KYUNGPOOK Math. J. 46(2006), 119-130

On Almost Continuity

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Abstract. A new class of functions is introduced in this paper. This class is called almost δ-precontinuity. This type of functions is seen to be strictly weaker than almost precontinuity. By using δ-preopen sets, many characterizations and properties of the said type of functions are investigated.

1. Introduction

The notion of δ-preopen set was introduced by Raychaudhuri and Mukherjee [20] in 1993. We introduce here a new type of functions strictly weaker than almost continuity. We call this the almost δ-precontinuous functions. We investigate properties of such functions.

Throughout the present paper, spaces mean topological spaces and $f : (X, \tau) \to (Y, \sigma)$ (or simply $f : X \to Y$) denotes a function f of a space (X, τ) into a space (Y, σ). Let S be a subset of a space X. The closure and the interior of S are denoted by $\mathrm{cl}(S)$ and $\mathrm{int}(S)$, respectively.

A subset S of a space X is said to be regular open [23] if $S = \mathrm{int}(\mathrm{cl}(S))$ and δ-open [24] if for each $x \in S$, there exists a regular open set W such that $x \in W \subset S$.

A subset S of a space X is said to be α-open [13] (resp. semi-open [8], preopen [10], γ-open [7], β-open [1] or semi-preopen [2]) if $S \subset \mathrm{int}(\mathrm{cl}(\mathrm{int}(S)))$ (resp. $S \subset \mathrm{cl}(\mathrm{int}(S))$, $S \subset \mathrm{int}(\mathrm{cl}(S))$, $S \subset \mathrm{int}(\mathrm{cl}(S)) \cup \mathrm{cl}(\mathrm{int}(S))$, $S \subset \mathrm{cl}(\mathrm{int}(\mathrm{cl}(S)))$).

The complement of a regular open set is said to be regular closed [23]. The complement of a semiopen set is said to be semiclosed [5]. The intersection of all semiclosed sets containing a subset A of X is called the semi-closure [5] of A and is denoted by s-$\mathrm{cl}(A)$. The union of all semiopen sets contained in a subset A of X is called the semi-interior of A and is denoted by s-$\mathrm{int}(A)$. A point $x \in X$ is called a δ-cluster (resp. θ-cluster) point of A [24] if $A \cap \mathrm{int}(\mathrm{cl}(U)) \neq \varnothing$ (resp. $A \cap \mathrm{cl}(U) \neq \varnothing$) for each open set U containing x. The set of all δ-cluster (resp. θ-cluster) points of A is called the δ-closure (resp. θ-closure) of A and is denoted by δ-$\mathrm{cl}(A)$ (resp. θ-$\mathrm{cl}(A)$). If δ-$\mathrm{cl}(A) = A$ (resp. θ-$\mathrm{cl}(A) = A$), then A is said to be δ-closed (resp.

Received October 4, 2004.
2000 Mathematics Subject Classification: 54C10.
Key words and phrases: δ-preopen set, almost continuity.

θ-closed). The complement of a δ-closed (resp. θ-closed) set is said to be δ-open (resp. θ-open).

A subset S of a topological space X is said to be δ-preopen [20] iff $S \subset \text{int}(\delta\text{-cl}(S))$. The complement of a δ-preopen set is called a δ-preclosed set [20]. The union (resp. intersection) of all δ-preopen (resp. δ-preclosed) sets, each contained in (resp. containing) a set S in a topological space X is called the δ-preinterior (resp. δ-preclosure) of S and it is denoted by δ-pint(S) (resp. δ-pcl(S)) [20].

The family of all δ-preopen (resp.regular open, preopen, β-open. α-open, semi-open, δ-open) sets of a space X will be denoted by $\delta PO(X)$ (resp. $RO(X)$, $PO(X)$, $\beta O(X)$, $\alpha O(X)$, $SO(X)$, $\delta O(X)$). The family of all δ-preclosed (resp. regular closed, δ-closed) sets in a space X is denoted by $\delta PC(X)$ (resp. $RC(X)$, $\delta C(X)$). The family of all δ-preopen (resp.regular open, δ-open) sets containing a point $x \in X$ will be denoted by $\delta PO(X, x)$ (resp. $RO(X, x)$, $\delta O(X, x)$).

Lemma 1 (Raychaudhuri and Mukherjee [20])**.** *Let A be a subset of a space X. Then*

(1) $\delta - \text{pcl}(X \backslash A) = X \backslash \delta - \text{pint}(A)$,
(2) $x \in \delta - \text{pcl}(A)$ *if and only if* $A \cap U \neq \varnothing$ *for each* $U \in \delta PO(X, x)$,
(3) A *is δ-preclosed in X if and only if* $A = \delta - \text{pcl}(A)$,
(4) $\delta - \text{pcl}(A)$ *is δ-preclosed in X.*

Lemma 2 (Noiri [17], [18])**.** *For a subset of a space Y, the following hold:*

(1) $\alpha - \text{cl}(V) = \text{cl}(V)$ *for every* $V \in \beta O(Y)$.
(2) $p - \text{cl}(F) = \text{cl}(V)$ *for every* $V \in SO(Y)$.
(3) $s - \text{cl}(V) = \text{int}(\text{cl}(V))$ *for every preopen set V of a space X.*

2. Characterizations

Definition 3. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be almost δ-precontinuous if for each $x \in X$ and each $V \in RO(Y)$ containing $f(x)$, there exists $U \in \delta PO(X)$ containing x such that $f(U) \subset V$.

