KYUNGPOOK Math. J. 46(2006), 139-144

The Invariant of Immersions under Isotwist Folding

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Abstract. In this paper, we will introduce the isotwist foldings of a manifold M into itself. The limits of the isotwist foldings of a manifold are obtained. Also the relations between conditional retraction and this type of the foldings are achieved. Finally the variant and invariant of the immersion under this type of foldings are deduced.

1. Introduction

The notion of isometric folding was introduced by S. A. Robertson who studied the stratification determined by the folds or singularities [13]. Then the theory of isometric foldings has been pushed and also different types of foldings are introduced by E. EL-Kholy and others [1], [2], [5]. The conditional foldings of manifolds are defined by M. El-Ghoul in [6], [7], [8], [9], [10]. Some applications on the folding of a manifold into it self was introduced by P. Di. Francesco in [12].

Let $\alpha : (a, b) \to E^3$ be a regular curve and let $t_0 \in (a, b)$. Set $L(t) = \int_{t_0}^{t} \left|\frac{d\alpha}{dt}\right| dt$. $s = L(t)$ is called arc length along α. If the curve α is parametrized by the arc length s then $\left|\frac{d\alpha}{ds}\right| = 1$. So let $\alpha : (a, b) \to E^3$ be a unit speed curve, then the principal normal vector field is a (unit) vector field $N(s) = T'(s)/\kappa(s)$. The binormal vector filed to $\alpha(s)$ is given by $B(s) = T(s) \times N(s)$. The torsion of α is real-valued function

$$\tau = \langle B'(s), N(s)\rangle ,$$

where $T(s)$ is the unit tangent to $\alpha(s)$ and $\kappa(s)$ is the curvature of $\alpha(s)$ [4].

Definition 1.1. Let α and β be two immersed curves in E^2, then α, β are called athwart immersions if they have not common tangent space at any points of α, β, otherwise they are called not athwort immersions if they have common tangent space at some points of α, β [3].

Received October 12, 2004.
2000 Mathematics Subject Classification: 54H10, 57H10.
Key words and phrases: twist, folding, manifold.

Definition 1.2. Let M be n-manifold and $A \subset M$, A is called a retract of M if there is a retraction $r : M \to A$, i.e., a continuous map with $r|_A = Id_A$ [14].

2. Isotwist folding

In this section we will give the definition of the isotwist folding of a manifold M into itself. The limits of the isotwist foldings of a manifold are obtained. Also the relations between conditional retractions and this type of the folding are achieved.

Definition 2.1. Let M be an n-dimensional Riemannian manifold. A topological folding $f_\tau : M \to M$ from M into itself is said to be isotwist folding if and only if f_τ preserves the sign of the twist (i.e., f_τ maps M_1 into M_2, where $M_1,\ M_2 \subset M$ such that $\text{sign}\tau(M_1) = \text{sign}\tau(M_2)$.

In the general case the limit of folding of an n-dimensional Riemannian manifold into itself is an Riemannian manifold of dimension $n-1$, i.e., $\lim_{r\to\infty} f_r(M^n) = M^{n-1}$ [12]. Now let α be unit speed curve in E^3 τ the torsion of α. Also $\bar{\alpha} \subset \alpha$, where $\tau > 0$ at each point $\bar{t} \in \bar{\alpha}$ and $\underline{\alpha} \subset \alpha$, where $\tau < 0$ at each point $\underline{t} \in \underline{\alpha}$.

