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Classes of Multivalent Functions Defined by Dziok-Srivastava Linear Operator and Multiplier Transformation

S. Sivaprasad Kumar and H. C. Taneja
Department of Applied Mathematics, Delhi College of Engineering, Delhi-110042, India
e-mail : sivpk71@yahoo.com *and* hctaneja47@rediffmail.com

V. Ravichandran
School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM Penang, Malaysia
e-mail : vravi@cs.usm.my

Abstract. In this paper, the authors introduce new classes of p-valent functions defined by Dziok-Srivastava linear operator and the multiplier transformation and study their properties by using certain first order differential subordination and superordination. Also certain inclusion relations are established and an integral transform is discussed.

1. Preliminaries

Let $\mathcal{H}$ be the class of functions analytic in $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}(a,n)$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$. Let $\mathcal{A}_p$ denote the class of all analytic functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (z \in \Delta) \tag{1.1}$$

and let $\mathcal{A} := \mathcal{A}_1$. For two functions $f(z)$ given by (1.1) and $g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) := z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k =: (g * f)(z). \tag{1.2}$$

For $\alpha_j \in \mathbb{C} \quad (j = 1, 2, \cdots, l)$ and $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, \cdots\}$ $(j = 1, 2, \cdots, m)$, the *generalized hypergeometric function* ${}_lF_m(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m; z)$ is defined by the

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infinite series

$$ {}_lF_m(\alpha_1,\cdots,\alpha_l;\beta_1,\cdots,\beta_m;z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n\cdots(\alpha_l)_n}{(\beta_1)_n\cdots(\beta_m)_n}\frac{z^n}{n!} $$

$$ (l \leq m+1;\ l,m \in \mathbb{N}_0 := \{0,1,2,\cdots\}) $$

where $(\lambda)_n$ is the Pochhammer symbol defined by

$$ (\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & (n=0); \\ \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1), & (n \in \mathbb{N} := \{1,2,3\cdots\}). \end{cases} $$

Corresponding to the function

$$ h_p(\alpha_1,\cdots,\alpha_l;\beta_1,\cdots,\beta_m;z) := z^p\ {}_lF_m(\alpha_1,\cdots,\alpha_l;\beta_1,\cdots,\beta_m;z), $$

the Dziok-Srivastava operator [7] (see also [18]) $H_p^{(l,m)}(\alpha_1,\cdots,\alpha_l;\beta_1,\cdots,\beta_m)$ is defined by the Hadamard product

$$ \begin{aligned} &H_p^{(l,m)}(\alpha_1,\cdots,\alpha_l;\beta_1,\cdots,\beta_m)f(z) \\ :=\ & h_p(\alpha_1,\cdots,\alpha_l;\beta_1,\cdots,\beta_m;z) * f(z) \\ =\ & z^p + \sum_{n=p+1}^{\infty} \frac{(\alpha_1)_{n-p}\cdots(\alpha_l)_{n-p}}{(\beta_1)_{n-p}\cdots(\beta_m)_{n-p}}\frac{a_n z^n}{(n-p)!}. \end{aligned} \tag{1.3} $$

Special cases of the Dziok-Srivastava linear operator includes the Hohlov linear operator [8], the Carlson-Shaffer linear operator [4], the Ruscheweyh derivative operator [16], the generalized Bernardi-Libera-Livingston integral operator (*cf.* [1], [9], [11]) and the Srivastava-Owa fractional derivative operators (*cf.* [14], [15]).

A function $f \in \mathcal{A}_p$ is said to be in the class $H_{a,c,p}(A,B)$ if it satisfies the following subordination:

$$ \frac{L_p(a+1,c)f(z)}{L_p(a,c)f(z)} \prec 1 + \frac{A-B}{a}\frac{z}{1+Bz} \quad (z \in \Delta;\ \ a \neq 0; -1 \leq B < A \leq 1) $$

where $L_p(a,c)$ is the familiar Carlson and Shaffer operator [4] given by $L_p(a,c) := H_p^{(2,1)}(a,1;c)$. This class $H_{a,c,p}(A,B)$ was introduced by Liu and Owa [10] and they have proved the following:

Theorem 1.1 (Liu and Owa [10, Theorem 1, p.1715])**.** *Let* $a \geq \frac{A-B}{1-B}$, *then*

$$ H_{a+1,c,p}(A,B) \subset H_{a,c,p}(A,B). $$

Theorem 1.2 (Liu and Owa [10, Theorem 2, p.1716])**.** *Let λ be a complex nuumber such that* $\Re\lambda > \frac{A-B}{1-B} - p$. *If* $f \in H_{a,c,p}(A,B)$, *then the function*

$$ F(z) = \frac{\lambda+p}{z^\lambda}\int_0^z t^{\lambda-1}f(t)dt $$

belongs to $H_{a,c,p}(A,B)$.

