The Minimum PAPR Code for OFDM Systems

Seog Geun Kang

ABSTRACT—In this letter, a block code that minimizes the peak-to-average power ratio (PAPR) of orthogonal frequency division multiplexing (OFDM) signals is proposed. It is shown that peak envelope power is invariant to cyclic shift and codeword inversion. The systematic encoding rule for the proposed code is composed of searching for a seed codeword, shifting the register elements, and determining codeword inversion. This eliminates the look-up table for one-to-one correspondence between the source and the coded data. Computer simulation confirms that OFDM systems with the proposed code always have the minimum PAPR.

Keywords—Wireless communications, OFDM, peak-to-average power ratio, peak envelope power, block codes.

I. Introduction

Due to its inherent merits, orthogonal frequency division multiplexing (OFDM) has been adopted or recommended as a powerful candidate for various wireless communications [1], [2]. One of the major drawbacks of OFDM is the high peak-to-average power ratio (PAPR) in the transmitted signals. There have been many researches to reduce the PAPR [3]-[8]. In block coding schemes [3]-[5], codewords having a relatively low PAPR are selectively used to transmit the source information. Since the scheme can be combined with an error control code [3], it is appropriate for high-quality wireless communications.

Most block coding schemes [3], [4] that have a unique encoding rule have a limitation in PAPR reduction. The scheme introduced in [5] can achieve the minimized PAPR, but has no generalized encoding rule. Hence, it may require a huge look-up table for one-to-one correspondence between the source and the coded data when the codeword length is increased, as pointed out in [8].

In this letter, a block code that guarantees the minimized PAPR in OFDM signals is proposed. It is also verified that the peak envelope power (PEP) is invariant to the cyclic shift of the register elements and codeword inversion. Based on such properties, a systematic encoding rule for the proposed minimum PAPR code (MPC) is presented, which consists of three steps: a search for an appropriate seed codeword, cyclic shift of the register elements, and codeword inversion. This rule enables the MPC encoder to generate all codewords using only a few seed codewords so that a look-up table is not necessary.

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II. Design of the Minimum PAPR Codes

1. Definition of Seed Codeword

The PEP of OFDM signals is generally computed as

$$\max_{0\le n\le N-1}|x_n|^2 = \max_{0\le n\le N-1}\left\{\frac{1}{N^2}\sum_k\sum_m X_k X_m^* e^{j2\pi n(k-m)/N}\right\}. \quad (1)$$

Thus, the minimum PEP is the solution of a min-max algorithm as shown below.

$$\min\left\{\max_{0\le n\le N-1}\left\{\frac{1}{N^2}\sum_\tau |X_\tau|^2\right\}\right\}. \quad (2)$$

An exhaustive search usually requires high computational complexity, but provides a full set of codewords of length N that has the minimum PEP.

Let $\boldsymbol{C}=\{C_k \mid 0\le k\le K\text{-}1\}$ be the set of entire codewords having the minimum PEP. $\boldsymbol{C}$ can be divided into L disjoint subsets of codewords, that is, a codeword $C_k\in\boldsymbol{C}_j$ cannot be the element of $\boldsymbol{C}_l$, $0\le j,\ l\le L\text{-}1,\ j\ne l$. Then, there are L seed codewords chosen from the L disjoint subsets. A seed codeword is, therefore, the representative of each subset.

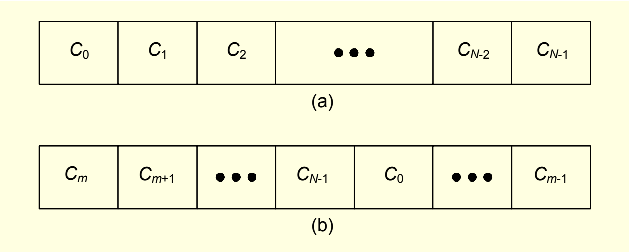

Fig. 1. The codeword stored in a shift register: (a) initial state and (b) after m cyclic shifts.