Theorem 2.1. *In the folding of a manifold into itself which preserve the sign of the twist must be a manifold of the same dimension.*

Proof. Let α be a unit speed curve in E^3 and $f_\tau : \alpha \to \alpha$ is isotorsion folding which preserves the sign of the torsion $f_\tau(\alpha_1) = \alpha_2$ such that $\tau(\alpha_1), \tau(\alpha_2) > 0$ $\alpha_1, \alpha_2 \subset \alpha$. Also let α be a curve with $\tau = 0$ at every point $p \in \alpha$ and $d : \alpha \to \alpha$ be a deformation such that $\tau\ |_p = 0$, $\tau < 0$ in the lower of p and $\tau > 0$ in the upper of p. Consider a sequence of foldings and $f_{\tau_1} : \bar{\alpha_d} \to \bar{\alpha}_d$ such that $f_1(\bar{\alpha}_{1,\tau>0}) = \bar{\alpha}_{2,\tau>0}$, $f_{\tau_1}(\underline{\alpha}_{1,\tau<0}) = \underline{\alpha}_{2,\tau<0}$ and $f_{\tau_2} : f_1(\alpha_d) \to f_1(\alpha_d)$ such that $f_2(\bar{\alpha}_{2,\tau>0}) = \bar{\alpha}_{3,\tau>0}, f_2(\underline{\alpha}_{2,\tau<0}) = \underline{\alpha}_{3,\tau<0}$, $\cdots$, $f_{\tau_r} : f_{\tau_{r-1}}(\alpha_d) \to f_{\tau_{r-1}}(\alpha_d)$, $f_{\tau_r}(\bar{\alpha}_{r,\tau>0}) = \bar{\alpha}_{r+1,\tau>0}$, $f_{\tau_r}(\underline{\alpha}_{r,\tau<0}) = \underline{\alpha}_{r+1,\tau<0}$. Thus the $\lim f_{\tau_r} = g_1$ at $\tau > 0$ and $\lim f_{\tau_r} = g_2$ at $\tau < 0$ such that g_1 is an upper geodesic, g_2 is an lower geodesic. So the α after the sequence of the folding is piecewise geodesics of the same dimension as α. □

Definition 2.2. Let α be equipped by a tiling $\Im(\alpha)$. Then the tiling folding of α is the folding $g : \alpha \to \alpha$ such that $g(\alpha_i) = \alpha_j$, where $\alpha_i, \alpha_j \in \Im(\alpha)$.

Now we will give the definition of the tiling of the curves and the classification of this tiling.

Definition 2.3. A tiling of a unit speed curve α is a collection $\Im = \{\alpha_i,\ i \in I = \{1, 2, 3, \cdots\}\}$ of segment curves (tiles) which cover the curve.

Classification of the tilings:

In the general case the arclength of tiles (segment curve) are different. In the following we give two types of the tilings:

(1) Equi-tiling: In this case all tiles have the same arclength i.e., $L(\alpha_i) = const$ for all $i \in I$.

(2) Uniform tiling: In this case $L(\alpha_i) = L(\alpha_{i+2})$ for all $i \in I$.

Definition 2.4. Let α be equipped by a tiling $\Im(\alpha)$. Then the tiling folding of α is the folding $g : \alpha \to \alpha$ such that $g(\alpha_i) = \alpha_j$, where $\alpha_i, \alpha_j \in \Im(\alpha)$.

Lemma 2.2. *Let α equipped by a tiling $\Im$ then the min tiling folding of α is either tile with $\tau \geq 0$, $\tau \leq 0$ or tile with different sign torsion.*

Example 2.1. In this example we introduce different types of *min* tiling folding. Let α be a unit speed curve in E^3 equipped by tiling, $\Im$, and if p is point in α thus that $\tau_p(\alpha) = 0$ and let $g : \alpha \to \alpha$ be a tiling folding of α into itself. Then the *min* tiling folding of α depends on the position of p.