Theorem 1.3 (Liu and Owa [10, Theorem 3, p.1717])**.** *Let* $f(z) \in \mathcal{A}_p$. *Then* $f \in H_{a,c,p}(A,B)$ *if and only if*

$$F(z) = \frac{a}{z^{a-p}} \int_0^z t^{a-p-1} f(t)dt \in H_{a+1,c,p}(A,B).$$

For two analytic functions f and F, we say that F is *superordinate* to f if f is subordinate to F. Recently Miller and Mocanu [13] considered certain second order differential superordinations. Using the results of Miller and Mocanu [13], Bulboacă have considered certain classes of first order differential superordinations [3] and superordination-preserving integral operators [2]. In the present investigation, we generalize the above-stated results of Liu and Owa [10] to a more general classes of p-valent functions which we define below using subordination and superordination.

Definition 1. A function $f \in \mathcal{A}(p,n)$ is said to be in the class $\mathcal{A}(p,n,\alpha_1 \cdots \alpha_l; \beta_1 \cdots \beta_m; \varphi)$ if it satisfies the following subordination:

$$\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} \prec \varphi(z), \tag{1.4}$$

and is said to be in $\overline{\mathcal{A}}(p,n,\alpha_1 \cdots \alpha_l; \beta_1 \cdots \beta_m; \varphi)$ if f satisfies the following superordination:

$$\varphi(z) \prec \frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \tag{1.5}$$

where $\varphi(z)$ is analytic in Δ and $\varphi(0) = 1$ and

$$H_p^{l,m}[\alpha_1]f(z) := H_p^{(l,m)}(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m)f(z).$$

To make the notation simple, we also write

$$\mathcal{A}(p,n,\alpha_1;\varphi) := \mathcal{A}(p,n,\alpha_1 \cdots \alpha_l; \beta_1 \cdots \beta_m; \varphi)$$

and

$$\overline{\mathcal{A}}(p,n,\alpha_1;\varphi) := \overline{\mathcal{A}}(p,n,\alpha_1 \cdots \alpha_l; \beta_1 \cdots \beta_m; \varphi).$$

Also we define the class $\mathcal{A}(p,n,\alpha_1;\varphi_1,\varphi_2)$ by the following:

$$\mathcal{A}(p,n,\alpha_1;\varphi_1,\varphi_2) := \overline{\mathcal{A}}(p,n,\alpha_1;\varphi_1) \cap \mathcal{A}(p,n,\alpha_1;\varphi_2).$$

For

$$\varphi(z) = 1 + \frac{A-B}{a}\frac{z}{1+Bz} \quad (z \in \Delta; \;\; -1 \le B < A \le 1),$$

the class $\mathcal{A}(p,n,a,1;c;\varphi)$ reduces to the class $H_{a,c,p}(A,B)$, introduced and studied by Liu and Owa [10].

Motivated by the multiplier transformation on $\mathcal{A}$, we define the operator $I_p(r,\lambda)$ on $\mathcal{A}_p$ by the following infinite series

$$I_p(r,\lambda)f(z) := z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda}\right)^r a_k z^k \quad (\lambda \geq 0). \tag{1.6}$$

The operator $I_p(r,\lambda)$ is closely related to the Sălăgean derivative operators [17]. The operator $I_\lambda^r := I_1(r,\lambda)$ was studied recently by Cho and Srivastava [5] and Cho and Kim [6]. The operator $I_r := I_1(r,1)$ was studied by Uralegaddi and Somanatha [19].

Definition 2. A function $f \in \mathcal{A}(p,n)$ is said to be in the class $\mathcal{A}(p,n,r,\lambda;\varphi)$ if it satisfies the following subordination:

$$\frac{I_p(r+1,\lambda)f(z)}{I_p(r,\lambda)f(z)} \prec \varphi(z) \quad (f \in \mathcal{A}(p,n)), \tag{1.7}$$

and is said to be in $\overline{\mathcal{A}}(p,n,r,\lambda;\varphi)$ if f satisfies the following superordination:

$$\varphi(z) \prec \frac{I_p(r+1,\lambda)f(z)}{I_p(r,\lambda)f(z)} \quad (f \in \mathcal{A}(p,n)), \tag{1.8}$$

where $\varphi(z)$ is analytic in Δ and $\varphi(0)=1$. Also we define the class $\mathcal{A}(p,n,r,\lambda;\varphi_1,\varphi_2)$ by the following:

$$\mathcal{A}(p,n,r,\lambda;\varphi_1,\varphi_2) := \overline{\mathcal{A}}(p,n,r,\lambda;\varphi_1) \cap \mathcal{A}(p,n,r,\lambda;\varphi_2).$$

In our present investigation of the above-defined classes, we need the following:

Definition 3 [13, Definition 2, p. 817]**.** Denote by $\mathcal{Q}$, the set of all functions $f(z)$ that are analytic and injective on $\overline{\Delta} - E(f)$, where

$$E(f) = \{\zeta \in \partial\Delta : \lim_{z\to\zeta} f(z) = \infty\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\Delta - E(f)$.

