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New Sufficient Conditions for Strongly Starlikeness and Strongly Convex Functions

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Abstract. The object of the present paper is to obtain new conditions for analytic function to be strongly starlike and strongly convex function defined in the open unit disk.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*$, the class of starlike functions if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \qquad (z \in \mathcal{U}). \tag{1.2}$$

A function $f(z) \in \mathcal{A}$ is said to be convex function if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \qquad (z \in \mathcal{U}). \tag{1.3}$$

Also we denote by $\mathcal{C}$ the class of all convex functions.
If $f(z) \in \mathcal{A}$ satisfies

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\alpha\pi}{2}, \qquad (z \in \mathcal{U}) \tag{1.4}$$

for some $0 < \alpha \leq 1$, then $f(z)$ said to be strongly starlike function of order α in $\mathcal{U}$, and this class denoted by $\overline{\mathcal{S}}^*(\alpha)$. Further, if $f(z) \in \mathcal{A}$ satisfies

$$\left|\arg \left(1 + \frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2}, \qquad (z \in \mathcal{U}) \tag{1.5}$$

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for some $0 < \alpha \le 1$, then we say that $f(z)$ is strongly convex function of order α in $\mathcal{U}$, and we denote by $\overline{\mathcal{C}}(\alpha)$ the class of all such functions.
Note that $\overline{\mathcal{S}}^*(1) \equiv \mathcal{S}^*$ and $\overline{\mathcal{C}}(1) \equiv \mathcal{C}$.

Nunokawa et al. [3] obtained the following result.

Lemma 1.1. *Let $p(z)$ be analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \neq 0$. If there exists two points $z_1, z_2 \in \mathcal{U}$ such that*

$$-\frac{\beta\pi}{2} = \arg p(z_1) < \arg p(z) < \arg p(z_2) = \frac{\alpha\pi}{2} \tag{1.6}$$

for $\alpha > 0$, $\beta > 0$, and for $|z| < |z_1| = |z_2|$, the we have

$$\frac{z_1 p(z_1)}{p(z_1)} = -i\frac{\alpha+\beta}{2}m \quad \text{and} \tag{1.7}$$

$$\frac{z_2 p(z_2)}{p(z_2)} = i\frac{\alpha+\beta}{2}m \tag{1.8}$$

where

$$m \ge \frac{1-|a|}{1+|a|} \quad \text{and} \quad a = i\tan\frac{\pi}{4}\left(\frac{\alpha-\beta}{\alpha+\beta}\right). \tag{1.9}$$

Making use of the above lemma, Takahashi and Nunokawa [5] define two classes of analytic functions $\mathcal{S}^*(\alpha,\beta)$ and $\mathcal{C}(\alpha,\beta)$, where $\mathcal{S}^*(\alpha,\beta)$ is the class of all functions $f(z) \in \mathcal{A}$ satisfying

$$-\frac{\beta\pi}{2} < \arg\frac{zf'(z)}{f(z)} < \frac{\alpha\pi}{2}, \qquad (z \in \mathcal{U}), \tag{1.10}$$

for some $0 \le \alpha < 1$, $0 \le \beta < 1$, and $\mathcal{C}(\alpha,\beta)$ is the class of all functions $f(z) \in \mathcal{A}$ satisfying

$$-\frac{\beta\pi}{2} < \arg\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{\alpha\pi}{2}, \qquad (z \in \mathcal{U}), \tag{1.11}$$

for some $0 \le \alpha < 1$, $0 \le \beta < 1$. We note that $\mathcal{S}^*(\alpha,\alpha) \equiv \overline{\mathcal{S}}^*(\alpha)$ and $\mathcal{C}(\alpha,\alpha) \equiv \overline{\mathcal{C}}(\alpha)$.

In this paper, applying the above lemma, we obtain new sufficient conditions for the function $f(z) \in \mathcal{A}$ to be strongly starlike function and strongly convex function of order α in $\mathcal{U}$. Also we shall obtain and improve the sufficient conditions for starlikeness given by Obradović and Owa [4], Nunokawa [2] and Lin [1].

