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On the Weighted L^1-convergence of Grünwald Interpolatory Operators

Jian Li Wang
Department of Mathematics, Shaoxing Arts and Science College, Shaoxing, Zhejiang 312000, China

Song Ping Zhou
Institute of Mathematics, Zhejiang Sci-Tech University, Xiasha Economic Development Area, Hangzhou, Zhejiang 310018, China
e-mail : szhou@zjip.com

Abstract. The present paper investigates the weighted L^1-convergence of Grünwald interpolatory operators based on the zeros of the second Chebyshev polynomials $U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}$. The approximation rate is sharp.

1. Introduction

Let $f \in C_{[-1,1]}$, taking $\{x_k^n\}_{k=1}^n = \{x_k\}_{k=1}^n$, the zeros of the second Chebyshev polynomials $U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}$, $x = \cos\theta$, as the nodes, then we define the famous Grünwald interpolatory operators as follows:

$$G_n(f,x) = \sum_{k=1}^n f(x_k) l_k^2(x),$$

where

$$l_k(x) = \frac{U_n(x)}{U_n'(x_k)(x - x_k)}, \quad k = 1, 2, \cdots, n.$$

G. Min (see [3] and [4]) proved that $G_n(f,x)$ uniformly converges to $f(x)$ in any closed interval $[a,b] \subset (-1,1)$, and it also converges to $f(x)$ in the L^1 norm (furthermore, [2] obtained the L^1-convergence rate). In order to analyse the nature of L^1-convergence by the Grünwald operators completely, the present paper will investigate the weighted case and establish the following

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Theorem. *For any* $f \in C_{[-1,1]}$, *the following estimates*

$$\int_{-1}^{1} |G_n(f,x) - f(x)| \, (\sqrt{1-x^2})^{\lambda} dx \le \begin{cases} C_\lambda \left[\omega_\varphi(f, \frac{1}{n}) + \frac{1}{n}\|f\|\right], & \lambda > 0, \\ C \left[\omega_\varphi(f, \frac{1}{n}) + \frac{\log n}{n}\|f\|\right], & \lambda = 0, \\ C_\lambda \left[\omega_\varphi(f, \frac{1}{n}) + \frac{1}{n^{1+\lambda}}\|f\|\right], & -1 \le \lambda < 0 \end{cases}$$

hold, where $\omega_\varphi(f,h)$ *is the Ditzian-Totik type modulus with* $\varphi(x) = \sqrt{1-x^2}$, $\|f\|$ *denotes the supremum norm on* $[-1,1]$, C *and* C_λ *denotes an absolute positive constant and a positive constant only depending on* λ *respectively, their values may be different even in the same line.*

2. Proof of Theorem

We establish some lemmas.

Lemma 1. *For any* $f \in C_{[-1,1]}$, $\lambda > -1$, *it holds that*

$$\int_{-1}^{1} |G_n(f,x)| \, (\sqrt{1-x^2})^{\lambda} dx \le C_\lambda \|f\|. \tag{1}$$

Proof. From [5], we have

$$\sum_{k=1}^{n} l_k^2(x) \le C \left[1 + \frac{\log n}{n}(1-x^2)^{-1}\right], \;\; x \in (-1,1).$$

Obviously, $G_n(f,x)$ is a polynomial of degree $\le 2n-1$. By using the inequality [1, (8.1.4)], we have

$$\begin{aligned} (2) \qquad & \int_{-1}^{1} |G_n(f,x)| \, (\sqrt{1-x^2})^{\lambda} dx \\ \le \;& C \int_{-1+(2n)^{-2}}^{1-(2n)^{-2}} |G_n(f,x)| \, (\sqrt{1-x^2})^{\lambda} dx \\ \le \;& C\|f\| \int_{-1+(2n)^{-2}}^{1-(2n)^{-2}} \left(\sum_{k=1}^{n} l_k^2(x)\right) (\sqrt{1-x^2})^{\lambda} dx \\ \le \;& C\|f\| + C\frac{\log n}{n}\|f\| \int_{-1+(2n)^{-2}}^{1-(2n)^{-2}} \left(1-x^2\right)^{-1} (\sqrt{1-x^2})^{\lambda} dx, \end{aligned}$$

and

$$\begin{aligned}(3)\qquad & \int_{-1+(2n)^{-2}}^{1-(2n)^{-2}} \left(1-x^2\right)^{-1} (\sqrt{1-x^2})^{\lambda} dx \\ \le\ & C\int_{-1+(2n)^{-2}}^{1-(2n)^{-2}} \left(\frac{1}{(1-x)^{1-\frac{\lambda}{2}}}+\frac{1}{(1+x)^{1-\frac{\lambda}{2}}}\right) dx \\ \le\ & \begin{cases} C_\lambda, & \lambda>0, \\ C\log n, & \lambda=0, \\ C_\lambda n^{-\lambda}, & -1<\lambda<0. \end{cases}\end{aligned}$$