Lemma 1 [cf. Miller and Mocanu [12, Theorem 3.4h, p.132]]**.** *Let $\psi(z)$ be univalent in the unit disk $\triangle$ and let ϑ and φ be analytic in a domain $D \supset \psi(\triangle)$ with $\varphi(w) \neq 0$, when $w \in \psi(\triangle)$. Set*

$$Q(z) := z\psi'(z)\varphi(\psi(z)), \quad h(z) := \vartheta(\psi(z)) + Q(z).$$

Suppose that

(1) $Q(z)$ *is starlike in* $\triangle$, *and*

(2) $\Re \frac{zh'(z)}{Q(z)} > 0$ *for* $z \in \triangle$.

If $q(z)$ *is analytic in* Δ, *with* $q(0) = \psi(0)$, $q(\Delta) \subset D$ *and*

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(\psi(z)) + z\psi'(z)\varphi(\psi(z)), \tag{1.9}$$

then $q(z) \prec \psi(z)$ *and* $\psi(z)$ *is the best dominant.*

Lemma 2 ([3]). *Let* $\psi(z)$ *be univalent in the unit disk* Δ *and* ϑ *and* φ *be analytic in a domain* D *containing* $\psi(\Delta)$. *Suppose that*

(1) $\Re[\vartheta'(\psi(z))/\varphi(\psi(z))] > 0$ *for* $z \in \Delta$,

(2) $z\psi'(z)\varphi(\psi(z))$ *is starlike in* Δ.

If $q(z) \in \mathcal{H}(\psi(0), 1) \cap \mathcal{Q}$, *with* $q(\Delta) \subseteq D$, *and* $\vartheta(q(z)) + zq'(z)\varphi(q(z))$ *is univalent in* Δ, *then*

$$\vartheta(\psi(z)) + z\psi'(z)\varphi(\psi(z)) \prec \vartheta(q(z)) + zq'(z)\varphi(q(z)), \tag{1.10}$$

implies $\psi(z) \prec q(z)$ *and* $\psi(z)$ *is the best subordinant.*

2. Results involving Dziok-Srivastava linear operator

By making use of Lemma 1, we first prove the following generalization of Theorem 1.1:

Theorem 2.1. *Let* $\psi(z)$ *be univalent in* Δ, $\psi(0) = 1$. *Assume that* $z\psi'/\psi$ *is starlike in* Δ *and* $\Re\{\alpha_1\psi(z)\} > 0$. *Let* $\chi(z)$ *be defined by*

$$\chi(z) := \frac{1}{\alpha_1 + 1}\left[\alpha_1\psi(z) + 1 + \frac{z\psi'(z)}{\psi(z)}\right] \quad (\alpha_1 \neq -1). \tag{2.1}$$

If $f \in \mathcal{A}(p, n, \alpha_1 + 1; \chi)$, *then* $f \in \mathcal{A}(p, n, \alpha_1; \psi)$. *If* $f \in \overline{\mathcal{A}}(p, n, \alpha_1 + 1; \chi)$,

$$0 \neq \frac{H_p^{l,m}[\alpha_1 + 1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} \in \mathcal{H}(1, 1) \cap \mathcal{Q}, \quad \frac{H_p^{l,m}[\alpha_1 + 2]f(z)}{H_p^{l,m}[\alpha_1 + 1]f(z)} \text{ is univalent in } \Delta, \tag{2.2}$$

then $f \in \overline{\mathcal{A}}(p, n, \alpha_1; \psi)$.

Proof. Define the function $q(z)$ by

$$q(z) := \frac{H_p^{l,m}[\alpha_1 + 1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}. \tag{2.3}$$

Then, clearly, $q(z)$ is analytic in Δ. Also by a simple computation, we find from (2.3) that

$$\frac{zq'(z)}{q(z)} = \frac{z(H_p^{l,m}[\alpha_1 + 1]f(z))'}{H_p^{l,m}[\alpha_1 + 1]f(z)} - \frac{z(H_p^{l,m}[\alpha_1]f(z))'}{H_p^{l,m}[\alpha_1]f(z)}. \tag{2.4}$$

By making use of the identity

$$z(H_p^{l,m}[\alpha_1]f(z))' = \alpha_1 H_p^{l,m}[\alpha_1+1]f(z) - (\alpha_1 - p)H_p^{l,m}[\alpha_1]f(z), \tag{2.5}$$

we have from (2.4), that

$$\frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)} = \frac{1}{\alpha_1+1}\left(\alpha_1 q(z) + 1 + \frac{zq'(z)}{q(z)}\right). \tag{2.6}$$