Definition 4. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be R-map [4] (resp. almost continuous [21], almost α-continuous [16], almost precontinuous [11], δ-continuous [14]) if $f^{-1}(V) \in RO(X)$ (resp. $f^{-1}(V) \in \tau$, $f^{-1}(V) \in \alpha O(X)$, $f^{-1}(V) \in PO(X)$, $f^{-1}(V) \in \delta O(X)$) for every $V \in RO(Y)$.

Remark 5. The following implications hold:

$$\text{al. contin.} \Rightarrow \text{al. } \alpha\text{-contin.} \Rightarrow \text{al. precontin.} \Rightarrow \text{al. } \delta\text{-precontin.}$$

The converses are not true in general.

Example 6. Let $X = \{a, b, c\}$ and $\tau = \{X, \varnothing, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$. Let $f : X \to X$ be a function defined by $f(a) = b$, $f(b) = a$, $f(c) = c$. Then, f is almost δ-precontinuous but not almost precontinuous.

The other examples can be seen in [11], [16].

Theorem 7. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following are equivalent*:

(1) f *is almost* δ*-precontinuous;*

(2) *for each* $x \in X$ *and each* $V \in \sigma$ *containing* $f(x)$*, there exists* $U \in \delta PO(X)$ *containing* x *such that* $f(U) \subset \mathrm{int}(\mathrm{cl}(V))$*;*

(3) $f^{-1}(F) \in \delta PC(X)$ *for every* $F \in RC(Y)$*;*

(4) $f^{-1}(V) \in \delta PO(X)$ *for every* $V \in RO(Y)$*.*

(5) $f(\delta - \mathrm{pcl}(A)) \subset \delta - \mathrm{cl}(f(A))$ *for every subset* A *of* X *;*

(6) $\delta - \mathrm{pcl}(f^{-1}(B)) \subset f^{-1}(\delta\text{-}\mathrm{cl}(B))$ *for every subset* B *of* Y*;*

(7) $f^{-1}(F) \in \delta PC(X)$ *for every* δ*-closed set* F *of* (Y, σ)*;*

(8) $f^{-1}(V) \in \delta PO(X)$ *for every* δ*-open set* V *of* (Y, σ)*;*

(9) $\delta - \mathrm{pcl}(f^{-1}(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(B))))) \subset f^{-1}(\mathrm{cl}(B))$ *for every subset* B *of* Y*;*

(10) $\delta - \mathrm{pcl}(f^{-1}(\mathrm{cl}(\mathrm{int}(F)))) \subset f^{-1}(F)$ *for every closed set* F *of* Y*;*

(11) $\delta - \mathrm{pcl}(f^{-1}(\mathrm{cl}(V))) \subset f^{-1}(\mathrm{cl}(V))$ *for every open set* V *of* Y*;*

(12) $f^{-1}(V) \subset \delta - \mathrm{pint}(f^{-1}(s - \mathrm{cl}(V)))$ *for every open set* V *of* Y*;*

(13) $f^{-1}(V) \subset \mathrm{int}(\delta - \mathrm{cl}(f^{-1}(s - \mathrm{cl}(V))))$ *for every open set* V *of* Y*;*

(14) $f^{-1}(V) \subset \delta - \mathrm{pint}(f^{-1}(\mathrm{int}(\mathrm{cl}(V)))))$ *for every open set* V *of* Y*;*

(15) $f^{-1}(V) \subset \mathrm{int}(\delta - \mathrm{cl}(f^{-1}(\mathrm{int}(\mathrm{cl}(V))))))$ *for every open set* V *of* Y*;*

(16) $\delta - \mathrm{pcl}(f^{-1}(V)) \subset f^{-1}(\mathrm{cl}(V))$ *for each* $V \in \beta O(Y)$*;*

(17) $\delta - \mathrm{pcl}(f^{-1}(V)) \subset f^{-1}(\mathrm{cl}(V))$ *for each* $V \in SO(Y)$*;*

(18) $f^{-1}(V) \subset \delta - \mathrm{pint}(f^{-1}(\mathrm{int}(\mathrm{cl}(V))))$ *for each* $V \in PO(Y)$*;*

(19) $\delta - \mathrm{pcl}(f^{-1}(V)) \subset f^{-1}(\alpha - \mathrm{cl}(V))$ *for each* $V \in \beta O(Y)$*;*

(20) $\delta - \mathrm{pcl}(f^{-1}(V)) \subset f^{-1}(p - \mathrm{cl}(V))$ *for each* $V \in SO(Y)$*;*

(21) $f^{-1}(V) \subset \delta - \mathrm{pint}(f^{-1}(s - \mathrm{cl}(V)))$ *for each* $V \in PO(Y)$*.*

Proof. (1)⇒(2). Let $x \in X$ and $V \in \sigma$ containing $f(x)$. We have $\mathrm{int}(\mathrm{cl}(V)) \in RO(Y)$. Since f is almost δ-precontinuous, then there exists $U \in \delta PO(X, x)$ such that $f(U) \subset \mathrm{int}(\mathrm{cl}(V))$.