2. Main theorem

Theorem 2.1. *Let $p(z)$ be analytic in $\mathcal{U}$ with $p(0)=1$, $p(z)\neq 0$ in $\mathcal{U}$ and suppose that*

$$\lambda(k,t_2)\leq Im\left(1+e^{i(\pi/4)(\alpha-\beta)}\frac{zp'(z)}{(p(z))^2}\right)\leq \mu(k,t_1),\qquad (z\ \in\ \mathcal{U}) \tag{2.1}$$

where

$$\lambda(k,t_2) = -\frac{t_2^{(1+4k)}(2k+1)(1+t_2^2)(1-|a|)}{(1+|a|)},\quad (t_2>0)\qquad \textit{and} \tag{2.2}$$

$$\mu(k,t_1) = \frac{t_1^{(1+4k)}(2k+1)(1+t_1^2)(1-|a|)}{(1+|a|)},\qquad (t_1>0) \tag{2.3}$$

then we have

$$-\frac{\beta\pi}{2}<\arg p(z)<\frac{\alpha\pi}{2},\quad (\ z\ \in\ \mathcal{U}) \tag{2.4}$$

for some $\alpha,\ \beta>0$ such that $\alpha+\beta=8k+4$, $k\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$.

Proof. Suppose that there exists points $z_1\in\mathcal{U}$ and $z_2\in\mathcal{U}$ such that

$$-\frac{\beta\pi}{2}=\arg p(z_1)<\arg p(z)<\arg p(z_2)=\frac{\alpha\pi}{2} \tag{2.5}$$

for $|z|<|z_1|=|z_2|$, then from the proof of the lemma 1.1 [2], we have

$$\frac{z_1p(z_1)}{p(z_1)}=-i.\frac{(\alpha+\beta)(1+t_1^2)}{4t_1}.m \text{ and } \frac{z_2p(z_2)}{p(z_2)}=i.\frac{(\alpha+\beta)(1+t_2^2)}{4t_2}.m \tag{2.6}$$

where

$$p(z_1) = (-it_1)^{(\alpha+\beta)/2}e^{i(\pi/4)(\alpha-\beta)}\quad (t_1>0), \tag{2.7}$$

$$p(z_2) = (it_2)^{(\alpha+\beta)/2}e^{i(\pi/4)(\alpha-\beta)}\quad (t_2>0), \tag{2.8}$$

and

$$m\geq\frac{1-|a|}{1+|a|}.$$

By making use of (2.6), (2.7) and (2.8), we have

$$\begin{aligned}
\operatorname{Im}\left(1+e^{i(\pi/4)(\alpha-\beta)}\frac{z_2p'(z_2)}{(p(z_2))^2}\right) &= \operatorname{Im}\left(\frac{e^{-i(\pi/4)(\alpha-\beta)}p(z_2)+\frac{z_2p'(z_2)}{p(z_2)}}{e^{-i(\pi/4)(\alpha-\beta)}p(z_2)}\right)\\
&= \operatorname{Im}\left(\frac{(it_2)^{(\alpha+\beta)/2}+i\frac{(\alpha+\beta)(1+t_2^2)}{4t_2}m}{(it_2)^{(\alpha+\beta)/2}}\right)\\
&= \operatorname{Im}\left(\frac{(it_2)^{4k+2}+i\frac{(2k+1)(1+t_2^2)}{t_2}m}{(it_2)^{4k+2}}\right)\\
&= -t_2^{(1+4k)}(2k+1)(1+t_2^2)m\\
&\leq -\frac{t_2^{(1+4k)}(2k+1)(1+t_2^2)(1-|a|)}{(1+|a|)}
\end{aligned}$$

and

$$\operatorname{Im}\left(1+e^{i(\pi/4)(\alpha-\beta)}\frac{z_1p'(z_1)}{(p(z_1))^2}\right)\geq\frac{t_1^{(1+4k)}(2k+1)(1+t_1^2)(1-|a|)}{(1+|a|)}$$

which cotntradicts the assumption (2.1) of the theorem. Therefore we must have

$$-\frac{\beta\pi}{2}<\arg p(z)<\frac{\alpha\pi}{2}\qquad(z\in\mathcal{U}). \tag{2.9}$$

