}} D^{-l_i}$$

$$y_i = p_i^{\frac{R+\nu-1}{1-R}}$$

$p = \frac{R-1}{R}$ and $q = 1-R$ in (2.6) and using (2.3), Also if $R > 1$ we get

(2.8) $$\left[\sum_{i=1}^{n} p_i^{\nu} D^{-l_i\left(\frac{R-1}{R}\right)}\right]^{\frac{R}{1-R}} \ge \frac{\left[\sum_{i=1}^{n} p_i^{R+\nu-1}\right]^{\frac{1}{1-R}}}{\sum_{i=1}^{n} p_i^{\nu}}$$

Dividing both sides of (2.8) by $\left(\sum_{i=1}^{n} p_i^{\nu}\right)^{\frac{R}{1-R}}$, we get

$$\left[\frac{\sum_{i=1}^{n} p_i^{\nu} D^{-l_i\left(\frac{R-1}{R}\right)}}{\sum_{i=1}^{n} p_i^{\nu}}\right]^{\frac{R}{1-R}} \geq \left[\frac{\sum_{i=1}^{n} p_i^{R+\nu-1}}{\sum_{i=1}^{n} p_i^{\nu}}\right]^{\frac{1}{1-R}}$$

Raising both sides to the power $\frac{1-R}{R}$, $R \neq 1$ also $\frac{R}{R-1} > 0$ for $R > 1$ and after suitable operations, we obtain the result (2.4). For $0 < R < 1$, the inequality (2.4) can be proved in a similar fashion

Theorem 2: For every code with lengths $l_1, l_2, ..., l_n$ satisfies (2.3). $L(P^{\nu}, R)$ can be made to satisfy the inequality

(2.9) $$L(P^{\nu}, R) < H_R(P^{\nu}, R) D^{\frac{1-R}{R}} + \frac{R}{R-1}\left(1 - D^{\frac{1-R}{R}}\right)$$

Proof: Let l_i be the positive integer satisfying the inequality

(2.10) $$-\log \frac{p_i^{R}}{\left(\frac{\sum_{i=1}^{n} p_i^{R+\nu-1}}{\sum_{i=1}^{n} p_i^{\nu}}\right)} \leq l_i < -\log \frac{p_i^{R}}{\left(\frac{\sum_{i=1}^{n} p_i^{R+\nu-1}}{\sum_{i=1}^{n} p_i^{\nu}}\right)} + 1$$

Consider the interval

(2.11) $$\delta_i = \left[-\log \frac{p_i^{R}}{\left(\frac{\sum_{i=1}^{n} p_i^{R+\nu-1}}{\sum_{i=1}^{n} p_i^{\nu}}\right)}, -\log \frac{p_i^{R}}{\left(\frac{\sum_{i=1}^{n} p_i^{R+\nu-1}}{\sum_{i=1}^{n} p_i^{\nu}}\right)} + 1\right]$$

of length 1. In every δ_i, there lies exactly one positive number l_i such that

(2.12) $$0 < -\log \frac{p_i^{R}}{\left(\frac{\sum_{i=1}^{n} p_i^{R+\nu-1}}{\sum_{i=1}^{n} p_i^{\nu}}\right)} \leq l_i < -\log \frac{p_i^{R}}{\left(\frac{\sum_{i=1}^{n} p_i^{R+\nu-1}}{\sum_{i=1}^{n} p_i^{\nu}}\right)} + 1$$

We will first show that sequence $\{l_1, l_2, ..., l_n\}$, thus defined satisfies (2.3), from (2.12) we have

$$-\log \frac{p_i^{R}}{\left(\dfrac{\sum_{i=1}^{n} p_i^{R+\nu-1}}{\sum_{i=1}^{n} p_i^{\nu}}\right)} \leq l_i$$

$$-\log \frac{p_i^{R}}{\left(\dfrac{\sum_{i=1}^{n} p_i^{R+\nu-1}}{\sum_{i=1}^{n} p_i^{\nu}}\right)} \leq -\log_D D^{-l_i}$$

(2.13) $$\frac{p_i^{R}}{\left(\dfrac{\sum_{i=1}^{n} p_i^{R+\nu-1}}{\sum_{i=1}^{n} p_i^{\nu}}\right)} \geq D^{-l_i}$$

Multiplying both sides by $\sum_{i=1}^{n} p_i^{\nu-1}$ and summing over $i = 1,2,...,n$, we get (2.3).

The last inequality in (2.12) gives

$$l_i < -\log \frac{p_i^{R}}{\left(\dfrac{\sum_{i=1}^{n} p_i^{R+\nu-1}}{\sum_{i=1}^{n} p_i^{\nu}}\right)} + 1$$

$$l_i < -\log \frac{p_i^{R}}{\left(\dfrac{\sum_{i=1}^{n} p_i^{R+\nu-1}}{\sum_{i=1}^{n} p_i^{\nu}}\right)} + \log_D D$$

i.e.,

$$D^{-l_i} < \frac{p_i^{R}}{\left(\dfrac{\sum_{i=1}^{n} p_i^{R+\nu-1}}{\sum_{i=1}^{n} p_i^{\nu}}\right)} D^{-1}$$

or
$$D^{-l_i\left(\frac{R-1}{R}\right)} < \left\{ \frac{p_i^{R}}{\left(\frac{\sum_{i=1}^{n} p_i^{R+\nu-1}}{\sum_{i=1}^{n} p_i^{\nu}} \right)} \right\}^{\frac{R-1}{R}} D^{\frac{1-R}{R}}$$

Multiplying both sides by $\frac{p_i^{\nu}}{\sum_{i=1}^{n} p_i^{\nu}}$ and summing over $i = 1,2,\dots,n$ and simplifying, gives (2.9). For $0 < R < 1$, the proof of the upper bound of $L(P^{\nu}, R)$ follows along the similar lines.

As $D \geq 2$, we have $\frac{R}{R-1}\left[1 - D^{\frac{(1-R)}{R}}\right] > 1$ from which it follows that the upper bound of $L(P^{\nu}, R)$ in (2.9) is greater than unity.

References:

(1)Boekee, E and Van Der Lubbe, J.C.A. "The R-norm information measure", Information and control, 45, 136-155 (1980).

(2) Feinstein, A. "Foundation of information theory", McGraw Hill, New York.

(3) Khan, A.B and Haseen, "Some noiseless coding theorems of entropy of order α of the power distribution P^{β}", Metron, 39 (3-4), 87-94 (1981).

(4) Khan, A.B, Autar, R and Haseen, "Noiseless coding theorems for generalized Non- additive entropy", Tamkang Journal of Mathematics, 12 (1), 16-19 (1981)

(5) Shannon, C.E, "The mathematical theory of communication", Bell.system Tech. Journal, 27,379-423,623-656 (1948).

(6) Shisha O." Inequalities", Academic press, New York(1967).