(2)⇒(1). Obvious.

(3)⇔(4). Obvious.

(1)⇒(4). Let $x \in X$ and $V \in RO(Y, f(x))$. Since f is almost δ-precontinuous, then there exists $U_x \in \delta PO(X, x)$ such that $f(U_x) \subset V$. We have $U_x \subset f^{-1}(V)$. Thus, $f^{-1}(V) = \cup U_x \in \delta PO(X)$.

(4)⇒(1). Obvious.

(1)⇒(5). Let A be a subset of X. Since $\delta-\mathrm{cl}(f(A))$ is δ-closed in Y, it is denoted by $\cap\{F_i : F_i \in RC(Y),\ i \in I\}$, where I is an index set. By (1)⇔(3), we have $A \subset f^{-1}(\delta-\mathrm{cl}(f(A))) = \cap\{f^{-1}(F_i) : i \in I\} \in \delta PC(X)$ and hence $\delta-\mathrm{pcl}(A) \subset f^{-1}(\delta-\mathrm{cl}(f(A)))$. Therefore, we obtain $f(\delta-\mathrm{pcl}(A)) \subset \delta-\mathrm{cl}(f(A))$.

(5)⇒(6). Let B be a subset of Y. We have $f(\delta-\mathrm{pcl}(f^{-1}(B))) \subset \delta-\mathrm{cl}(f(f^{-1}(B))) \subset \delta-\mathrm{cl}(B)$ and hence $\delta-\mathrm{pcl}(f^{-1}(B)) \subset f^{-1}(\delta-\mathrm{cl}(B))$.

(6)⇒(7). Let F be any δ-closed set of (Y,σ). We have $\delta-\mathrm{pcl}(f^{-1}(F)) \subset f^{-1}(\delta-\mathrm{cl}(F)) = f^{-1}(F)$ and hence $f^{-1}(F)$ is δ-preclosed in (X,τ).

(7)⇒(8). Let V be any δ-open set of (Y,σ). We have $f^{-1}(Y\backslash V) = X\backslash f^{-1}(V) \in \delta PC(X)$ and hence $f^{-1}(V) \in \delta PO(X)$.

(8)⇒(1). Let V be any regular open set of (Y,σ). Since V is δ-open in (Y,σ), $f^{-1}(V) \in \delta PO(X)$ and hence, by (1)⇔(4), f is almost δ-precontinuous.

(1)⇒(9). Let B be any subset of Y. Assume that $x \in X\backslash f^{-1}(\mathrm{cl}(B))$. Then $f(x) \in Y\backslash \mathrm{cl}(B)$ and there exists an open set V containing $f(x)$ such that $V \cap B = \varnothing$; hence $\mathrm{int}(\mathrm{cl}(V)) \cap \mathrm{cl}(\mathrm{int}(\mathrm{cl}(B))) = \varnothing$. Since f is almost δ-precontinuous, there exists $U \in \delta PO(X,x)$ such that $f(U) \subset \mathrm{int}(\mathrm{cl}(V))$. Therefore, we have $U \cap f^{-1}(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(B)))) = \varnothing$ and hence $x \in X\backslash\delta-\mathrm{pcl}(f^{-1}(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(B)))))$. Thus we obtain $\delta-\mathrm{pcl}(f^{-1}(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(B))))) \subset f^{-1}(\mathrm{cl}(B))$.

(9)⇒(10). Let F be any closed set of Y. Then we have

$$\begin{aligned}\delta-\mathrm{pcl}(f^{-1}(\mathrm{cl}(\mathrm{int}(F))) = \ & \delta-\mathrm{pcl}(f^{-1}(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(F)))))\\ & \subset f^{-1}(\mathrm{cl}(F)) = f^{-1}(F).\end{aligned}$$

(10)⇒(11). For any open set V of Y, $cl(V)$ is regular closed in Y and we have

$$\delta-\mathrm{pcl}(f^{-1}(\mathrm{cl}(V)) = \delta-\mathrm{pcl}(f^{-1}(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(V))))) \subset f^{-1}(\mathrm{cl}(V)).$$

(11)⇒(12). Let V be any open set of Y. Then $Y\backslash \mathrm{cl}(V)$ is open in Y and by using Lemma 2 we have

$$\begin{aligned}& X\backslash\delta-\mathrm{pint}(f^{-1}(s\text{-}\mathrm{cl}(V)))\\ = \ & \delta-\mathrm{pcl}(f^{-1}(Y\backslash\mathrm{int}(\mathrm{cl}(V)))) \subset f^{-1}(\mathrm{cl}(Y\backslash\mathrm{cl}(V))) \subset X\backslash f^{-1}(V).\end{aligned}$$

Therefore, we obtain $f^{-1}(V) \subset \delta-\mathrm{pint}(f^{-1}(s-\mathrm{cl}(V)))$.

(12)⇒(13). Let V be any open set of Y. We obtain $f^{-1}(V) \subset \delta-\mathrm{pint}(f^{-1}(s-\mathrm{cl}(V))) \subset \mathrm{int}(\delta-\mathrm{cl}(f^{-1}(s-\mathrm{cl}(V))))$.

