KYUNGPOOK Math. J. 46(2006), 89-96

Rank-preserver of Matrices over Chain Semiring

Seok-Zun Song and Kyung-Tae Kang
Department of Mathematics, Cheju National University, Jeju 690-756, Korea
e-mail : szsong@cheju.ac.kr *and* kangkt@cheju.ac.kr

Abstract. For a rank-1 matrix A, there is a factorization as $A = \mathbf{a}\mathbf{b}^t$, the product of two vectors $\mathbf{a}$ and $\mathbf{b}$. We characterize the linear operators that preserve rank and some equivalent condition of rank-1 matrices over a chain semiring. We also obtain a linear operator T preserves the rank of rank-1 matrices if and only if it is a form (P, Q, B)-operator with appropriate permutation matrices P and Q, and a matrix B with all nonzero entries.

1. Introduction and preliminaries

There are many papers on linear operators that preserve the rank of matrices over several semirings([1]-[6]). Matrices over a chain semiring also have been the subject of research by many authors (see [2], [5]). Beasley and Pullman [1] defined the perimeter of a Boolean rank-1 matrix in order to characterize the linear operators preserving Boolean rank. Song [5] obtained characterization of linear operators that preserve column rank over the fuzzy scalars.

In this article, we consider the rank-1 matrices over a chain semiring and their perimeters. We also characterize the linear operators that preserve the rank and perimeter of the rank-1 matrices over a chain semiring.

A *semiring* is essentially a ring in which only the zero is required to have an additive inverse.

Let $\mathbb{K}$ be any set of two or more elements. If $\mathbb{K}$ is totally ordered by $<$ (i.e., $x < y$ or $y < x$ for all distinct elements x, y in $\mathbb{K}$), then define $x + y$ as $\max(x, y)$ and xy as $\min(x, y)$ for all $x, y \in \mathbb{K}$. If $\mathbb{K}$ has a universal lower bound and a universal upper bound, then $\mathbb{K}$ becomes a semiring, and called a *chain* semiring. The following are interesting examples of a chain semiring.

Let $\mathbb{H}$ be any nonempty family of sets nested by inclusion, $0 = \cap_{x\in\mathbb{H}}\, x$, and $1 = \cup_{x\in\mathbb{H}}\, x$. Then $\mathbb{S} = \mathbb{H} \cup \{0, 1\}$ is a chain semiring.

Let α, w be real numbers with $\alpha < w$. Define $\mathbb{S} = \{\beta \in \mathbb{R} : \alpha \leq \beta \leq w\}$. Then $\mathbb{S}$ is a chain semiring. It is isomorphic to the chain semiring in the previous example with $\mathbb{H} = \{[\alpha, \beta] : \alpha \leq \beta \leq w\}$. Furthermore, if we choose the real numbers 0 and 1 as α and w in the previous example, then $m \times n$ matrices over $\mathbb{F} \equiv \{\beta : 0 \leq \beta \leq 1\}$ is called *fuzzy* matrices.

Received September 20, 2004.
2000 Mathematics Subject Classification: 15A03, 15A04.
Key words and phrases: linear operator, perimeter, (P, Q, B)-operator.

In particular, if we take $\mathbb{H}$ to be a singleton set, say $\{a\}$, and denote $\emptyset$ by 0 and $\{a\}$ by 1, the resulting chain semiring $\mathbb{B} = \{0, 1\}$ is called a *binary Boolean algebra*, and it is a subsemiring of every chain semiring.

Let $\mathcal{M}_{m,n}(\mathbb{K})$ denote the set of all $m \times n$ matrices with entries in a chain semiring $\mathbb{K}$. Then addition, multiplication by scalars, and product of matrices on $\mathcal{M}_{m,n}(\mathbb{K})$ are defined as if $\mathbb{K}$ were a field.

The *rank* or *factor rank*, $r(A)$, of a nonzero matrix $A \in \mathcal{M}_{m,n}(\mathbb{K})$ is defined as the least integer k for which there exist $m \times k$ and $k \times n$ matrices B and C with $A = BC$. The rank of a zero matrix is zero. We note that $A \in \mathcal{M}_{m,n}(\mathbb{K})$ is a matrix with rank 1 if and only if there exist nonzero vectors $\mathbf{a} \in \mathcal{M}_{m,1}(\mathbb{K})$ and $\mathbf{b} \in \mathcal{M}_{n,1}(\mathbb{K})$ such that $A = \mathbf{a}\mathbf{b}^t$. But the following example shows that these vectors $\mathbf{a}$ and $\mathbf{b}$ are not uniquely determined by A.

Example 1.1. Let

$$A = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

be a fuzzy matrix in $\mathcal{M}_{2,n}(\mathbb{F})$ with $n \geq 2$, where $0 < \alpha_i < 1$ for all i. Then we have

$$A = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix} = \begin{bmatrix} \gamma \\ 0 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix} = \cdots,$$

where $\alpha_i \leq \gamma < 1$.

Let $\Delta_{m,n} = \{(i,j) |\ 1 \leq i \leq m,\ 1 \leq j \leq n\}$ and $\mathbb{E}_{m,n} = \{E_{ij} |\ (i,j) \in \Delta_{m,n}\}$, where E_{ij} is the matrix whose (i,j)th entry is 1 and whose other entries are all 0. We call E_{ij} a *cell*.