Since $f \in \mathcal{A}(p, n, \alpha_1 + 1; \chi)$, we have from (2.6) that

$$\alpha_1 q(z) + \frac{zq'(z)}{q(z)} \prec \alpha_1 \psi(z) + \frac{z\psi'(z)}{\psi(z)}$$

and this can be written as (1.9), by defining

$$\vartheta(w) := \alpha_1 w \text{ and } \varphi(w) := \frac{1}{w}.$$

Note that $\varphi(w) \neq 0$ and $\vartheta(w)$, $\varphi(w)$ are analytic in $\mathbb{C} - \{0\}$. Set

$$Q(z) := \frac{z\psi'(z)}{\psi(z)} \tag{2.7}$$

$$h(z) := \vartheta(\psi(z)) + Q(z) = \alpha_1\psi(z) + \frac{z\psi'(z)}{\psi(z)}. \tag{2.8}$$

By the hypothesis of Theorem 2.1, $Q(z)$ is starlike and

$$\Re\left\{\frac{zh'(z)}{Q(z)}\right\} = \Re\left\{\alpha_1\psi(z) + 1 + \frac{z\psi''(z)}{\psi'(z)} - \frac{z\psi'(z)}{\psi(z)}\right\} > 0.$$

By an application of Lemma 1, we obtain that $q(z) \prec \psi(z)$ or

$$\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} \prec \psi(z),$$

which shows that $f \in \mathcal{A}(p, n, \alpha_1; \psi)$.

The other half of the Theorem 2.1 follows by a similar application of Lemma 2. □

Using Theorem 2.1, we obtain the following "sandwich result":

Corollary 1. *Let $\psi_i(z)$ be univalent in Δ, $\psi_i(0) = 1$ $(i = 1, 2)$. Further assume that $z\psi_i'(z)/\psi_i(z)$ is starlike univalent in Δ and $\Re\{\alpha_1\psi_i(z)\} > 0$ $(i = 1, 2)$. If $f \in \mathcal{A}(p, n, \alpha_1 + 1; \chi_1, \chi_2)$ satisfies* (2.2)*, then $f \in \mathcal{A}(p, n, \alpha_1; \psi_1, \psi_2)$, where*

$$\chi_i(z) := \frac{1}{\alpha_1+1}\left[\alpha_1\psi_i(z) + 1 + \frac{z\psi_i'(z)}{\psi_i(z)}\right] \quad (i = 1,\ 2; \alpha_1 \neq -1).$$

Theorem 2.2. *Let ψ be univalent in Δ, $\psi(0) = 1$, and λ be a complex number. Assume that $z\psi'/(\lambda+p-\alpha_1+\alpha_1\psi)$ is starlike in Δ and $\Re\{\lambda + p - \alpha_1 + \alpha_1\psi(z)\} > 0$. Define the functions F and h by*

$$F(z) := \frac{\lambda + p}{z^\lambda}\int_0^z t^{\lambda-1}f(t)dt, \tag{2.9}$$

$$h(z) := \psi(z) + \frac{z\psi'(z)}{\lambda + p - \alpha_1 + \alpha_1\psi(z)}.$$

If $f \in \mathcal{A}(p, n, \alpha_1; h)$, then $F \in \mathcal{A}(p, n, \alpha_1; \psi)$. If $f \in \overline{\mathcal{A}}(p, n, \alpha_1; h)$,

$$0 \neq \frac{H_p^{l,m}[\alpha_1 + 1]F(z)}{H_p^{l,m}[\alpha_1]F(z)} \in \mathcal{H}(1, 1) \cap \mathcal{Q} \text{ and } \frac{H_p^{l,m}[\alpha_1 + 1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} \text{ is univalent in } \Delta, \tag{2.10}$$

then the function $F \in \overline{\mathcal{A}}(p, n, \alpha_1; \psi)$.

Proof. From the definition of $F(z)$ and (2.5), we obtain that

$$\begin{aligned}(\lambda + p)H_p^{l,m}[\alpha_1]f(z) &= \lambda H_p^{l,m}[\alpha_1]F(z) + z(H_p^{l,m}[\alpha_1]F(z))' \\ &= \alpha_1 H_p^{l,m}[\alpha_1 + 1]F(z) + (\lambda + p - \alpha_1)H_p^{l,m}[\alpha_1]F(z). \end{aligned} \tag{2.11}$$

Define the function $q(z)$ by

$$q(z) := \frac{H_p^{l,m}[\alpha_1 + 1]F(z)}{H_p^{l,m}[\alpha_1]F(z)}. \tag{2.12}$$

Then, clearly, $q(z)$ is analytic in Δ. Using (2.11) and (2.12), we have

$$(\lambda + p)\frac{H_p^{l,m}[\alpha_1]f(z)}{H_p^{l,m}[\alpha_1]F(z)} = \lambda + p - \alpha_1 + \alpha_1 q(z). \tag{2.13}$$

Upon logarithmic differentiation of (2.13) and using (2.5), and (2.12), we get

$$\frac{H_p^{l,m}[\alpha_1 + 1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} = q(z) + \frac{zq'(z)}{\lambda + p - \alpha_1 + \alpha_1 q(z)}. \tag{2.14}$$

