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CONVECTION IN A HORIZONTAL POROUS LAYER UNDERLYING A FLUID LAYER IN THE PRESENCE OF NON LINEAR MAGNETIC FIELD ON BOTH LAYERS

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Abstract : A linear stability analysis applied to a system consist of a horizontal fluid layer overlying a layer of a porous medium affected by a vertical magnetic field on both layers. Flow in porous medium is assumed to be governed by Darcy's law. The Beavers-Joseph condition is applied at the interface between the two layers. Numerical solutions are obtained for stationary convection case using the method of expansion of Chebyshev polynomials. It is found that the spectral method has a strong ability to solve the multi-layered problem and that the magnetic field has a strong effect in his model.

1. INTRODUCTION

The onset of convection in a system consisting of a horizontal fluid layer overlying a layer of a porous layer when the system is heated from below has been considered first by Sun (1973) who showed that the critical Rayleigh number in the porous layer decreases continuously as the thickness of the fluid layer is increased. He used the shooting method to solve the linear stability equations. Nield (1977) formulated the problem with surface-tension effects at a deformable upper surface and obtained asymptotic solutions for small wave numbers for a constant heat-flux boundary condition. Sun and Nield used Darcy's law in formulating the equations for porous layer and Somerton and Catton (1982) used the Brinkman term in the equation of motion to solve the problem using Galerkin method. Chen and Chen (1988) considered the problem with temperature and salinity gradients existing in both layers. Their investigation assumed stationary instability from the outset and they used a

Key Words :

shooting technique based on fourth order Runge-Kutta approximations for integration of all differential equations. Chen et. al (1991) studied the problem with anisotropic permeability and thermal diffusivity in the porous layer. Flow in porous layer was assumed to be governed by Darcy's law. The linearized stability equations were solved using shooting method. In the present study, we shall emulate the work of Chen and Chen (1988) in the presence of a vertical magnetic field. i.e. we shall consider the onset of thermal convection in a horizontal porous layer, affected by a vertical magnetic field, superposed by a fluid layer. The flow in porous layer is assumed to be governed by Darcy's law. The linear stability equations are solved using expansion of Chebyshev polynomials. This method has been used by Abdullah (1991) in the study of the Benard problem in the presence of a non-linear magnetic fluid and by Lindsay and Ogden (1992) in the implementation of spectral methods resistant to the generation of spurious eigenvalues. Lamb (1994) used this method to investigate an eigenvalue problem arising from a model discussing the instability in the earth's core. The method possesses excellent convergence characteristics and effectively exhibits exponential convergence rather than finite power convergence.

2. Mathematical Formulation

Let $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ be two horizontal layers such that the bottom of the layer $\boldsymbol{L}_1$ touches the top of the layer $\boldsymbol{L}_2$. A right handed system of Cartesian coordinates $(x_i,\ \ i=1,2,3)$ is chosen so that the interface is the plane $x_3=0$, the top boundary of $\boldsymbol{L}_1$ is $x_3=d_f$ and the lower boundary of $\boldsymbol{L}_2$ is $x_3=-d_m$. suppose that the upper layer $\boldsymbol{L}_1$ is filled with an incompressible thermally and electrically conducting viscous fluid field whereas the lower layer $\boldsymbol{L}_2$ is occupied by a porous medium permeated by the fluid and is subjected to a constant vertical magnetic field.

Gravity acts in the negative direction and the porous medium is heated at its lower boundary. Convection takes place in which temperature driven buoyancy effects are damped by viscous effects. A stationary fluid with a thermal gradient in the x_3 direction (the so called "conduction solution") is one possible solution to this problem and so it is natural to investigate its stability.

The fluid flow in the porous layer $\boldsymbol{L}_2$, with thickness d_m , is governed by Darcy's law, whereas the fluid flow in the upper layer $\boldsymbol{L}_1$, with thickness d_f , is governed by Navier-Stokes equations. Convection is driven by the temperature dependence of the fluid density. Typically, the Oberbeck-Boussinesq approximation is made in which local thermal equilibrium, heating from viscous dissipation, radiation effects ect. are ignored as are variations in fluid density except where they occure in the momentum equation. Let T denote the Kelvin temperature of the fluid and T_0 be a constant reference Kelvin temperature. For the purpose of this work, the fluid density ρ_f is related to T by

$$\rho_f = \rho_0[1-\alpha(T-T_0)] \tag{2.1}$$

where ρ_0 is the density of the fluid at T_0 and α (suppose constant) is the coefficient of volume expansion of the fluid.