(1) In the first case let the point p lies in the boundary of tiles $\bar{\alpha}_1$, where $\tau > 0$ and $\underline{\alpha}_1$, where $\tau < 0$. Then the *min* tiling folding of α is

$$\lim_{r\to\infty} g_r(\alpha) = \begin{cases} \bar{\alpha}_1, & \text{where } \tau(\bar{\alpha}_1) \geq 0 \text{ for all } t \in \bar{\alpha}_1 \\ \underline{\alpha}_1, & \text{where } \tau(\underline{\alpha}_1) \leq 0 \text{ for all } t \in \underline{\alpha}_1 \end{cases}, \text{ see Figure 1.}$$

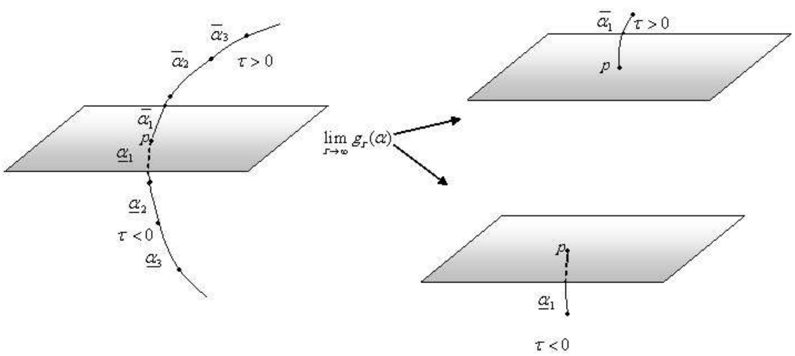

Figure 1.

(2) In the second case let α_0 be the tile that the point p lies inside it. Then the *min* tiling folding of α is $\lim_{r\to\infty} g_r(\alpha) = \alpha_0$, where the torsion of α_0 has different signs, see Figure 2.

Now in the following theorem we will discuss the relation between the retraction and the isotorsion folding of curves in E^3. In the general case the retraction of a

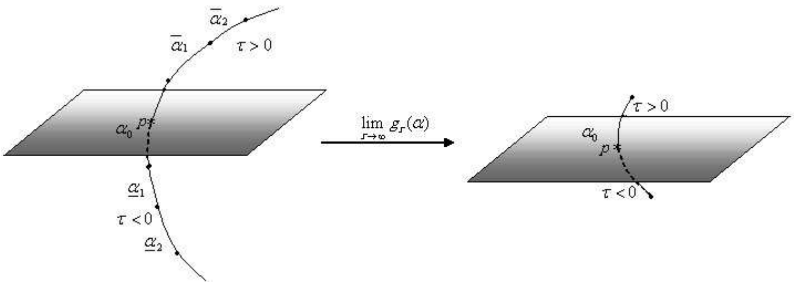

Figure 2.

manifold M does not coincide with the folding of M, but under some restrictions we obtain the following.

Theorem 2.3. *Let α be a unit speed curve The retraction of α restricted by the torsion coincide with isotorsion folding of α.*

Proof. Let α be unit speed curve such that the torsion τ at some point $p \in \alpha$ equal 0 i.e., $\tau_p = 0$ otherwise the torsion is $\tau > 0$ or $\tau < 0$. So we can equipped α by tiling $\Im$ such that the point p lies between two tiles. Let $r : \alpha \to A$ be the retraction of α restricted by the torsion where $A \subset \alpha$, then $r_1 \mid_{\tau>0} (\bar{\alpha}) = \bar{A}_1$, $r_1 \mid_{\tau<0} (\underline{\alpha}) = \underline{A}_1$ and $r_2 \mid_{\tau>0} (\bar{A}_1) = \bar{A}_2$, $r_2 \mid_{\tau<0} (\bar{A}_1) = \underline{A}_2, \cdots, r_n \mid_{\tau>0} (\bar{A}_{n-1}) = \bar{A}_n$, $r_n \mid_{\tau<0} (\bar{A}_{n-1}) = \underline{A}_n$. Then $\lim r_n \mid_{\tau>0} (\bar{A}_{n-1}) = \bar{A}$ and $\lim r_n \mid_{\tau<0} (\bar{A}_{n-1}) = \underline{A}$. See Figure 3,