2. PEP Invariance to Cyclic Shift and Codeword Inversion

Let a binary word $C_s = [c_0\ c_1 \cdots c_{N-1}]$ be a seed codeword of length N, and fed to an N-bit shift register. Actually, the accurate notation of the seed codeword is $C_{j,s} \in \boldsymbol{C}_j$. We shorten the index hereafter if it does not result in confusion. Here, a seed codeword is a codeword whose binary phase shift keying (BPSK)-mapped real sequence $\mathbf{X}_s = [X_0\ X_1 \cdots X_{N-1}]$ has the minimum PAPR after an inverse discrete Fourier transform (IDFT). Consider a new codeword C_s^m produced by m cyclic shifts on C_s as shown in Fig. 1(b). The corresponding real sequence is $\mathbf{X}_s^m = [X_m\ X_{m+1} \cdots X_{N-1}\ X_0\ X_1 \cdots X_{m-1}]$. Then, the transmitted signal of the new sequence is

$$x_n^m = \frac{1}{N}\mathbf{X}_s^m \mathbf{W}_n, \tag{3}$$

where $\mathbf{W}_n = [1\ e^{j2\pi n/N}\ e^{j4\pi n/N} \cdots e^{j2\pi n(N-1)/N}]^T$ is an $N \times 1$ column vector of the $N \times N$ IDFT matrix $\mathbf{W}_N$. The superscript T denotes the transpose of the matrix. The vector $\mathbf{W}_n$ can be represented as

$$\mathbf{W}_n = e^{-j2\pi nm/N}\mathbf{W}_n^m, \tag{4}$$

where $\mathbf{W}_n^m = [\ e^{j2\pi nm/N}\ e^{j2\pi n(m+1)/N} \cdots e^{j2\pi n(N-1)/N}\ 1 \cdots e^{j2\pi n(m-1)/N}]^T$. Hence, the signal in (3) is modified as

$$x_n^m = \frac{1}{N}e^{-j2\pi nm/N}\ \mathbf{X}^m \mathbf{W}_n^m = e^{-j2\pi nm/N}\ x_n, \tag{5}$$

where x_n is the transmitted signal of the seed. Note that a cyclic shift of the register elements results in a phase shift after IDFT. As the phase shift does not alter the PEP of the OFDM signals, both the new and seed codewords have the same PEP. Thus, N–1 codewords having the minimum PAPR exist for every seed.

Consider a codeword $\overline{C}_s = [\overline{c_0}\ \overline{c_1} \cdots \overline{c_{N-1}}]$ produced by codeword inversion of elements of seed C_s. The modulated signal after BPSK mapping of $\overline{C}_s$ is

$$\overline{x}_n = \frac{1}{N}\overline{\mathbf{X}}_s\ \mathbf{W}_n. \tag{6}$$

Since $\overline{X}_k = -X_k$ with the BPSK signal mapper, the transmitted signal is computed as

$$\overline{x}_n = -\frac{1}{N}\mathbf{X}_s \mathbf{W}_n = e^{-j\pi}x_n. \tag{7}$$

As inverting the sign of a codeword is the same as a rotation by $e^{j\pi}$, it does not change the power distribution of the transmitted signals. Thus, the PEP is invariant to the codeword inversion. It is clear that we have an additional set of N codewords with the inversion. Thus, a seed codeword is the representative of a disjoint subset consisting of a total of $2N$ codewords of length N. This implies that a set of seed codewords can be used to find all the codewords of an MPC. In addition, the invariance properties are still valid although the signal mapper generates a complex sequence.

3. A Systematic Encoding Rule for the MPC

Assume that there exist N_s seed codewords. Then, it is possible to produce an MPC consisting of $2NN_s$ codewords. This results in an $(N, \log_2(2NN_s))$ block coding scheme as shown in Fig. 2(a). The source binary information of length $\log_2(2NN_s)$ bits is classified into three parts by their functions. First, $\log_2 N_s$ least significant bits (LSB) are exploited to select one of the N_s seed codewords. If only one seed is available, that is, $\log_2 N_s = 0$, no seed selection procedure is required. It can be shown that such a special case occurs in a (4, 3) MPC. The next $\log_2 N$ bits determine the number of cyclic shifts of the register elements. Finally, the register elements are inverted to generate a coded bit sequence if the most significant bit (MSB) of the source data is ‘1’. Otherwise, the