(13)⇒(1). Let x be any point of X and V any open set of Y containing $f(x)$. Then $x \in f^{-1}(\mathrm{int}(\mathrm{cl}(V))) \subset \mathrm{int}(\delta-\mathrm{cl}(f^{-1}(s-\mathrm{cl}(\mathrm{int}(\mathrm{cl}(V)))))) = \mathrm{int}(\delta-\mathrm{cl}(f^{-1}(\mathrm{int}(\mathrm{cl}(V)))))$. Thus, $f^{-1}(\mathrm{int}(\mathrm{cl}(V))) \in \delta PO(X)$. Take $U = f^{-1}(\mathrm{int}(\mathrm{cl}(V)))$. We obtain $x \in U$ and $f(U) \subset \mathrm{int}(\mathrm{cl}(V))$. Therefore, f is almost δ-precontinuous.

(12)⇔(14) and (13)⇔(15). Obvious.

(1)⇒(16). Let V be any β-open set of Y. It follows from [2, Theorem 2.4] that $\mathrm{cl}(V)$ is regular closed in Y. Since f is almost δ-precontinuous, by (1)⇔(3), $f^{-1}(\mathrm{cl}(V))$ is δ-preclosed in X. Therefore, we obtain $\delta-\mathrm{pcl}(f^{-1}(V)) \subset f^{-1}(\mathrm{cl}(V))$.

(16)⇒(17). This is obvious since $SO(Y) \subset \beta O(Y)$.

(17)⇒(1). Let F be any regular closed set of Y. Then F is semi-open in Y and hence $\delta - \mathrm{pcl}(f^{-1}(F)) \subset f^{-1}(\mathrm{cl}(F)) = f^{-1}(F)$. This shows that $f^{-1}(F)$ is δ-preclosed. Therefore, by (1)⇔(3), f is almost δ-precontinuous.

(1)⇒(18). Let V be any preopen set of Y. Then $V \subset \mathrm{int}(\mathrm{cl}(V))$ and $\mathrm{int}(\mathrm{cl}(V))$ is regular open in Y. Since f is almost δ-precontinuous, by (1)⇔(4), $f^{-1}(\mathrm{int}(\mathrm{cl}(V)))$ is δ-preopen in X and hence we obtain that $f^{-1}(V) \subset f^{-1}(\mathrm{int}(\mathrm{cl}(V))) \subset \delta - \mathrm{pint}(f^{-1}(\mathrm{int}(\mathrm{cl}(V))))$.

(18)⇒(1). Let V be any regular open set of Y. Then V is preopen and $f^{-1}(V) \subset \delta - \mathrm{pint}(f^{-1}(\mathrm{int}(\mathrm{cl}(V)))) = \delta - \mathrm{pint}(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is δ-preopen in X and hence, by (1)⇔(4), f is almost δ-precontinuous.

(16)⇔(19), (17)⇔(20), (18)⇔(21). Obvious. □

Lemma 8 (Raychaudhuri and Mukherjee [20]). *A set S in X is δ-preopen if and only if $S \cap G \in \delta PO(X)$ for every δ-open set G of X.*

Lemma 9 (Raychaudhuri and Mukherjee [20]). *Let A and X_0 be subsets of a space (X, τ). If $A \in \delta PO(X)$ and $X_0 \in \delta O(X)$, then $A \cap X_0 \in \delta PO(X_0)$.*

Theorem 10. *If $f : (X, \tau) \to (Y, \sigma)$ is almost δ-precontinuous and A is δ-open in (X, τ), then the restriction $f|_A : (A, \tau_A) \to (Y, \sigma)$ is almost δ-precontinuous.*

Proof. Let V be any regular open set of Y. By Theorem 7, we have $f^{-1}(V) \in \delta PO(X)$ and hence $(f|_A)^{-1}(V) = f^{-1}(V) \cap A \in \delta PO(A)$ by Lemma 9. Thus, it follows that $f|_A$ is almost δ-precontinuous. □

Lemma 11 (Raychaudhuri and Mukherjee [20]). *Let A and X_0 be subsets of a space (X, τ). If $A \in \delta PO(X_0)$ and $X_0 \in \delta O(X)$, then $A \in \delta PO(X)$.*

Theorem 12. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function and $\{U_i : i \in I\}$ a cover of X by δ-open sets of (X, τ). If $f|_{U_i} : (U_i, \tau_{U_i}) \to (Y, \sigma)$ is almost δ-precontinuous for each $i \in I$, then f is almost δ-precontinuous.*

Proof. Let V be any regular open set of (Y, σ). Then, we have

$$f^{-1}(V) = X \cap f^{-1}(V) = \cup\{U_i \cap f^{-1}(V) : i \in I\} = \cup\{(f|_{U_i})^{-1}(V) : i \in I\}.$$

Since $f|_{U_i}$ is almost δ-precontinuous, $(f|_{U_i})^{-1}(V) \in \delta PO(U_i)$ for each $i \in I$. By Lemma 11, for each $i \in I$, $(f|_{U_i})^{-1}(V)$ is δ-preopen in X and hence $f^{-1}(V)$ is δ-preopen in X. Therefore, f is almost δ-precontinuous. □

Theorem 13. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function and $g : (X, \tau) \to (X \times Y, \tau \times \sigma)$ the graph function defined by $g(x) = (x, f(x))$ for every $x \in X$. Then g is almost δ-precontinuous if and only if f is almost δ-precontinuous.*

Proof. (⇒). Let $x \in X$ and $V \in RO(Y)$ containing $f(x)$. Then, we have $g(x) = (x, f(x)) \in X \times V \in RO(X \times Y)$. Since g is almost δ-precontinuous, there exists a δ-preopen set U of X containing x such that $g(U) \subset X \times V$. Therefore, we obtain $f(U) \subset V$ and hence f is almost δ-precontinuous.