Let $\boldsymbol{V}_m, \boldsymbol{H}_m, \boldsymbol{B}_m, \boldsymbol{J}_m$ and $\boldsymbol{E}_m$ be respectively the velocity of the fluid in the porous medium layer, magnetic field, magnetic induction, current density and electric field. the incompressibility of the fluid and the non-existence of magnetic monopoles require that $\boldsymbol{V}_m$ and $\boldsymbol{B}_m$ are both solenoidal vectors. Hence

$$div\ \boldsymbol{V}_m = 0, \tag{2.2}$$

$$div\ \boldsymbol{B}_m = 0, \tag{2.3}$$

Suppose that the magnetization in the fluid is directly proportional to the applied field and that the fluid behaves like an Ohmic conductor so that $\boldsymbol{H}_m, \boldsymbol{B}_m, \boldsymbol{J}_m$ and $\boldsymbol{E}_m$ are connected by the relations

$$\boldsymbol{H}_m = \rho\phi\,\boldsymbol{B}_m, \qquad \phi = \frac{1}{B}\frac{\partial \xi}{\partial B}, \qquad \xi = \xi(\rho, B)$$
$$\boldsymbol{J}_m = \sigma(\boldsymbol{E}_m + \boldsymbol{V}_m \times \boldsymbol{B}_m) \tag{2.4}$$

and the Maxwell equations

$$curl\,\boldsymbol{E}_m = -\frac{\partial \boldsymbol{B}_m}{\partial t},$$
$$\boldsymbol{J}_m = \frac{1}{4\pi} curl\,\boldsymbol{H}_m, \tag{2.5}$$

where μ_m (constant) is the magnetic permeability, σ is the electrical conductivity and the displacement current has been neglected in the second of these Maxwell equations as is customary in situation when free charge is instantaneously dispersed. On taking the curl of equations $(2.5)_2$ and replacing the electric field by the Maxwell equation $(2.5)_1$, the magnetic field $\boldsymbol{H}_m$ is now readily seen to satisfy the partial differential equation

$$\eta_m curl\,curl\,\boldsymbol{H}_m = -\frac{\partial \boldsymbol{H}_m}{\partial t} + curl(\boldsymbol{V}_m \times \boldsymbol{H}_m) \tag{2.6}$$

where $\eta_m = (4\pi\mu_m\sigma)^{-1}$ is the electrical resistivity. Equation (2.6) is now reworked using standard vector identities to yield

$$\frac{\partial \boldsymbol{H}_m}{\partial t} = (\boldsymbol{H}_m \cdot \nabla)\boldsymbol{V}_m - (\boldsymbol{V}_m \cdot \nabla)\boldsymbol{H}_m + \eta_m curl\,\boldsymbol{J}_m. \tag{2.7}$$

The relation (2.4) and (2.5) can be used to recast the Lorentz force $\boldsymbol{J} \times \boldsymbol{B}$ into

$$\boldsymbol{J}_m \times \boldsymbol{B}_m = \frac{1}{4\pi}(curl\,\boldsymbol{H}_m) \times \left(\frac{\boldsymbol{H}_m}{\rho\phi}\right) = \frac{1}{4\pi}\left[\boldsymbol{H}_m \cdot \left(\nabla \frac{\boldsymbol{H}_m}{\rho\phi}\right) - \nabla\left[\frac{\boldsymbol{H}_m^2}{2\rho\phi}\right]\right] \tag{2.8}$$

The field equations for this problem are written separately for the overlying fluid layer and porous medium layer. The governing equations for the fluid layer are

$$\rho_0\left(\frac{\partial \boldsymbol{V}_f}{\partial t}+\boldsymbol{V}_f.\nabla \boldsymbol{V}_f\right)=-\nabla P_f+\mu\nabla^2\boldsymbol{V}_f+\rho_f\boldsymbol{g}+\frac{1}{4\pi}\boldsymbol{H}_f\cdot\nabla\left(\frac{\boldsymbol{H}_f}{\rho\phi}\right),$$
$$(\rho c_p)_f\left(\frac{\partial T_f}{\partial t}+\boldsymbol{V}_f.\nabla T_f\right)=k_f\nabla^2 T_f, \tag{2.9}$$
$$\frac{\partial \boldsymbol{H}_f}{\partial t}=(\boldsymbol{H}_f\cdot\nabla)\boldsymbol{V}_f-(\boldsymbol{V}_f\cdot\nabla)\boldsymbol{H}_f+\eta_f curl\,\boldsymbol{J}_f.$$