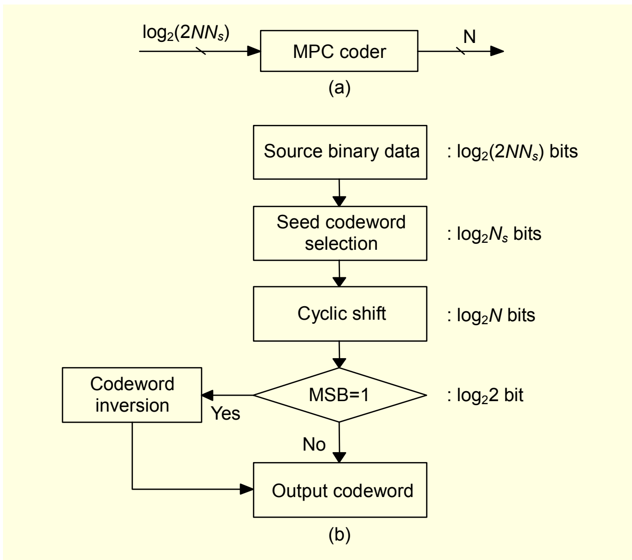

Fig. 2. (a) The MPC encoder. (b) the systematic encoding rule for the MPC.

elements are the output codeword. Such an encoding rule is illustrated in Fig. 2(b).

III. Numerical Analysis

To verify if the proposed code really provides the minimum PAPR characteristic, computer simulation for 8- and 16-subcarrier OFDM systems with BPSK signal mapper is carried out. As a performance measure, a complementary cumulative distribution function (CCDF) of the PAPR defined as CCDF(P_{ref}) = 1 − Pr($PAPR > P_{ref}$) [6] is used. Here, P_{ref} is a reference value of the PAPR. Pr($PAPR > P_{ref}$) is the probability that the PAPR of the OFDM signals exceeds the reference value.

Example 1: *An 8-subcarrier OFDM*

There are two seed codewords of length 8: one is '00001011' and the other is '00001101'. Thus, $2NN_s$ = 32 codewords constituting an (8, 5) MPC exist. That is, 5-bit source information is encoded to generate a codeword of length 8.

Consider that a 5-bit source word is '11010' as an example. The LSB chooses one of two possible seeds. Assume that '00001011' is selected as the seed codeword. Since the next $\log_2 N$ bits are '101', we have '01100001' by five cyclic shifts of the register elements. As the MSB of the source word is '1', the register elements are inverted to '10011110' and mapped with BPSK to produce a real sequence '1 -1 -1 1 1 1 1 -1'. CCDFs of an uncoded 8-subcarrier and the (8, 5) MPC coded OFDM are plotted in Fig. 3.

It is observed that the uncoded 8-subcarrier OFDM system has eight levels of PAPR in the transmitted signals. The theoretical maximum and minimum values are 9.03 and 1.76 dB, respectively. Signals having the PAPR of 3.01 dB are produced most frequently. Unlike the uncoded scheme, the MPC coded OFDM system has a unique PEP so that it shows only one PAPR level. The PAPR is 1.76 dB, which is the minimum value achievable in the uncoded system.

Example 2: *A 16-subcarrier OFDM*

In the case of a 16-subcarrier OFDM, eight 16-bit seed codewords are found by an exhaustive search. The seed words are presented in Table 1. By the cyclic shift and codeword inversion, the set of seeds can be expanded to 256 codewords to form a (16, 8) MPC. CCDFs of an uncoded 16-subcarrier and the (16, 8) MPC coded OFDM are presented in Fig. 4.

It can be observed that the possible range of the PAPR in the uncoded OFDM is from 1.76 to 12.04 dB. As can be expected, with an increase in the codeword length by a factor of 2, the maximum PAPR value is increased by 3.01 dB. While such minimum and maximum values rarely occur, the signals having a PAPR of 6.02 dB are produced most frequently. The probability that the PAPR exceeds 6.0 dB is around 0.25. Thus,

Table 1. Seed codewords for the (16, 8) MPC code.