Together with (2) and (3), (1) is proved. □

Write $P_n(x)$ as the best polynomial approximant of degree n to $f(x)$, and

$$\sigma_k(x)=(x-x_k)\frac{1-x^2}{1-x_k^2}l_k^2(x),\ k=1,2,\cdots,n.$$

Lemma 2. *For any $f\in C_{[-1,1]}$, $\lambda>-1$, we have*

$$(4)\qquad \int_{-1}^{1}|f(x)-P_n(x)|\,(\sqrt{1-x^2})^{\lambda}dx \le C_\lambda\omega_\varphi\left(f,\frac{1}{n}\right),$$

and

$$(5)\qquad \int_{-1}^{1}|G_n(f,x)-G_n(P_n,x)|\,(\sqrt{1-x^2})^{\lambda}dx \le C_\lambda\omega_\varphi\left(f,\frac{1}{n}\right).$$

Proof. Using [1, Theorem 7.21], we know

$$\|f-P_n\|\le C\omega_\varphi\left(f,\frac{1}{n}\right),$$

which means (4) holds. Applying (1) and noting that $G_n(f,x)$ is a positive linear operator, we have

$$\begin{aligned}& \int_{-1}^{1}|G_n(f,x)-G_n(P_n,x)|\,(\sqrt{1-x^2})^{\lambda}dx \\ \le\ & \|f-P_n\|\int_{-1}^{1}G_n(1,x)(\sqrt{1-x^2})^{\lambda}dx \le C_\lambda\omega_\varphi\left(f,\frac{1}{n}\right),\end{aligned}$$

therefore, (5) holds. □

Lemma 3. *If f is a polynomial of degree $2n+1$, then we have the following identity:*

$$G_n(f,x)-f(x) = (G_n(f,1)-f(1))\,\frac{1+x}{2}\left(\frac{U_n(x)}{U_n(1)}\right)^2 + (G_n(f,-1)-f(-1))\,\frac{1-x}{2}\left(\frac{U_n(x)}{U_n(-1)}\right)^2 + \sum_{k=1}^{n} f(x_k)\frac{3x_k}{1-x_k^2}\sigma_k(x) - \sum_{k=1}^{n} f'(x_k)\sigma_k(x). \tag{6}$$

Proof. Write

$$H_n(x) = G_n(f,x)-f(x)-\Bigg((G_n(f,1)-f(1))\,\frac{1+x}{2}\left(\frac{U_n(x)}{U_n(1)}\right)^2 + (G_n(f,-1)-f(-1))\,\frac{1-x}{2}\left(\frac{U_n(x)}{U_n(-1)}\right)^2 + \sum_{k=1}^{n} f(x_k)\frac{3x_k}{1-x_k^2}\sigma_k(x) - \sum_{k=1}^{n} f'(x_k)\sigma_k(x)\Bigg).$$

Since $G_n(f,x)$ is a polynomial of degree $\leq 2n-2$, $H_n(x)$ is a polynomial of degree $2n+1$. We check that

$$H_n(x_k)=0,\quad k=1,2,\cdots,n;\quad H_n(\pm 1)=0.$$

In view of that $l_k(x_k)=1$, we see that

$$\left(l_k^2(x)\right)'\Big|_{x=x_j}=0,\quad j\neq k,$$

$$\left(l_k^2(x)\right)'\Big|_{x=x_k} = 2l_k'(x_k)=\frac{2}{U_n'(x_k)}\sum_{j\neq k}\frac{U_n(x)}{(x-x_k)(x-x_j)}\Bigg|_{x=x_k} = \frac{U_n''(x_k)}{U_n'(x_k)}=\frac{3x_k}{1-x_k^2},$$

hence

$$G_n'(f,x_k)-f'(x_k) = f(x_k)\frac{3x_k}{1-x_k^2}-f'(x_k) = f(x_k)\frac{3x_k}{1-x_k^2}\sigma_k'(x_k)-f'(x_k)\sigma_k'(x_k),$$

thus $H_n'(x_k)=0$, $k=1,2,\cdots,n$. A polynomial of degree $2n+1$ vanishes at $2n+2$ points (multiplicity calculated) must equal to zero. Lemma 3 is proved. □

It is not difficult to deduce that

Lemma 4. *For $p > q$,*

$$\int_0^\pi \frac{|\sin(n+1)\theta|^p}{\sin^q\theta} d\theta \simeq \begin{cases} \log n, & q = 1, \\ n^{q-1}, & q > 1. \end{cases}$$