($\Leftarrow$). Let $x \in X$ and W be a regular open set of $X \times Y$ containing $g(x)$. There exist $U_1 \in RO(X)$ and $V \in RO(Y)$ such that $(x, f(x)) \in U_1 \times V \subset W$. Since f is almost δ-precontinuous, there exists $U_2 \in \delta PO(X)$ such that $x \in U_2$ and $f(U_2) \subset V$. Put $U = U_1 \cap U_2$, then we obtain $x \in U \in \delta PO(X)$ and $g(U) \subset U_1 \times V \subset W$. This shows that g is almost δ-precontinuous. □

Definition 14. The δ-prefrontier of a subset A of X, denoted by $\delta - pfr(A)$, is defined by $\delta - pfr(A) = \delta - \mathrm{pcl}(A) \cap \delta - \mathrm{pcl}(X \backslash A) = \delta - \mathrm{pcl}(A) \backslash \delta - \mathrm{pint}(A)$.

Theorem 15. *The set of all points x of X at which a function $f : X \to Y$ is not almost δ-precontinuous is identical with the union of the δ-prefrontiers of the inverse images of regular open sets containing $f(x)$.*

Proof. Let x be a point of X at which f is not almost δ-precontinuous Then, there exists a regular open set V of Y containing $f(x)$ such that $U \cap (X \backslash f^{-1}(V)) \neq \varnothing$ for every $U \in \delta PO(X, x)$. Therefore, we have $x \in \delta - \mathrm{pcl}(X \backslash f^{-1}(V)) = X \backslash \delta - \mathrm{pint}(f^{-1}(V))$ and $x \in f^{-1}(V)$. Thus, we obtain $x \in \delta - pfr(f^{-1}(V))$.

Conversely, suppose that f is almost δ-precontinuous at $x \in X$ and let V be a regular open set containing $f(x)$. Then there exists $U \in \delta PO(X, x)$ such that $U \subset f^{-1}(V)$; hence $x \in \delta - \mathrm{pint}(f^{-1}(V))$. Therefore, it follows that $x \in X \backslash \delta - pfr(f^{-1}(V))$. This completes the proof. □

Definition 16. A space X is said to be δ-pre-T_2 [6] if for any distinct points x, y of X, there exist disjoint δ-preopen sets U, V of X such that $x \in U$ and $y \in V$.

Definition 17. A function $f : X \to Y$ is said to be weakly δ-precontinuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in \delta PO(X, x)$ such that $f(U) \subset \mathrm{cl}(V)$.

Theorem 18. *If for each pair of distinct points x_1 and x_2 in a space X, there exists a function f of X into a Hausdorff space Y such that*

(1) *$f(x_1) \neq f(x_2)$,*

(2) *f is weakly δ-precontinuous at x_1 and*

(3) *almost δ-precontinuous at x_2,*

then X is δ-pre-T_2.

Proof. Since Y is Hausdorff, there exist open sets V_1 and V_2 of Y such that $f(x_1) \in V_1$, $f(x_2) \in V_2$ and $V_1 \cap V_2 = \varnothing$; hence $\mathrm{cl}(V_1) \cap \mathrm{int}(\mathrm{cl}(V_2)) = \varnothing$. Since f is weakly δ-precontinuous at x_1, there exists $U_1 \in \delta PO(X, x_1)$ such that $f(U_1) \subset \mathrm{cl}(V_1)$. Since f is almost δ-precontinuous at x_2, there exists $U_2 \in \delta PO(X, x_2)$ such that $f(U_2) \subset \mathrm{int}(\mathrm{cl}(V_2))$. Therefore, we obtain $U_1 \cap U_2 = \varnothing$. This shows that X is δ-pre-T_2. □

Let $f : X \to Y$ be a function. The subset $\{(x, f(x)) : x \in X\} \subset X \times Y$ is called the graph of f and is denoted by $G(f)$.

Definition 19. A function $f : X \to Y$ has a (δ_p, r)-graph if for each $(x, y) \in X \times Y \backslash G(f)$, there exist $U \in \delta PO(X, x)$ and a regular open set V of Y containing

y such that $(U \times V) \cap G(f) = \varnothing$.