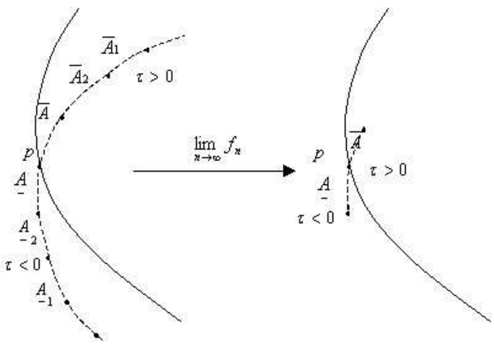

Figure 3.

where $\bar{A}$, $\underline{A}$ are two tiles around the point p, $\tau(\bar{A}) > 0, \tau(\underline{A}) < 0$.

From Theorem 1.1, we obtain that the the retraction of α restricted by the torsion coincide with isotorsion folding of α. $\square$

3. Immersion curves and isotorsion folding

In this section, we discuss whether the isotorsion folding preserves the type of immersions or not. We know that if f is a folding from n-dimensional Riemannian manifold M into itself then there exist an induced folding f_* from TM into itself where TM is the tangent space of M.

Theorem 3.1. *Let α, β be not athwart immersed curves in E^2 then the isotorsion folding f_τ of α into itself, β into itself preserves the immersion if $f_*(t_{com}) = t_{com}$ where t_{com} is a common tangent at some points.*

Proof. Let $f_\tau : \alpha \to \alpha$, $\tau(\alpha) = \tau(f_\tau(\alpha))$ and $f_\tau : \beta \to \beta$, $\tau(\beta) = \tau(f_\tau(\beta))$ and suppose the two immersed manifolds have common tangent at some points. So there exist an induced folding $f_* : T\alpha \to T\alpha$ such that $f_*(t_{com}) = t_{com}$, see Figure 4.

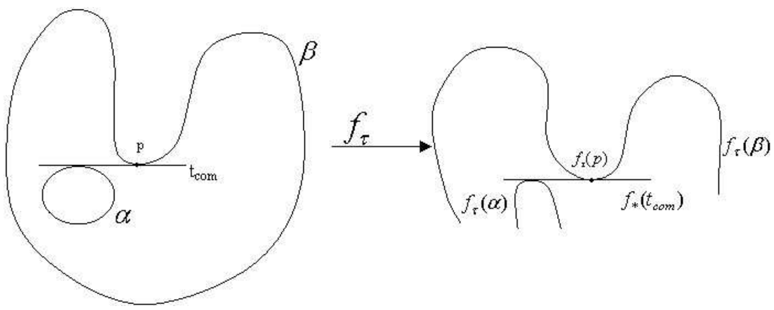

Figure 4.

Thus $f_\tau(\alpha)$, $f_\tau(\beta)$ are remain not athwart immersions, but if $\bar{f_\tau}$ is the type of isotorsion folding for which $\bar{f_*}(t_{com}) \neq t_{com}$. See Figure 5,

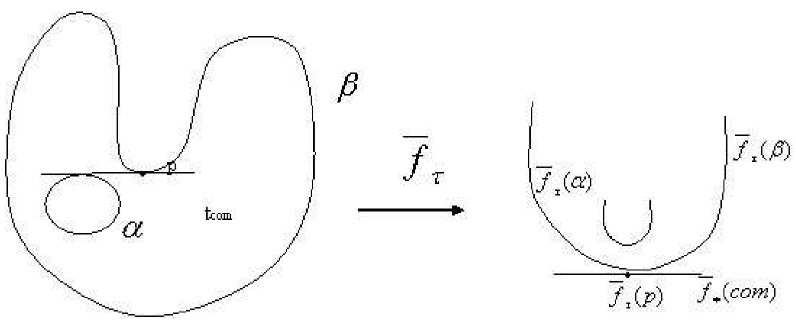

Figure 5.

then $\bar{f}_\tau(\alpha)$, $\bar{f}_\tau(\beta)$ are athwart [1] immersions. □

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