□

Putting $p(z)=zf'(z)/f(z)$ in (2.1), we obtain

Corollary 2.2. *Let $f(z)\in\mathcal{A}$, $zf'(z)/f(z)\neq 0$ in $\mathcal{U}$ and suppose that*

$$\begin{aligned}
\lambda(k,t_2) &\leq Im\left(1+e^{i(\pi/4)(\alpha-\beta)}\left(\frac{(zf'(z))'/f'(z)}{zf'(z)/f(z)}-1\right)\right)\\
&\leq \mu(k,t_1)\qquad(z\in\mathcal{U})
\end{aligned} \tag{2.10}$$

where $\lambda(k,t_2)$ and $\mu(k,t_1)$ as in (2.2) *and* (2.3) *respectively, then $f(z)\in\mathcal{S}^*(\alpha,\beta)$.*

Putting $\alpha=\beta$ and $t_1=t_2$ (say $t>0$) in Corollary 2.2, we easily obtain

Corollary 2.3. *Let $f(z)\in\mathcal{A}$, $zf'(z)/f(z)\neq 0$ in $\mathcal{U}$ and suppose that*

$$\left|Im\left(\frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)}\right)\right|\leq\frac{\alpha}{2}t^{(\alpha-1)}(1+t^2)\quad(z\in\mathcal{U}) \tag{2.11}$$

then $f(z)\in\overline{\mathcal{S}}^*(\alpha)$.

From Corollary 2.3, we easily obtain

Corollary 2.4. *If $f(z) \in \mathcal{A}$ satisfying*

$$\left| \frac{zf''(z)}{f'(z)} + 1 \right| \leq \frac{\alpha}{2} t^{(\alpha-1)}(1+t^2) \left| \frac{zf'(z)}{f(z)} \right| \quad (z \in \mathcal{U}) \tag{2.12}$$

then $f(z) \in \overline{\mathcal{S}}^*(\alpha)$.

Putting $\alpha = 1$ in Corollary 2.4, we have

Corollary 2.5. *If $f(z) \in \mathcal{A}$ satisfying*

$$\left| \frac{zf''(z)}{f'(z)} + 1 \right| \leq \frac{1+t^2}{2} \left| \frac{zf'(z)}{f(z)} \right| \quad (z \in \mathcal{U}) \tag{2.13}$$

then $f(z) \in \mathcal{S}^*$.

If we put $t = \sqrt{6}/2$ in Corollary 2.5, we obtain

Corollary 2.6 (see [4]). *If $f(z) \in \mathcal{A}$ satisfying*

$$\left| \frac{zf''(z)}{f'(z)} + 1 \right| \leq \frac{5}{4} \left| \frac{zf'(z)}{f(z)} \right| \quad (z \in \mathcal{U}) \tag{2.14}$$

then $f(z) \in \mathcal{S}^*$.

If we put $t = \sqrt{2}$ in Corollary 2.5, we obtain

Corollary 2.7 (see [1]). *If $f(z) \in \mathcal{A}$ satisfying*

$$\left| \frac{zf''(z)}{f'(z)} + 1 \right| \leq \frac{3}{2} \left| \frac{zf'(z)}{f(z)} \right| \quad (z \in \mathcal{U}) \tag{2.15}$$

then $f(z) \in \mathcal{S}^*$.

If we put $t = \sqrt{2\log 4 - 1}$ in Corollary 2.5, we obtain

Corollary 2.8 (see [2]). *If $f(z) \in \mathcal{A}$ satisfying*

$$\left| \frac{zf''(z)}{f'(z)} + 1 \right| \leq \log 4 \left| \frac{zf'(z)}{f(z)} \right| \quad (z \in \mathcal{U}) \tag{2.16}$$

then $f(z) \in \mathcal{S}^*$.