Lemma 5. *For any $\lambda > -1$, we have*

(7)
$$\int_{-1}^1 \left| (G_n(P_n,1) - P_n(1)) \frac{1+x}{2} \left(\frac{U_n(x)}{U_n(1)}\right)^2 \right| (\sqrt{1-x^2})^\lambda dx$$
$$\leq \begin{cases} C_\lambda n^{-1}\|f\|, & \lambda > 0, \\ Cn^{-1}\log n\|f\|, & \lambda = 0, \\ C_\lambda n^{-1-\lambda}\|f\|, & -1 < \lambda < 0, \end{cases}$$

and

(8)
$$\int_{-1}^1 \left| (G_n(P_n,-1) - P_n(-1)) \frac{1-x}{2} \left(\frac{U_n(x)}{U_n(-1)}\right)^2 \right| (\sqrt{1-x^2})^\lambda dx$$
$$\leq \begin{cases} C_\lambda n^{-1}\|f\|, & \lambda > 0, \\ Cn^{-1}\log n\|f\|, & \lambda = 0, \\ C_\lambda n^{-1-\lambda}\|f\|, & -1 < \lambda < 0. \end{cases}$$

Proof. It is easy to check that $U_n(\pm 1) = n + 1$ and

$$\sum_{k=1}^n l_k^2(\pm 1) = \frac{3n-1}{2}.$$

Then we use that $\|P_n\| \leq 2\|f\|$ to yield that

$$\begin{aligned} &\int_{-1}^1 \left| (G_n(P_n,1) - P_n(1)) \frac{1+x}{2} \left(\frac{U_n(x)}{U_n(1)}\right)^2 \right| (\sqrt{1-x^2})^\lambda dx \\ \leq\ & 2\|f\| \left(1 + \sum_{k=1}^n l_k^2(1)\right) \frac{1}{(n+1)^2} \int_{-1}^1 U_n^2(x)(\sqrt{1-x^2})^\lambda dx \\ \leq\ & C\|f\|n^{-1} \int_{-1}^1 U_n^2(x)(\sqrt{1-x^2})^\lambda dx \\ \leq\ & C\|f\|n^{-1} \int_0^\pi \frac{\sin^2 n\theta}{\sin^{1-\lambda}\theta} d\theta. \end{aligned}$$

Applying Lemma 4, with simple calculations for different cases $\lambda > 0$, $\lambda = 0$ and $-1 < \lambda < 0$, leads to (7). (8) can be proved similarly. □

Lemma 6. *For any $\lambda > -2$, we have*

$$\int_{-1}^{1}\left|\sum_{k=1}^{n} P_n(x_k)\frac{3x_k}{1-x_k^2}\sigma_k(x)\right|(\sqrt{1-x^2})^{\lambda}dx \tag{9}$$
$$\leq \begin{cases} C_\lambda n^{-1}\|f\|, & \lambda > -\frac{1}{2}, \\ Cn^{-1}\log n\|f\|, & \lambda = -\frac{1}{2}, \\ C_\lambda n^{-\lambda-\frac{3}{2}}\|f\|, & -2 < \lambda < -\frac{1}{2}. \end{cases}$$

Proof. We have (cf. [2], (22))

$$\int_{-1}^{1}\left(\sum_{k=1}^{n} P_n(x_k)\frac{3x_k}{1-x_k^2}\sigma_k(x)\right)^2\sqrt{1-x^2}dx \leq \frac{180\pi}{n^2}\|f\|^2.$$

Note that $\sum_{k=1}^{n} P_n(x_k)\frac{3x_k}{1-x_k^2}\sigma_k(x)$ is a polynomial of degree $\leq 2n+1$, in a similar way to the proof of Lemma 1, we have

$$\begin{aligned} &\int_{-1}^{1}\left|\sum_{k=1}^{n} P_n(x_k)\frac{3x_k}{1-x_k^2}\sigma_k(x)\right|(\sqrt{1-x^2})^{\lambda}dx \\ \leq\ & C\int_{-1+n^{-2}}^{1-n^{-2}}\left|\sum_{k=1}^{n} P_n(x_k)\frac{3x_k}{1-x_k^2}\sigma_k(x)\right|(\sqrt{1-x^2})^{\lambda}dx \\ \leq\ & \left\{\int_{-1+n^{-2}}^{1-n^{-2}}\left(\sum_{k=1}^{n} P_n(x_k)\frac{3x_k}{1-x_k^2}\sigma_k(x)\right)^2\sqrt{1-x^2}dx\right\}^{\frac{1}{2}}\left\{\int_{-1+n^{-2}}^{1-n^{-2}}(\sqrt{1-x^2})^{2\lambda-1}dx\right\}^{\frac{1}{2}} \\ \leq\ & Cn^{-1}\|f\|\left\{\int_{-1+n^{-2}}^{1-n^{-2}}(\sqrt{1-x^2})^{2\lambda-1}dx\right\}^{\frac{1}{2}} \\ \leq\ & Cn^{-1}\|f\|\left\{\int_{-1+n^{-2}}^{1-n^{-2}}\left(\frac{1}{(1-x)^{\frac{1}{2}-\lambda}}+\frac{1}{(1+x)^{\frac{1}{2}-\lambda}}\right)dx\right\}^{\frac{1}{2}}, \end{aligned}$$

then (9) can be verified easily. □

Lemma 7. *For any $\lambda > -1$, we have*

$$\int_{-1}^{1}\left|\sum_{k=1}^{n} P_n'(x_k)\sigma_k(x)\right|(\sqrt{1-x^2})^{\lambda}dx \leq C_\lambda\omega_\varphi\left(f, \frac{1}{n}\right). \tag{10}$$