Since $f \in \mathcal{A}(p, n, \alpha_1; h)$, we have, from (2.14),

$$q(z) + \frac{zq'(z)}{\lambda + p - \alpha_1 + \alpha_1 q(z)} \prec \psi(z) + \frac{z\psi'(z)}{\lambda + p - \alpha_1 + \alpha_1\psi(z)}$$

and this can be written as (1.9), by defining

$$\vartheta(w) := w \text{ and } \varphi(w) := \frac{1}{\lambda + p - \alpha_1 + \alpha_1 w}.$$

Note that $\varphi(w) \neq 0$ and $\vartheta(w),\ \varphi(w)$ are analytic in $\mathbb{C} - \{\frac{\alpha_1 - p - \lambda}{\alpha_1}\}$. Set

$$\begin{aligned} Q(z) &:= \frac{z\psi'(z)}{\lambda + p - \alpha_1 + \alpha_1\psi(z)} \\ h(z) &:= \vartheta(\psi(z)) + Q(z) = \psi(z) + \frac{z\psi'(z)}{\lambda + p - \alpha_1 + \alpha_1\psi(z)}. \end{aligned}$$

By the hypothesis of Theorem 2.2, $Q(z)$ is starlike and

$$\Re\left\{\frac{zh'(z)}{Q(z)}\right\} = \Re\left\{\lambda + p - \alpha_1 + \alpha_1\psi(z) + 1 + \frac{z\psi''(z)}{\psi'(z)} - \frac{\alpha_1 z\psi'(z)}{\lambda + p - \alpha_1 + \alpha_1\psi(z)}\right\} > 0.$$

By an application of Lemma 1, we obtain that

$$q(z) \prec \psi(z)$$

or

$$\frac{H_p^{l,m}[\alpha_1 + 1]F(z)}{H_p^{l,m}[\alpha_1]F(z)} \prec \psi(z),$$

which shows that $F \in \mathcal{A}(p, n, \alpha_1; \psi)$. The second half of the Theorem 2.2 follows by a similar application of Lemma 2. □

Using Theorem 2.2, we have the following result:

Corollary 2. *Let ψ_i be univalent in Δ, $\psi_i(0) = 1$ $(i = 1, 2)$ and λ be a complex number. Assume that $z\psi_i'/(\lambda + p - \alpha_1 + \alpha_1\psi_i)$ is starlike in Δ and $\Re\{\lambda + p - \alpha_1 + \alpha_1\psi_i(z)\} > 0$ $(i = 1, 2)$. If $f \in \mathcal{A}(p, n, \alpha_1; h_1, h_2)$ satisfies* (2.10)*, then the function F defined by* (2.9) *belongs to $\mathcal{A}(p, n, \alpha_1; \psi_1, \psi_2)$ where*

$$h_i(z) := \psi_i(z) + \frac{z\psi_i'(z)}{\lambda + p - \alpha_1 + \alpha_1\psi_i(z)} \quad (i = 1, 2).$$

Theorem 2.3. *Let $f(z) \in \mathcal{A}(p, n)$ and $\alpha_1 \neq -1$. Define F by*

$$F(z) := \frac{\alpha_1}{z^{\alpha_1 - p}} \int_0^z t^{\alpha_1 - p - 1} f(t)dt. \tag{2.15}$$

Then $f \in \mathcal{A}(p, n, \alpha_1; \varphi)$ if and only if $F \in \mathcal{A}\left(p, n, \alpha_1 + 1; \frac{1+\alpha_1\varphi}{1+\alpha_1}\right)$. Also $f \in \overline{\mathcal{A}}(p, n, \alpha_1; \varphi)$ if and only if $F \in \overline{\mathcal{A}}\left(p, n, \alpha_1 + 1; \frac{1+\alpha_1\varphi}{1+\alpha_1}\right)$.

Proof. From (2.15), we have

$$\alpha_1 f(z) = (\alpha_1 - p)F(z) + zF'(z). \tag{2.16}$$

By convoluting (2.16) with $h_p(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m; z)$ and using the fact that $z(f*g)'(z) = f(z) * zg'(z)$, we obtain

$$\alpha_1 H_p^{l,m}[\alpha_1]f(z) = (\alpha_1 - p)H_p^{l,m}[\alpha_1]F(z) + z(H_p^{l,m}[\alpha_1]F(z))'$$

and by using (2.5), we get

$$H_p^{l,m}[\alpha_1]f(z) = H_p^{l,m}[\alpha_1+1]F(z) \tag{2.17}$$

and

$$\begin{aligned}\alpha_1 H_p^{l,m}[\alpha_1+1]f(z) &= z(H_p^{l,m}[\alpha_1]f(z))' + (\alpha_1 - p)H_p^{l,m}[\alpha_1]f(z)\\ &= z(H_p^{l,m}[\alpha_1+1]F(z))' + (\alpha_1 - p)H_p^{l,m}[\alpha_1+1]F(z)\\ &= (\alpha_1+1)H_p^{l,m}[\alpha_1+2]F(z) - (\alpha_1+1-p)H_p^{l,m}[\alpha_1+1]F(z)\\ &\quad + (\alpha_1 - p)H_p^{l,m}[\alpha_1+1]F(z)\\ &= (\alpha_1+1)H_p^{l,m}[\alpha_1+2]F(z) - H_p^{l,m}[\alpha_1+1]F(z). \qquad (2.18)\end{aligned}$$