No.	Seed codeword	No.	Seed codeword
0	0000011010110111	4	0000101111001101
1	0000011101101011	5	0000110100111011
2	0000101100111101	6	0000110111001011
3	0000101101100111	7	0000111001101101

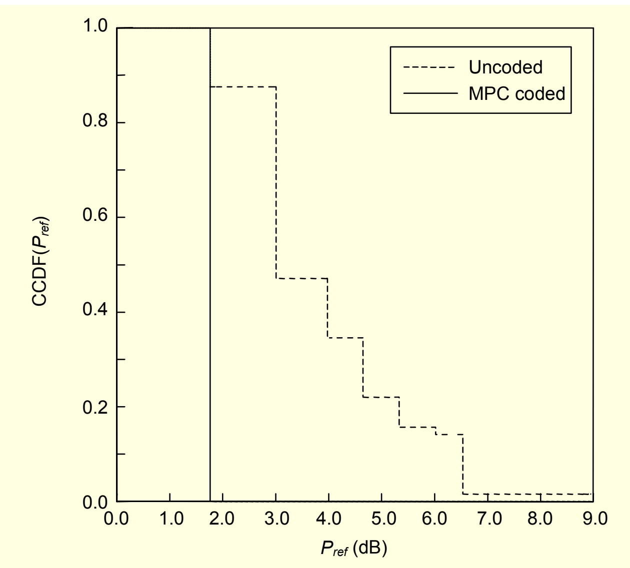

Fig. 3. CCDFs of an uncoded 8-subcarrier and the (8, 5) MPC coded OFDM.

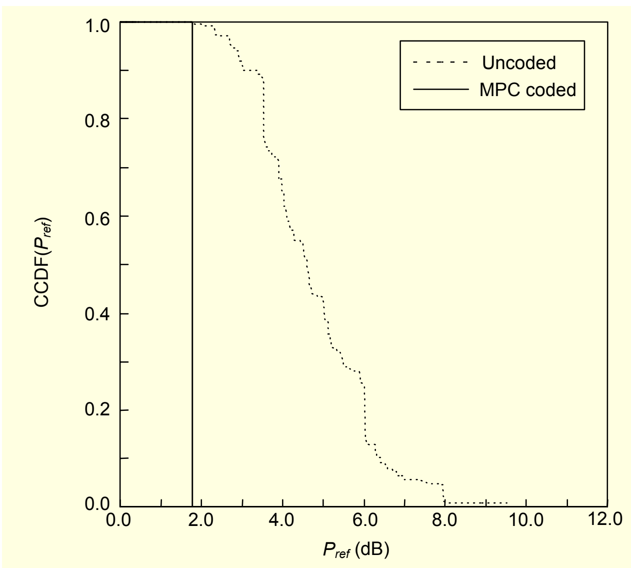

Fig. 4. CCDFs of an uncoded 16-subcarrier and the (16, 8) MPC coded OFDM.

a quarter of the OFDM signals have a PAPR of 6.0 dB or more. Compared to that, the coded OFDM has a constant PEP in the transmitted signals, and the PAPR is only 1.76 dB. The PAPR can be reduced by 10.2 dB at maximum. It is, therefore, clear that the MPC guarantees the minimum PAPR.

IV. Conclusions and Further Studies

In this work, a block code that has an extreme PAPR performance is introduced. And it is shown that the PEP of the OFDM signals is invariant to cyclic shift and codeword inversion. Based on such properties, a systematic encoding rule that does not depend on the codeword length is also presented. Due to the encoding rule, the look-up table to map the source data can be eliminated in the MPC encoder. Thus, the required memory size can be reduced significantly. Simulation results confirm that OFDM systems with the proposed code always have the minimum PAPR.

Since the sets of seed codewords provided in this letter are found by an exhaustive search, it must be the same for a fixed codeword length. Hence, no further investigations to find the set of seed codewords other than that presented in this letter are needed. However, a further study on the way of finding the set of seed codewords efficiently is needed.

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