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A Complete Convergence Theorem for Randomly Weighted Sums of Random Variables

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확률변수의 랜덤 가중합에 대한 완전수렴

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Abstract

A complete convergence theorem for randomly weighted sums of random variables is obtained. No conditions are imposed on the joint distributions of the random variables.

요 약

확률변수의 랜덤 가중합에 대한 완전수렴 정리를 얻는다.

Key words: Complete convergence, weighted sums, random weights, random variables, random elements, almost sure convergence

1. Introduction

The concept of complete convergence of a sequence of random variables was introduced by

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Hsu and Robbins (1947) as follows. A sequence $\{U_n, n\geq 1\}$ of random variables converges completely to the constant θ if

$$\sum_{n=1}^{\infty} P(|U_n-\theta|>\varepsilon)<\infty \text{ for all } \varepsilon>0.$$

In view of the Borel-Cantelli lemma, this implies that $U_n\rightarrow\theta$ almost surely (a.s.). The converse is true if $\{U_n, n\geq 1\}$ are independent random variables. Hsu and Robbins (1947) proved that the sequence of arithmetic means of independent and identically distributed random variables converges completely to the expected value if the variance of the summands is finite. Erdos (1949) proved the converse. We refer to Gut (1994) for a survey on results on complete convergence related to strong laws and published before the 1990s.

The result of Hsu-Robbins-Erdos has been generalized and extended in several directions. Some of these generalizations are in a Banach space setting. A sequence of Banach space valued random elements is said to converge completely to the 0 element of the Banach space if the corresponding sequence of norms converges completely to 0.

In this paper, we obtain a complete convergence theorem for randomly weighted sums of random variables.

2. Main Result

To prove our main result, we will need the following lemma which was proved by Sung (1988).

Lemma 1. Let $\{X_n, n\geq 1\}$ be a sequence of random variables which are stochastically dominated by a random variable X with $E|X|^r<\infty$ for some $r>0$. That is,

$P(|X_n|>x)\leq P(|X|>x)$ for all $x>0$ and $n\geq 1$.

Then the followings hold.

(i) $\sum_{n=1}^{\infty}\frac{1}{n^{\beta/r}}E|X_n|^{\beta}I(|X_n|\leq n^{1/r})<\infty$ for $0<r<\beta$.

(ii) $\sum_{n=1}^{\infty}\frac{1}{n^{\alpha/r}}E|X_n|^{\alpha}I(|X_n|>n^{1/r})<\infty$ for $0<\alpha<r$.

We state and prove our main result.

Theorem 1. Let $\{X_n, n\geq 1\}$ be a sequence of random variables which are stochastically dominated by a random variable X with $E|X|^r < \infty$ for some $r > 1$. Let $\{A_{ni}, 1\leq i\leq n, n\geq 1\}$ be an array of random variables such that

$$\max_{1\leq i\leq n}|A_{ni}| = O(\frac{1}{n^\alpha}) \text{ a.s. for some } \alpha\geq 1+\frac{1}{r}. \tag{1}$$

Then $\sum_{i=1}^{n} A_{ni}X_i \to 0$ completely.

Proof. Define $X_n' = X_n I(|X_n|\leq n^{1/r})$ and $X_n'' = X_n I(|X_n| > n^{1/r})$ for $n\geq 1$. Then $X_n' + X_n'' = X_n$ for $n\geq 1$. To prove the result, it suffices to show that for every $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P(|\sum_{i=1}^{n} A_{ni}X_i'| > \varepsilon) < \infty \tag{2}$$

and

$$\sum_{n=1}^{\infty} P(|\sum_{i=1}^{n} A_{ni}X_i''| > \varepsilon) < \infty. \tag{3}$$

To prove (2), choose a real number s such that $r < s$. By the Holder inequality and (1), we have that

$$|\sum_{i=1}^{n} A_{ni}X_i'| \leq (\sum_{i=1}^{n}|A_{ni}|^{\frac{s}{s-1}})^{\frac{s-1}{s}} (\sum_{i=1}^{n}|X_i'|^s)^{\frac{1}{s}} \leq C\frac{n^{\frac{s-1}{s}}}{n^\alpha}(\sum_{i=1}^{n}|X_i'|^s)^{\frac{1}{s}} \text{ a.s.}$$

It follows that

$$\begin{aligned}
\sum_{n=1}^{\infty} P(|\sum_{i=1}^{n} A_{ni}X_i'| > \varepsilon) &\leq \frac{1}{\varepsilon^s}\sum_{n=1}^{\infty} E|\sum_{i=1}^{n} A_{ni}X_i'|^s \\
&\leq C\sum_{n=1}^{\infty}\frac{1}{n^{\alpha s - s+1}}\sum_{i=1}^{n} E|X_i'|^s \\
&= C\sum_{i=1}^{\infty} E|X_i'|^s \sum_{n=i}^{\infty}\frac{1}{n^{\alpha s - s+1}} \\
&\leq C\sum_{i=1}^{\infty}\frac{1}{i^{\alpha s - s}} E|X_i'|^s \\
&\leq C\sum_{i=1}^{\infty}\frac{1}{i^{s/r}} E|X_i'|^s < \infty.
\end{aligned}$$

The last inequality follows by Lemma 1 (i). Hence (2) holds.

Next, we will prove that (3) holds. By (1) and Lemma 1, we have that

$$\begin{aligned}\sum_{n=1}^{\infty} P(|\sum_{i=1}^{n} A_{ni}X_i''|>\varepsilon) &\le \frac{1}{\varepsilon}\sum_{n=1}^{\infty} E|\sum_{i=1}^{n} A_{ni}X_i''| \\ &\le \frac{1}{\varepsilon}\sum_{n=1}^{\infty} E(\max_{1\le i\le n}|A_{ni}|\sum_{i=1}^{n}|X_i''|) \\ &\le C\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}\sum_{i=1}^{n} E|X_i''|<\infty.\end{aligned}$$

Theorem 1 can be generalized to Banach space setting.

Corollary 1. Let $\{X_n, n\ge 1\}$ be a sequence of random elements which are stochastically dominated by a random variable X with $E|X|^r<\infty$ for some $r>1$. Let $\{A_{ni}, 1\le i\le n, n\ge 1\}$ be an array of random variables such that $\max_{1\le i\le n}|A_{ni}| = O(\frac{1}{n^{\alpha}})$ a.s. for some $\alpha\ge 1+\frac{1}{r}$.

Then $\sum_{i=1}^{n} A_{ni}X_i \to 0$ completely.

Proof. Note that $||\sum_{i=1}^{n} A_{ni}X_i||\le \sum_{i=1}^{n}|A_{ni}|\,||X_{ni}||$. Hence Corollary 1 follows from Theorem 1.

Remark 1. Ordonez Cabrera (1988) proved Corollary 1 when $\alpha>1+\frac{1}{r}$. So Corollary 1 is an extension of Ordonez Cabrera's theorem.

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