where T_f is the Kelvin temperature of the fluid layer, P_f is the hydrostatic pressure, $\boldsymbol{g}$ is the acceleration due to gravity, μ is the dynamic viscosity of the fluid, $(\rho c_p)_f$ is the heat capacity per unit volume of the fluid at constant pressure and k_f is the thermal conductivity of the fluid. The governing equations for porous medium are given by

$$\frac{\rho_0}{\phi}\frac{\partial \boldsymbol{V}_m}{\partial t}=-\nabla P_m-\frac{\mu}{k}\boldsymbol{V}_m+\rho_f\boldsymbol{g}+\frac{1}{4\pi}\boldsymbol{H}_m\cdot\nabla\left(\frac{\boldsymbol{H}_m}{\rho\phi}\right),$$
$$(\rho c)_m\frac{\partial T_m}{\partial t}+(\rho c_p)_f\boldsymbol{V}_m.\nabla T_m=k_m\nabla^2 T_m, \tag{2.10}$$
$$\frac{\partial \boldsymbol{H}_m}{\partial t}=(\boldsymbol{H}_m\cdot\nabla)\boldsymbol{V}_m-(\boldsymbol{V}_m\cdot\nabla)\boldsymbol{H}_m+\eta_m curl\,\boldsymbol{J}_m.$$

where T_m is the Kelvin temperature of the porous medium layer, $\boldsymbol{V}_m$ is the solenoidal seepage velocity, P_m is the hydrostatic pressure, k is the permeability of the porous medium, ϕ is its porosity, k_m is the overall thermal conductivity of the porous medium and $(\rho c)_m$ is the overall heat capacity per unit volume of porous medium at constant pressure. In fact

$$(\rho c)_m=\phi(\rho c_p)_f+(1-\phi)(\rho c_p)_m$$

where $(\rho c_p)_m$ is the heat capacity per unit volume of porous substrate. The convection problem is completed by the specification of boundary conditions at the upper surface of the viscous fluid layer, at the interface between the fluid and porous medium layers and at the lower boundary of the porous medium layer. Many combination of boundary conditions are possible but for comparison with Chen and Chen (1988) we shall assume that $x_3 = d_f$ is rigid and held at constant temperature T_u , whereas $x_3 = -d_m$ is assumed to be impenetrable and at constant temperature T_m . In terms of w_f and w_m the axial velocity components of the fluid in $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ respectively, these requirements leads to the three conditions

$$T_f(d_f) = T_u, \qquad w_f(d_f) = 0, \qquad \frac{\partial w_f(d_f)}{\partial x_3} = 0, \qquad \frac{\partial H(d_f)}{\partial x_3} = 0. \tag{2.11}$$

on the top boundary of $\boldsymbol{L}_1$ and the conditions

$$T_m(-d_m) = T_l, \qquad w_m(-d_m) = 0, \qquad \frac{\partial H(-d_m)}{\partial x_3} = 0. \tag{2.12}$$

on the lower boundary of $\boldsymbol{L}_2$ where H is the third component of the magnetic field. The fluid/ porous –medium interface boundary conditions are based on the assumption that temperature, heat flux and normal fluid velocity are continuous across the interface. Thus

$$\begin{aligned} T_m(0) &= T_f(0), & k_m \frac{\partial T_m(0)}{\partial x_3} &= k_f \frac{\partial T_f(0)}{\partial x_3}, \\ w_m(0) &= w_f(0), & -P_f(0) + 2\mu \frac{\partial w_f(0)}{\partial x_3} &= -P_m(0), \\ H_m(0) &= H_f(0), & k_m \frac{\partial H_m(0)}{\partial x_3} &= k_f \frac{\partial H_f(0)}{\partial x_3}. \end{aligned} \tag{2.13}$$

this leaves two final conditions to be specified on the interface. One of these is related to the magnetic field which is

$$\frac{\partial H_f(0)}{\partial x_3} = 0, \qquad \frac{\partial H_m(0)}{\partial x_3} = 0 \tag{2.14}$$

and the final one is due to Beavers and Joseph (1967) which has the form

$$\frac{\partial u_f(0)}{\partial x_3} = \frac{\alpha_{BJ}}{\sqrt{K}}(u_f - u_m), \qquad \frac{\partial v_f(0)}{\partial x_3} = \frac{\alpha_{BJ}}{\sqrt{K}}(v_f - v_m), \tag{2.15}$$

where u_f, v_f are the limiting tangential components of the fluid velocity as the interface is approached from the fluid layer L_1, whereas u_m, v_m are the same limiting components of tangential fluid velocity as the interface is approached from the porous layer L_2.