Lemma 20. *A function $f : X \to Y$ has a (δ_p, r)-graph if and only if for each $(x, y) \in X \times Y$ such that $y \neq f(x)$, there exist a δ-preopen set U and a regular open set V containing x and y, respectively, such that $f(U) \cap V = \varnothing$.*

Theorem 21. *If $f : X \to Y$ is an almost δ-precontinuous function and Y is Hausdorff, then f has a (δ_p, r)-graph.*

Proof. Let $(x, y) \in X \times Y$ such that $y \neq f(x)$. Then there exist open sets V and W such that $y \in V$, $f(x) \in W$ and $V \cap W = \varnothing$; hence $\mathrm{int}(\mathrm{cl}(V)) \cap \mathrm{int}(\mathrm{cl}(W)) = \varnothing$. Since f is almost δ-precontinuous, there exists $U \in \delta PO(X, x)$ such that $f(U) \subset \mathrm{int}(\mathrm{cl}(W))$. This implies that $f(U) \cap \mathrm{int}(\mathrm{cl}(V)) = \varnothing$. Therefore, f has a (δ_p, r)-graph. □

Definition 22. A space X is said to be δ-pre-compact [6] if every δ-preopen cover of X has a finite subcover.

Theorem 23. *If $f : (X, \tau) \to (Y, \sigma)$ has a (δ_p, r)-graph, then $f(K)$ is δ-closed in (Y, σ) for each subset K which is δ-pre-compact relative to (X, τ).*

Proof. Suppose that $y \notin f(K)$. Then $(x, y) \notin G(f)$ for each $x \in K$. Since $G(f)$ is (δ_p, r)-graph, there exist $U_x \in \delta PO(X)$ containing x and a regular open set V_x of Y containing y such that $f(U_x) \cap V_x = \varnothing$. The family $\{U_x : x \in K\}$ is a cover of K by δ-preopen sets. Since K is δ-pre-compact relative to (X, τ), there exists a finite subset K_0 of K such that $K \subset \cup\{U_x : x \in K_0\}$. Set $V = \cap\{V_x : x \in K_0\}$. Then V is a regular open set in Y containing y. Therefore, we have $f(K) \cap V \subset [\bigcup_{x \in K_0} f(U_x)] \cap V \subset \bigcup_{x \in K_0} [f(U_x) \cap V] = \varnothing$. It follows that $y \notin \delta - \mathrm{cl}(f(K))$. Therefore, $f(K)$ is δ-closed in (Y, σ). □

Corollary 24. *If $f : (X, \tau) \to (Y, \sigma)$ is an almost δ-precontinuous function and Y is Hausdorff, then $f(K)$ is δ-closed in (Y, σ) for each subset K which is δ-pre-compact relative to (X, τ).*

Theorem 25. *If $f : X \to Y$ is almost δ-precontinuous, $g : X \to Y$ is δ-continuous and Y is Hausdorff, then the set $\{x \in X : f(x) = g(x)\}$ is δ-preclosed in X.*

Proof. Let $A = \{x \in X : f(x) = g(x)\}$ and $x \in X \backslash A$. Then $f(x) \neq g(x)$. Since Y is Hausdorff, there exist open sets V and W of Y such that $f(x) \in V$, $g(x) \in W$ and $V \cap W = \varnothing$; hence $\mathrm{int}(\mathrm{cl}(V)) \cap \mathrm{int}(\mathrm{cl}(W)) = \varnothing$. Since f is almost δ-precontinuous, there exists $G \in \delta PO(X, x)$ such that $f(G) \subset \mathrm{int}(\mathrm{cl}(V))$. Since g is δ-continuous, there exists an δ-open set H of X containing x such that $g(H) \subset \mathrm{int}(\mathrm{cl}(W))$. Now, put $U = G \cap H$, then $U \in \delta PO(X, x)$ and $f(U) \cap g(U) \subset \mathrm{int}(\mathrm{cl}(V)) \cap \mathrm{int}(\mathrm{cl}(W)) = \varnothing$. Therefore, we obtain $U \cap A = \varnothing$ and hence $x \in X \backslash \delta - \mathrm{pcl}(A)$. This shows that A is δ-preclosed in X. □

Theorem 26. *If $f_1 : X_1 \to Y$ is weakly δ-precontinuous, $f_2 : X_2 \to Y$ is almost δ-precontinuous and Y is Hausdorff, then the set $\{(x_1, x_2) \in X_1 \times X_2 : f(x_1) = f(x_2)\}$*

is δ-preclosed in $X_1 \times X_2$.

Proof. Let $A = \{(x_1, x_2) \in X_1 \times X_2 : f(x_1) = f(x_2)\}$ and $(x_1, x_2) \in (X_1 \times X_2) \backslash A$. Then $f(x_1) \neq f(x_2)$ and there exist open sets V_1 and V_2 of Y such that $f(x_1) \in V_1$, $f(x_2) \in V_2$ and $V_1 \cap V_2 = \varnothing$; hence $\mathrm{cl}(V_1) \cap \mathrm{int}(\mathrm{cl}(V_2)) = \varnothing$. Since f_1 (resp, f_2) is weakly δ-precontinuous (resp. almost δ-precontinuous), there exists $U_1 \in \delta PO(X_1, x_1)$ such that $f_1(U_1) \subset \mathrm{cl}(V_1)$ (resp. $U_2 \in \delta PO(X_2, x_2)$ such that $f_2(U_2) \subset \mathrm{int}(\mathrm{cl}(V_2))$). Therefore, we obtain $(x_1, x_2) \in U_1 \times U_2 \subset (X_1 \times X_2) \backslash A$ and $U_1 \times U_2 \in \delta PO(X_1 \times X_2)$. This shows that A is δ-preclosed in $X_1 \times X_2$. □

Let $\{X_i : i \in I\}$ and $\{Y_i : i \in I\}$ be any two families of spaces with the same index set I. For each $i \in I$, let $f_i : X_i \to Y_i$ be a function. The product space $\prod_{i \in I} X_i$ will be denoted by $\prod X_i$ and the product function $\prod f_i : \prod X_i \to \prod Y_i$ is simply denoted by $f : \prod X_i \to \prod Y_i$.