Therefore, from (2.17) and (2.18), we have

$$\frac{H_p^{l,m}[\alpha_1+2]F(z)}{H_p^{l,m}[\alpha_1+1]F(z)} = \frac{1}{\alpha_1+1}\left[1 + \alpha_1\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}\right],$$

and the desired results follow at once. □

Using Theorem 2.3, we have

Corollary 3. *Let $f(z) \in \mathcal{A}(p,n)$ and $\alpha_1 \neq -1$. Then $f \in \mathcal{A}(p,n,\alpha_1;\varphi_1,\varphi_2)$ if and only if F given by* (2.15) *is in $\mathcal{A}\left(p,n,\alpha_1+1;\frac{1+\alpha_1\varphi_1}{1+\alpha_1},\frac{1+\alpha_1\varphi_2}{1+\alpha_1}\right)$.*

3. Results involving multiplier transformation

By making use of Lemma 1, we prove the following generalization of Theorem 1.1:

Theorem 3.1. *Let $\psi(z)$ be univalent in Δ, $\psi(0) = 1$, $\Re\psi(z) > 0$ and $z\psi'/\psi$ be starlike in Δ. Let $\chi(z)$ be defined by*

$$\chi(z) := \frac{1}{p+\lambda}\left[(p+\lambda)\psi(z) + \frac{z\psi'(z)}{\psi(z)}\right].$$

If $f \in \mathcal{A}(p,n,r+1,\lambda;\chi)$, then $f \in \mathcal{A}(p,n,r,\lambda;\psi)$. If $f \in \overline{\mathcal{A}}(p,n,r+1,\lambda;\chi)$,

$$0 \neq \frac{I_p(r+1,\lambda)f(z)}{I_p(r,\lambda)f(z)} \in \mathcal{H}(1,1)\cap\mathcal{Q}, \frac{I_p(r+2,\lambda)f(z)}{I_p(r+1,\lambda)f(z)} \text{ is univalent in } \Delta, \tag{3.1}$$

then $f \in \overline{\mathcal{A}}(p,n,r,\lambda;\psi)$.

Proof. Define the function $q(z)$ by

$$q(z) := \frac{I_p(r+1,\lambda)f(z)}{I_p(r,\lambda)f(z)}. \tag{3.2}$$

Then, clearly, $q(z)$ is analytic in Δ. Also by a simple computation, we find from (3.2) that

$$\frac{zq'(z)}{q(z)} = \frac{z(I_p(r+1,\lambda)f(z))'}{I_p(r+1,\lambda)f(z)} - \frac{z(I_p(r,\lambda)f(z))'}{I_p(r,\lambda)f(z)}. \tag{3.3}$$

By making use of the identity

$$(p+\lambda)I_p(r+1,\lambda)f(z) = z[I_p(r,\lambda)f(z)]' + \lambda I_p(r,\lambda)f(z), \tag{3.4}$$

we have from (3.3), that

$$\frac{I_p(r+2,\lambda)f(z)}{I_p(r+1,\lambda)f(z))} = \frac{1}{p+\lambda}\left((p+\lambda)q(z) + \frac{zq'(z)}{q(z)}\right). \tag{3.5}$$

Since $f \in \mathcal{A}(p,n,r+1,\lambda;\chi)$ and in view of (3.5), we have

$$(p+\lambda)q(z) + \frac{zq'(z)}{q(z)} \prec (p+\lambda)\psi(z) + \frac{z\psi'(z)}{\psi(z)}.$$

The first result follows by an application of Lemma 1. Similarly the second result follows from Lemma 2. □

Using Theorem 3.1, we obtain the following "sandwich result":

Corollary 4. *Let $\psi_i(z)$ be univalent in Δ, $\psi_i(0) = 1$, $\Re\psi_i(z) > 0$ and $z\psi_i'(z)/\psi_i(z)$ be starlike univalent in Δ for $i = 1, 2$. Define*

$$\chi_i(z) := \frac{1}{p+\lambda}\left[(p+\lambda)\psi_i(z) + \frac{z\psi_i'(z)}{\psi_i(z)}\right] \quad (i = 1, 2).$$

If $f \in \mathcal{A}(p,n,r+1,\lambda;\chi_1,\chi_2)$ satisfies (3.1), then $f \in \mathcal{A}(p,n,r,\lambda;\psi_1,\psi_2)$.