Suppose that the static solution is now perturbed so that the velocity, pressure, temperature and magnetic field in the fluid and porous layers are respectively

$$v_f, \qquad P_f + p_f, \qquad T_0 - (T_0 - T_u)\frac{x_3}{d_f} + \theta_f, \qquad He_3 + h_m,$$

and (2.16)

$$v_m, \qquad P_m + p_m, \qquad T_0 - \left(T_l - \frac{T_u}{T_0}\right)\frac{x_3}{d_m} + \theta_m.$$

In the fluid layer we shall introduce the non-dimensional spatial coordinates $\hat{x}_f$, time $\hat{t}_f$, perturbed velocity $\hat{v}_f$, pressure $\hat{p}_f$, and temperature $\hat{\theta}_f$ by the definitions

$$x = d_f \hat{x}_f, \qquad t_f = \frac{d_f^2}{\lambda_f}\hat{t}_f, \qquad v_f = \frac{\lambda_f}{d_f}\hat{v}_f,$$
$$p_f = \frac{\mu\lambda_f}{d_f^2}\hat{p}_f, \qquad \theta_f = |T_0 - T_u|\hat{\theta}_f. \tag{2.17}$$

where λ_f is the thermal diffusivity of the fluid phase defined by $\lambda_f = \frac{k_f}{(\rho c_p)_f}$.

A similar procedure is applied to the porous medium layer in which non-dimensional spatial coordinates $\hat{x}_m$, time $\hat{t}_m$, perturbed velocity $\hat{v}_m$, pressure $\hat{p}_m$ magnetic field $\hat{h}_m$ and temperature $\hat{\theta}_m$ are introduced by the definitions

$$\begin{aligned} \boldsymbol{x}_m &= d_m \hat{\boldsymbol{x}}_m, & t_m &= \frac{d_m^2}{\lambda_m}\hat{t}_m, & v_m &= \frac{\lambda_m}{d_m}\hat{v}_m, \\ p_m &= \frac{\mu\lambda_f}{K}\hat{p}_m, & \boldsymbol{h}_m &= \frac{H_m\lambda_m}{\eta_m}\hat{\boldsymbol{h}}_m, & \theta_m &= |T_0 - T_u|\hat{\theta}_m. \end{aligned} \tag{2.18}$$

where λ_m is the thermal diffusivity of the porous medium layer defined by

$$\lambda_m = \frac{k_m}{(\rho c_p)_f}.$$

Where $\beta = sign(T_0 - T_u) = sign(T_l - T_0)$ and hat superscript has been dropped although the variables are non-dimensional. By taking the *curl curl* of the momentum equation in each layer then taking the third component of the equations in each layer. We now look for a solution of the form

$$\psi(t, \boldsymbol{x}) = \psi(x_3)\exp[i(rx_1 + qx_2) + \sigma\, t]$$

It follows from equations (2.33) and (2.34) that

$$\begin{aligned} \frac{\sigma_f}{P_{r_f}}(D_f^2 - a_f^2)w_f - \sigma_f Q P_{m_f}^{-1} D_f h_f &= (D_f^2 - a_f^2)^2 w_f - Ra_f a_f^2 \theta_f - Q D_f^2 w_f, \\ \sigma_f \theta_f &= w_f + (D_f^2 - a_f^2)\theta_f, \\ \sigma_f P_{m_f}^{-1} h_f &= (D_f^2 - a_f^2)h_f + D_f w_f, \\ -\frac{Da}{\phi}\frac{\sigma_m}{P_{r_m}}(D_m^2 - a_m^2)w_m - \sigma_m Q P_{m_m}^{-1} D_m h_m &= (D_m^2 - a_m^2)^2 w_m + Ra_m a_m^2 \theta_m - Q D_m^2 w_m, \\ G_m \sigma_m \theta_m &= w_m + (D_m^2 - a_m^2)\theta_m, \\ \sigma_m P_{m_m}^{-1} h_m &= (D_m^2 - a_m^2)h_m + D_m w_m. \end{aligned} \tag{2.19}$$