Theorem 27. *If a function $f : X \to \prod Y_i$ is almost δ-precontinuous, then $p_i \circ f : X \to Y_i$ is almost δ-precontinuous for each $i \in I$, where p_i is the projection of $\prod Y_i$ onto Y_i.*

Proof. Let V_i be any regular open set of Y_i. Since p_i is continuous open, it is an R-map and hence $p_i^{-1}(V_i) \in RO(\prod Y_i)$. By Theorem 7, $f^{-1}(p_i^{-1}(V_i)) = (p_i \circ f)^{-1}(V_i) \in \delta PO(X)$. This shows that $p_i \circ f$ is almost δ-precontinuous for each $i \in I$. □

Theorem 28. *The product function $f : \prod X_i \to \prod Y_i$ is almost δ-precontinuous if and only if $f_i : X_i \to Y_i$ is almost δ-precontinuous for each $i \in I$.*

Proof. (Necessity). Let k be an arbitrarily fixed index and V_k any regular open set of Y_k. Then $\prod Y_j \times V_k$ is regular open in $\prod Y_i$, where $j \in I$ and $j \neq k$, and hence $f^{-1}(\prod Y_j \times V_k) = \prod Y_j \times f_k^{-1}(V_k)$ is δ-preopen in $\prod X_i$. Thus, $f_k^{-1}(V_k)$ is δ-preopen in X_k and hence f_k is almost δ-precontinuous.

(Sufficiency). Let $\{x_i\}$ be any point of $\prod X_i$ and W any regular open set of $\prod Y_i$ containing $f(\{x_i\})$. There exists a finite subset I_0 of I such that $V_k \in RO(Y_k)$ for each $k \in I_0$ and $\{f_i(x_i)\} \in \prod\{V_k : k \in I_0\} \times \prod\{Y_j : j \in I \backslash I_0\} \subset W$. For each $k \in I_0$, there exists $U_k \in \delta PO(X_k)$ containing x_k such that $f_k(U_k) \subset V_k$. Thus, $U = \prod\{U_k : k \in I_0\} \times \prod\{X_j : j \in I \backslash I_0\}$ is a δ-preopen set of $\prod X_i$ containing $\{x_i\}$ and $f(U) \subset W$. This shows that f is almost δ-precontinuous. □

3. Functions

Definition 29. Let (X, τ) be a topological space. The collection of all regular open sets forms a base for a topology τ_s. It is called the semiregularization. In case when $\tau = \tau_s$, the space (X, τ) is called semi-regular [23].

Theorem 30. *Let (X, τ) be a semi-regular space. Then a function $f : (X, \tau) \to (Y, \sigma)$ is almost precontinuous if and only if it is almost δ-precontinuous.*

Definition 31. A function $f : X \to Y$ is said to be δ-almost continuous [20] if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in \delta PO(X, x)$ such that $f(U) \subset V$.

Definition 32. A function $f : X \to Y$ is said to be δ-preirresolute if for each $x \in X$ and each δ-preopen set V of Y containing $f(x)$, there exists $U \in \delta PO(X, x)$ such that $f(U) \subset V$.

Definition 33. A function $f : X \to Y$ is said to be almost δ-preopen if $f(U) \subset \mathrm{int}(\mathrm{cl}(f(U)))$ for every δ-preopen set U of X.

Theorem 34. *If $f : X \to Y$ is an almost δ-preopen and weakly δ-precontinuous function, then f is almost δ-precontinuous.*

Proof. Let $x \in X$ and let V be an open set of Y containing $f(x)$. Since f is weakly δ-precontinuous, there exists $U \in \delta PO(X, x)$ such that $f(U) \subset \mathrm{cl}(V)$. Since f is almost δ-preopen, $f(U) \subset \mathrm{int}(\mathrm{cl}(f(U))) \subset \mathrm{int}(\mathrm{cl}(V))$ and hence f is almost δ-precontinuous. □

Definition 35. A space X is said to be

(1) almost regular [22] if for any regular closed set F of X and any point $x \in X \backslash F$ there exist disjoint open sets U and V such that $x \in U$ and $F \subset V$,

(2) semi-regular if for any open set U of X and each point $x \in U$ there exists a regular open set V of X such that $x \in V \subset U$.