Theorem 3.2. *Let ψ be univalent in Δ, $\psi(0) = 1$, δ be a complex number, $z\psi'/(\delta - \lambda + (p+\lambda)\psi)$ be starlike in Δ and $\Re(\delta - \lambda + (p+\lambda)\psi(z)) > 0$. Define the function F by*

$$\begin{aligned} F(z) &:= \frac{\delta+p}{z^\delta}\int_0^z t^{\delta-1}f(t)dt, \\ h(z) &:= \psi(z) + \frac{z\psi'(z)}{\delta - \lambda + (p+\lambda)\psi(z)}. \end{aligned} \tag{3.6}$$

If $f \in \mathcal{A}(p,n,r,\lambda;h)$, then $F \in \mathcal{A}(p,n,r,\lambda;\psi)$. If $f \in \overline{\mathcal{A}}(p,n,r,\lambda;h)$,

$$0 \neq \frac{I_p(r+1,\lambda)F(z)}{I_p(r,\lambda)F(z)} \in \mathcal{H}(1,1)\cap\mathcal{Q}, \frac{I_p(r+1,\lambda)f(z)}{I_p(r,\lambda)f(z)} \text{ is univalent in } \Delta, \tag{3.7}$$

then $F \in \overline{\mathcal{A}}(p, n, r, \lambda; \psi)$.

Proof. From the definition of $F(z)$ and

$$z(I_p(r,\lambda)F(z))' = (p+\lambda)I_p(r+1,\lambda)F(z) - \lambda I_p(r,\lambda)F(z), \tag{3.8}$$

we have

$$\begin{aligned}(\delta+p)I_p(r,\lambda)f(z) &= \delta I_p(r,\lambda)F(z) + z(I_p(r,\lambda)F(z))' \\ &= (p+\lambda)I_p(r+1,\lambda)F(z) + (\delta-\lambda)I_p(r,\lambda)F(z).\end{aligned} \tag{3.9}$$

Define the function $q(z)$ by

$$q(z) := \frac{I_p(r+1,\lambda)F(z)}{I_p(r,\lambda)F(z)}. \tag{3.10}$$

Then, clearly, $q(z)$ is analytic in Δ. Using (3.9) and (3.10), we have

$$(\delta+p)\frac{I_p(r,\lambda)f(z)}{I_p(r,\lambda)F(z)} = \delta - \lambda + (p+\lambda)q(z). \tag{3.11}$$

Upon logarithmic differentiation of (3.11) and using (3.4), (3.8) and (3.10), we get

$$\frac{I_p(r+1,\lambda)f(z)}{I_p(r,\lambda)f(z)} = q(z) + \frac{zq'(z)}{\delta-\lambda+(p+\lambda)q(z)}. \tag{3.12}$$

For $f \in \mathcal{A}(p, n, r, \lambda, h)$, we have from (3.11),

$$q(z) + \frac{zq'(z)}{\delta-\lambda+(p+\lambda)q(z)} \prec \psi(z) + \frac{z\psi'(z)}{\delta-\lambda+(p+\lambda)\psi(z)}.$$

The first part of our result now follows by an application of Lemma 1. Similarly the second part follows from Lemma 2. □

Using Theorem 2.2, we have the following result:

Corollary 5. *Let* ψ_i *be univalent in* Δ, $\psi_i(0) = 1$ *and* δ *be a complex number. Assume that* $z\psi_i'/(\delta-\lambda+(p+\lambda)\psi_i)$ *is starlike in* Δ *and* $\Re\{\delta - \lambda + (p+\lambda)\psi_i(z)\} > 0$ *for* $i = 1,\ 2$. *Define the functions* h_i *by*

$$h_i(z) := \psi_i(z) + \frac{z\psi_i'(z)}{\delta-\lambda+(p+\lambda)\psi_i(z)} \quad (i = 1, 2).$$

If $f \in \mathcal{A}(p, n, r, \lambda; h_1, h_2)$, *then* F *defined by* (3.6) *belongs to* $\mathcal{A}(p, n, r, \lambda; \psi_1, \psi_2)$.

Theorem 3.3. *Let* $f(z) \in \mathcal{A}(p, n)$. *Then* $f \in \mathcal{A}(p, n, r, \lambda; \varphi)$ *if and only if*

$$F(z) := \frac{p+\lambda}{z^\lambda}\int_0^z t^{\lambda-1}f(t)dt \in \mathcal{A}(p, n, r+1, \lambda; \varphi). \tag{3.13}$$

Also $f \in \overline{\mathcal{A}}(p,n,r,\lambda;\varphi)$ *if and only if* $F \in \overline{\mathcal{A}}\,(p,n,r+1,\lambda;\varphi)\,.$