where the parameters $\varepsilon_T, \hat{d}$ and $\hat{k}$ are defined by

$$\varepsilon_T = \frac{\hat{d}}{\hat{k}}, \qquad \hat{d} = \frac{d_m}{d_f}, \qquad \hat{k} = \frac{k_m}{k_f}.$$

and $G_m = \dfrac{(\rho c)_m}{(\rho c_p)_f}$, and

$$P_{r_f} = \frac{\nu}{\lambda_f}, \qquad Ra_f = \frac{g\alpha d_f^3 |T_0 - T_u|}{\nu \lambda_f}, \qquad P_{m_f} = \frac{\eta_f}{\lambda_f}.$$

$$P_{r_m} = \frac{\nu}{\lambda_m}, \qquad Da = \frac{K}{d_m^2}, \qquad Ra_f = \frac{g\alpha K d_f |T_l - T_0|}{\nu \lambda_m},$$

$$P_{m_m} = \frac{\eta_m}{\lambda_m}, \qquad Q = \frac{\mu H_m^2 d_m^2}{4\rho_0 \pi \nu \eta_m}.$$

where $a_m^2 = r_m^2 + q_m^2$, $a_f^2 = r_f^2 + q_f^2$ are non-dimensional wave numbers in the porous medium and fluid layers respectively and where $D(\ ) = \dfrac{\partial}{\partial x_3}(\ )$, $a_f = \hat{d} a_m$, $\sigma_f = \dfrac{\hat{d}^2}{\hat{k}} \sigma_m$. The final boundary conditions are:

Upper boundary $x_3 = 1$

$$w_f = 0, \qquad D_f w_f = 0, \qquad \theta_f = 0, \qquad D_f h_f = 0. \tag{2.20}$$

Middle boundary $x_3 = 0$

$$\theta_f = \varepsilon_T \theta_m, \qquad D_f \theta_f = D_m \theta_m, \qquad w_m = \varepsilon_T w_f,$$

$$h_f = \frac{\hat{k}}{\hat{n}} h_m, \qquad D_f h_f = \frac{1}{\varepsilon_T \hat{n}} D_m h_m,$$

$$\varepsilon_T \hat{d}\left(D_f w_f - \frac{\hat{d}\sqrt{Da}}{\alpha_{BJ}} D_f^2 w_f \right) = D_m w_m, \tag{2.21}$$

$$\hat{d}^3 \varepsilon_T Da\left(D_f^3 w_f - 3a_f^2 D_f w_f - \frac{\sigma_f}{P_{r_f}} D_f w_f \right) = -\left(\frac{Da}{\phi} \frac{\sigma_m}{P_{r_m}} + 1 \right) D_m w_m.$$

Lower boundary $x_3 = -1$

$$w_m = 0, \qquad \theta_m = 0, \qquad D_m h_m = 0. \tag{2.22}$$

3. RESULTS AND DISCUSSION

The eigenvalue problem consist of a eight order ordinary differential equation in the fluid layer and a six order ordinary differential equation in the porous layer with 14 boundary conditions. This problem is solved using spectral method based on series expansion of Chebyshev polynomials. In producing the results, σ_f and σ_m are set to zero identically which corresponds to the stationary convection instability, where the relation between σ_f and σ_m are given by

$$\sigma_f = \frac{\hat{d}^2}{\hat{k}}\sigma_m$$

Numerical results and stability curves are obtained for the problem, with thermal conductivity ratio $\hat{k} = 1.43$, Darcy number = 4×10^{-6}, Beavers-Joseph constant $\alpha_{BJ} = 0.1$ and for a variety reciprocal depth ratio ranging from 0.33 to 0.1. the results of this paper are illustrated in figures (1) – (4). They are qualitatively and quantitively similar to those produced by Bukhari (1997) in the absence of magnetic field in both layers. Bukhari has showed that the numerical results produced by Chen and Chen (1988) have a large rounding error due to the method used which is a 4th order Runge-Kutta method and he showed that the spectral methods have a strong ability to solve the multi-layered problems and produces accurate results.

Figure (1) shows the relation between a_m and Ra_m for different values of the depth ratio of $\hat{d}$ when the Chandrasekhar number $Q = 100$. it is clear from the figure that the Rayleigh number in the porous layer decreases continuously as the thickness of the layer increases. The results corresponding to $Q = 500, 1000, 10000$ are displayed in figures (2)- (4) respectively. It is clear from these figures that the

Rayleigh number increases as the Chandrasekhar number increases. i.e. the magnetic field has a stabilizing on the system.

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