Proof. From [6, (32)], we have

$$\int_{-1}^{1}\left(\sum_{k=1}^{n} P_n'(x_k)\sigma_k(x)\right)^2 \frac{dx}{\sqrt{1-x^2}} \le C\omega_\varphi^2\left(f,\frac{1}{n}\right),$$

because $\lambda > -1,\ 1+2\lambda > -1$, by Hölder inequality, we get

$$\begin{aligned}
&\int_{-1}^{1}\left|\sum_{k=1}^{n} P_n'(x_k)\sigma_k(x)\right|(\sqrt{1-x^2})^\lambda dx \\
\le\ & \left\{\int_{-1}^{1}\left(\sum_{k=1}^{n} P_n'(x_k)\sigma_k(x)\right)^2 \frac{dx}{\sqrt{1-x^2}}\right\}^{\frac{1}{2}}\left\{\int_{-1}^{1}(\sqrt{1-x^2})^{2\lambda+1}dx\right\}^{\frac{1}{2}} \\
\le\ & C_\lambda\omega_\varphi\left(f,\frac{1}{n}\right).
\end{aligned}$$

□

Proof of Theorem. Altogether, with the above lemmas, we can proceed the proof of our theorem now. Since $G_n(f,x)$ is linear, by applying lemma 3, we then have

$$\begin{aligned}
& G_n(f,x)-f(x) \\
=\ & G_n(f,x)-G_n(P_n,x)+G_n(P_n,x)-P_n(x)+P_n(x)-f(x) \\
=\ & (G_n(f,x)-G_n(P_n,x))+(P_n(x)-f(x))+(G_n(P_n,1)-P_n(1))\frac{1+x}{2}\left(\frac{U_n(x)}{U_n(1)}\right)^2 \\
& +(G_n(P_n,-1)-P_n(-1))\frac{1-x}{2}\left(\frac{U_n(x)}{U_n(-1)}\right)^2 \\
& +\sum_{k=1}^{n} P_n(x_k)\frac{3x_k}{1-x_k^2}\sigma_k(x)-\sum_{k=1}^{n} P_n'(x_k)\sigma_k(x),
\end{aligned}$$

with (4), (5), (7)-(10), we can get the required result. □

3. Remarks

For $\lambda \le -1$, there exist an $f_0(x) \in C_{[-1,1]}$ such that

$$\int_{-1}^{1}|G_n(f_0,x)-f_0(x)|(\sqrt{1-x^2})^\lambda dx \not\longrightarrow 0,\ n\to\infty. \tag{11}$$

In fact, take $f_0(x)=1$, then we have

$$G_n(f_0,x)-f_0(x)=\frac{3}{2}\frac{n-1}{(n+1)^2}U_n^2(x)+\sum_{k=1}^{n}\frac{3x_k}{1-x_k^2}\sigma_k(x).$$

When $\lambda \geq -1$, by Lemma 4 we have

$$\begin{aligned}\int_{-1}^{1} \frac{3}{2}\frac{n-1}{(n+1)^2}U_n^2(x)(\sqrt{1-x^2})^{\lambda}dx &\geq \frac{C}{n}\int_0^{\pi}\frac{\sin^2 n\theta}{\sin^{1-\lambda}\theta}d\theta \\ &\geq \begin{cases} C, & \lambda=-1, \\ C_{\lambda}n^{-1-\lambda}, & -1<\lambda<0, \\ C\log n/n, & \lambda=0, \\ C_{\lambda}n^{-1}, & \lambda>0, \end{cases}\end{aligned}$$

together with (9), we have

$$\int_{-1}^{1}\left|\sum_{k=1}^{n}\frac{3x_k}{1-x_k^2}\sigma_k(x)\right|(\sqrt{1-x^2})^{\lambda}dx = o\left(\int_{-1}^{1}\frac{3}{2}\frac{n-1}{(n+1)^2}U_n^2(x)(\sqrt{1-x^2})^{\lambda}dx\right),$$

that means, (11) holds for $\lambda=-1$.

The above estimates also show that our results are sharp for $\lambda>-1$.

We conclude that the result of [2] is a special case $\lambda=0$ of our theorem.

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