Putting $p(z) = 1 + zf''(z)/f'(z)$ in (2.1), we have

Corollary 2.9. *Let $f(z) \in \mathcal{A}$, $f'(z) \neq 0$ in $\mathcal{U}$ and suppose that*

$$\lambda(k,t_2) \leq Im\left(1 + e^{i(\pi/4)(\alpha-\beta)}\left(\frac{(zf'(z))'' - zf''(z)}{(zf'(z))'}\right)\right) \leq \mu(k,t_1) \quad (z \in \mathcal{U}) \tag{2.17}$$

where $\lambda(k,t_2)$ and $\mu(k,t_1)$ as in (2.2) and (2.3) respectively, then $f(z) \in \mathcal{C}(\alpha,\beta)$.

Putting $\alpha = \beta$ and $t_1 = t_2$ (say $t > 0$) in Corollary 2.9 we easily obtain

Corollary 2.10. *Let $f(z) \in \mathcal{A}$, $f'(z) \neq 0$ in $\mathcal{U}$ and suppose that*

$$\left| Im\left(1 + \frac{(zf'(z))'' - zf''(z)}{(zf'(z))'}\right)\right| \leq \frac{\alpha}{2} t^{(\alpha-1)}(1+t^2), \quad (z \in \mathcal{U}) \tag{2.18}$$

then $f(z) \in \overline{\mathcal{C}}(\alpha)$.

From Corollary 2.10, we easily obtain

Corollary 2.11. *Let $f(z) \in \mathcal{A}$, $f'(z) \neq 0$ in $\mathcal{U}$ and suppose that*

$$\left| 1 + \frac{(zf'(z))'' - zf''(z)}{(zf'(z))'}\right| \leq \frac{\alpha}{2} t^{(\alpha-1)}(1+t^2), \quad (z \in \mathcal{U}) \tag{2.19}$$

then $f(z) \in \overline{\mathcal{C}}(\alpha)$.

Putting $\alpha = 1$ in Corollary 2.11, we have

Corollary 2.12. *Let $f(z) \in \mathcal{A}$, $f'(z) \neq 0$ in $\mathcal{U}$ and suppose that*

$$\left| 1 + \frac{(zf'(z))'' - zf''(z)}{(zf'(z))'}\right| \leq \frac{1+t^2}{2}, \quad (z \in \mathcal{U}) \tag{2.20}$$

then $f(z) \in \mathcal{C}$.

Putting $p(z) = f(z)/z$ in (2.1), we obtain

Corollary 2.13. *If $f(z) \in \mathcal{A}$ satisfying*

$$\lambda(k, t_2) \leq Im\left(1 + e^{i(\pi/4)(\alpha-\beta)}\left(\frac{z^2 f'(z)}{f^2(z)} - \frac{z}{f(z)}\right)\right) \leq \mu(k, t_1) \quad (z \in \mathcal{U}) \tag{2.21}$$

where $\lambda(k, t_2)$ and $\mu(k, t_1)$ as in (2.2) *and* (2.3) *respectively, then we have*

$$-\frac{\beta\pi}{2} < \arg\frac{f(z)}{z} < \frac{\alpha\pi}{2} \qquad (\alpha, \beta > 0;\ z \in \mathcal{U}). \tag{2.22}$$

Putting $p(z) = f'(z)$ in (2.1), we obtain

Corollary 2.14. *If $f(z) \in \mathcal{A}$ satisfying*

$$\lambda(k, t_2) \leq Im\left(1 + e^{i(\pi/4)(\alpha-\beta)}\left(\frac{zf''(z)}{(f'(z))^2}\right)\right) \leq \mu(k, t_1) \quad (z \in \mathcal{U}) \tag{2.23}$$

where $\lambda(k, t_2)$ and $\mu(k, t_1)$ as in (2.2) *and* (2.3) *respectively, then we have*

$$-\frac{\beta\pi}{2} < \arg f'(z) < \frac{\alpha\pi}{2} \qquad (\alpha, \beta > 0;\ z \in \mathcal{U}). \tag{2.24}$$

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