Theorem 36. *If $f : X \to Y$ is a weakly δ-precontinuous function and Y is almost regular, then f is almost δ-precontinuous.*

Proof. Let $x \in X$ and let V be any open set of Y containing $f(x)$. By the almost regularity of Y, there exists a regular open set G of Y such that $f(x) \in G \subset \mathrm{cl}(G) \subset \mathrm{int}(\mathrm{cl}(V))$ [22, Theorem 2.2]. Since f is weakly δ-precontinuous, there exists $U \in \delta PO(X, x)$ such that $f(U) \subset \mathrm{cl}(G) \subset \mathrm{int}(\mathrm{cl}(V))$. Therefore, f is almost δ-precontinuous. □

Theorem 37. *If $f : X \to Y$ is an almost δ-precontinuous function and Y is semi-regular, then f is δ-almost continuous.*

Proof. Let $x \in X$ and let V be an open set of Y containing $f(x)$. By the semi-regularity of Y, there exists a regular open set G of Y such that $f(x) \in G \subset V$. Since f is almost δ-precontinuous, there exists $U \in \delta PO(X, x)$ such that $f(U) \subset \mathrm{int}(\mathrm{cl}(G)) = G \subset V$ and hence f is δ-almost continuous. □

Theorem 38. *Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then the following hold*:

(1) *If f is almost δ-precontinuous and g is an R-map, then the composition $g \circ f : X \to Z$ is almost δ-precontinuous,*

(2) *If f is δ-preirresolute and g is almost δ-precontinuous, the composition $g \circ f : X \to Z$ is almost δ-precontinuous.*

Definition 39. A function $f : X \to Y$ is said to be faintly δ-precontinuous if for each $x \in X$ and each θ-open set V of Y containing $f(x)$, there exists $U \in \delta PO(X,x)$ such that $f(U) \subset V$.

Theorem 40. *Let $f : X \to Y$ be a function. Suppose that Y is regular. Then, the following properties are equivalent*:

(1) *f is δ-almost continuous,*

(2) *$f^{-1}(\delta - \mathrm{cl}(B))$ is δ-preclosed in X for every subset B of Y,*

(3) *f is almost δ-precontinuous,*

(4) *f is weakly δ-precontinuous,*

(5) *f is faintly δ-precontinuous.*

Proof. (1)⇒(2). Since $\delta-\mathrm{cl}(B)$ is closed in Y for every subset B of Y, $f^{-1}(\delta-\mathrm{cl}(B))$ is δ-preclosed in X.

(2)⇒(3). For any subset B of Y, $f^{-1}(\delta - \mathrm{cl}(B))$ is δ-preclosed in X and hence we have $\delta - \mathrm{pcl}(f^{-1}(B)) \subset \delta - \mathrm{pcl}(f^{-1}(\delta - \mathrm{cl}(B))) = f^{-1}(\delta\text{-}\mathrm{cl}(B))$. It follows that f is almost δ-precontinuous

(3)⇒(4). This is obvious.

(4)⇒(5). Let A be any subset of X. Let $x \in \delta - \mathrm{pcl}(A)$ and V be any open set of Y containing $f(x)$. There exists $U \in \delta PO(X,x)$ such that $f(U) \subset \mathrm{cl}(V)$. Since $x \in \delta - \mathrm{pcl}(A)$, we have $U \cap A \neq \varnothing$ and hence $\varnothing \neq f(U) \cap f(A) \subset \mathrm{cl}(V) \cap f(A)$. Therefore, we have $f(x) \in \theta - \mathrm{cl}(f(A))$ and hence $f(\delta - \mathrm{pcl}(A)) \subset \theta - \mathrm{cl}(f(A))$.

Let B be any subset of Y. We have $f(\delta - \mathrm{pcl}(f^{-1}(B))) \subset \theta - \mathrm{cl}(B)$ and $\delta - \mathrm{pcl}(f^{-1}(B)) \subset f^{-1}(\theta - \mathrm{cl}(B))$.

Let F be any θ-closed set of Y. It follows that $\delta - \mathrm{pcl}(f^{-1}(F)) \subset f^{-1}(\theta - \mathrm{cl}(F)) = f^{-1}(F)$. Therefore $f^{-1}(F)$ is δ-preclosed in X and hence f is faintly δ-precontinuous.

(5)⇒(1). Let V be any open set of Y. Since Y is regular, V is θ-open in Y. By the faint δ-precontinuity of f, $f^{-1}(V)$ is δ-preopen in X. Therefore, f is δ-almost continuous. □

Recall that a space (X,τ) is said to be (1) submaximal [3] if every dense subset of X is open in X, (2) extremally disconnected [3, 15] if $\mathrm{cl}(U) \in \tau$ for every $U \in \tau$.

Definition 41. A function $f : X \to Y$ is said to be faintly continuous [9] (resp. faintly semi-continuous [19], faintly precontinuous [19], faintly β-continuous [12], [19], faintly α-continuous [12]) if $f^{-1}(V)$ is open (resp. semi-open, preopen, β-open, α-open) in X for each θ-open set V of of Y.

Theorem 42. *If (X,τ) is submaximal extremally disconnected semi-regular and (Y,σ) is regular, then the following are equivalent for a function $f : (X,\tau) \to (Y,\sigma)$:*

(1) *f is faintly continuous,*

(2) *f is faintly α-continuous,*

(3) *f is faintly semi-continuous,*

(4) *f is faintly precontinuous,*

(5) *f is faintly δ-precontinuous,*

(6) *f is faintly γ-continuous,*

(7) *f is faintly β-continuous,*

(8) *f is δ-almost continuous,*

(9) *f is almost δ-precontinuous,*

(10) *f is weakly δ-precontinuous.*

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