Proof. From (3.13), we have

$$(p+\lambda)f(z) = \lambda F(z) + zF'(z). \tag{3.14}$$

By convoluting (3.14) with $\phi_p(k,\lambda;z) := z^p + \sum_{k=p+1}^{\infty}\left(\frac{k+\lambda}{p+\lambda}\right)^r z^k$ and using the fact that $z(f*g)'(z) = f(z)*zg'(z)$, we obtain

$$(p+\lambda)I_p(r,\lambda)f(z) = \lambda I_p(r,\lambda)F(z) + z(I_p(r,\lambda)F(z))'$$

and by using (3.4), we get

$$I_p(r,\lambda)f(z) = I_p(r+1,\lambda)F(z) \tag{3.15}$$

and

$$\begin{aligned}(p+\lambda)I_p(r+1,\lambda)f(z) &= z(I_p(r,\lambda)f(z))' + \lambda I_p(r,\lambda)f(z) \\ &= z(I_p(r+1,\lambda)F(z))' + \lambda I_p(r+1,\lambda)F(z) \\ &= (p+\lambda)I_p(r+2,\lambda)F(z).\end{aligned} \tag{3.16}$$

Therefore, from (3.15) and (3.16), we have

$$\frac{I_p(r+2,\lambda)F(z)}{I_p(r+1,\lambda)F(z)} = \frac{I_p(r+1,\lambda)f(z)}{I_p(r,\lambda)f(z)},$$

and the desired result follows at once. □

Using Theorem 3.3, we have

Corollary 6. *Let* $f(z) \in \mathcal{A}(p,n)$. *Then* $f \in \overline{\mathcal{A}}(p,n,r,\lambda;\varphi_1,\varphi_2)$ *if and only if* $F \in \mathcal{A}\,(p,n,r+1,\lambda;\varphi_1,\varphi_2)\,.$

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References

[1] S. D. Bernardi, *Convex and starlike univalent functions*, Trans. Amer. Math. Soc., **135**(1969), 429–446.

[2] T. Bulboacă, *A class of superordination-preserving integral operators*, Indag. Math. (N.S.), **13(3)**(2002), 301–311.

[3] T. Bulboacă, *Classes of first-order differential superordinations*, Demonstratio Math., **35(2)**(2002), 287–292.

[4] B. C. Carlson and D. B. Shaffer, *Starlike and prestarlike hypergeometric functions*, SIAM J. Math. Anal., **15(4)**(1984), 737–745.

[5] N. E. Cho and H. M. Srivastava, *Argument estimates of certain analytic functions defined by a class of multiplier transformations*, Math. Comput. Modelling, **37(1-2)**(2003), 39–49.

[6] N. E. Cho and T. H. Kim, *Multiplier transformations and strongly close-to-convex functions*, Bull. Korean Math. Soc., **40(3)**(2003), 399–410.

[7] J. Dziok and H. M. Srivastava, *Certain subclasses of analytic functions associated with the generalized hypergeometric function*, Integral Transforms Spec. Funct., **14(1)**(2003), 7–18.

[8] Ju. E. Hohlov, *Operators and operations on the class of univalent functions*, Izv. Vyssh. Uchebn. Zaved. Mat. 1978, **10(197)**(1978), 83–89.

[9] R. J. Libera, *Some classes of regular univalent functions*, Proc. Amer. Math. Soc., **16**(1965), 755–758.

[10] J.-L. Liu and S. Owa, *On a class of multivalent functions involving certain linear operator*, Indian J. Pure Appl. Math., **33(11)**(2002), 1713–1722.

[11] A. E. Livingston, *On the radius of univalence of certain analytic functions*, Proc. Amer. Math. Soc., **17**(1966), 352–357.

[12] S. S. Miller and P. T. Mocanu, Differential Subordinations: Theory and Applications, Monographs and Textbooks in Pure and Applied Mathematics, No. 225, Marcel Dekker, New York, (2000).

[13] S. S. Miller and P. T. Mocanu, *Subordinants of differential superordinations*, Complex Var. Theory Appl., **48(10)**(2003), 815–826.

[14] S. Owa, *On the distortion theorems I*, Kyungpook Math. J., **18(1)**(1978), 53–59.

[15] S. Owa and H. M. Srivastava, *Univalent and starlike generalized hypergeometric functions*, Canad. J. Math., **39(5)**(1987), 1057–1077.

[16] St. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc., **49**(1975), 109–115.

[17] G. Ş. Sălăgean, Subclasses of univalent functions, in *Complex analysis—fifth Romanian-Finnish seminar, Part 1* (Bucharest, 1981), 362–372, Lecture Notes in Math., 1013, Springer, Berlin.

[18] H. M. Srivastava, Some families of fractional derivative and other linear operators associated with analytic, univalent, and multivalent functions, in *Analysis and its applications* (Chennai, 2000), 209–243, Allied Publ., New Delhi.

[19] B. A. Uralegaddi and C. Somanatha, Certain classes of univalent functions, in *Current topics in analytic function theory*, 371–374, World Sci. Publishing, River Edge, NJ, (1992).