For any vector $\mathbf{u} \in \mathcal{M}_{m,1}(\mathbb{K})$, we define $|\mathbf{u}|$ to be the number of nonzero entries in $\mathbf{u}$. Let $A = [a_{ij}]$ be any matrix in $\mathcal{M}_{m,n}(\mathbb{K})$. Then we define $A^* = [a^*_{ij}]$ to be the matrix in $\mathcal{M}_{m,n}(\mathbb{B})$ whose (i,j)th entry is 1 if and only if $a_{ij} \neq 0$. It follows from the definition that

$$(AB)^* = A^*B^* \quad \text{and} \quad (B + C)^* = B^* +_{\mathbb{B}} C^* \tag{1.1}$$

for all $A \in \mathcal{M}_{m,n}(\mathbb{K})$ and all $B,\ C \in \mathcal{M}_{n,r}(\mathbb{K})$.

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are in $\mathcal{M}_{m,n}(\mathbb{K})$, we say that A *dominates* B (written $A \geq B$ or $B \leq A$) if $a_{ij} = 0$ implies $b_{ij} = 0$ for all i, j. Then we can easily obtain that $A \geq B$ if and only if $(A + B)^* = A^*$ for all matrices $A, B \in \mathcal{M}_{m,n}(\mathbb{K})$.

Lemma 1.2. *For any factorization* $\mathbf{a}\mathbf{b}^t$ *of a rank-*1 *matrix* $A \in \mathcal{M}_{m,n}(\mathbb{K})$*,* $|\mathbf{a}|$ *and* $|\mathbf{b}|$ *are uniquely determined by* A.

Proof. It follows from (1.1) that $A^* = \mathbf{a}^*(\mathbf{b}^*)^t$ is a rank-1 matrix in $\mathcal{M}_{m,n}(\mathbb{B})$. Then we can easily show that $|\mathbf{a}^*|$ and $|\mathbf{b}^*|$ are uniquely determined by A^*. Therefore $|\mathbf{a}|$ and $|\mathbf{b}|$ are uniquely determined by A. □

For any rank-1 matrix $A \in \mathcal{M}_{m,n}(\mathbb{K})$, define the *perimeter* of A, $P(A)$, as $|\mathbf{a}| + |\mathbf{b}|$ for arbitrary factorization $A = \mathbf{a}\mathbf{b}^t$. Even though the factorizations of

A are not unique, Lemma 1.2 shows that the perimeter of A is unique, and that $P(A) = P(A^*)$.

2. Rank and perimeter preservers

A mapping $T : \mathcal{M}_{m,n}(\mathbb{K}) \to \mathcal{M}_{m,n}(\mathbb{K})$ is called a *linear operator* if $T(\alpha A + \beta B) = \alpha T(A) + \beta T(B)$ for all $A, B \in \mathcal{M}_{m,n}(\mathbb{K})$ and for all $\alpha,\ \beta \in \mathbb{K}$.

In this article, we characterize the set of linear operators preserving the rank and the perimeter of every rank-1 matrix over any chain semiring. These are motivated by analogous results for the set of linear operators that preserve all ranks in $\mathcal{M}_{m,n}(\mathbb{K})$. However, we obtain results and proofs in view of the perimeter analog.

For matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ in $\mathcal{M}_{m,n}(\mathbb{K})$, the *Hadamard* (or *Schur*) product $A \circ B$ of A and B is the m by n matrix whose (i, j)-entry equals $a_{ij}\, b_{ij}$.

Suppose that T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{K})$. Then

(1) T is a (P, Q, B)-*operator* if there exist $m\times m$ and $n\times n$ permutation matrices P and Q, respectively and a matrix $B \in \mathcal{M}_{m,n}(\mathbb{K})$ with all nonzero entries and $r(B) = 1$ such that $T(A) = P(A \circ B)Q$ for all A in $\mathcal{M}_{m,n}(\mathbb{K})$, or $m = n$ and $T(A) = P(A^t \circ B)Q$ for all A in $\mathcal{M}_{m,n}(\mathbb{K})$.

(2) T *preserve rank* 1 if $r(T(A)) = 1$ whenever $r(A) = 1$ for all $A \in \mathcal{M}_{m,n}(\mathbb{K})$.

(3) T *preserve perimeter* k of rank-1 matrices if $P(T(A)) = k$ whenever $P(A) = k$ for all $A \in \mathcal{M}_{m,n}(\mathbb{K})$ with $r(A) = 1$.

Theorem 2.1. *If T is a (P, Q, B)-operator on $\mathcal{M}_{m,n}(\mathbb{K})$, then T preserves both rank and perimeter of rank-1 matrices.*

Proof. If T is a (P, Q, B)-operator on $\mathcal{M}_{m,n}(\mathbb{K})$, there exist $m \times m$ and $n \times n$ permutation matrices P and Q, respectively such that $T(A) = P(A \circ B)Q$, or $m = n$ and $T(A) = P(A^t \circ B)Q$ for all A in $\mathcal{M}_{m,n}(\mathbb{K})$, where $B \in \mathcal{M}_{m,n}(\mathbb{K})$ is a matrix with all nonzero entries and $r(B) = 1$. Then we can write $B = \mathbf{c}\mathbf{d}^t$, where none of entries $\mathbf{c}$ or $\mathbf{d}$ is zero. Let A be a rank-1 matrix in $\mathcal{M}_{m,n}(\mathbb{K})$ with a factorization $A = \mathbf{a}\mathbf{b}^t$. For the case $T(A) = P(A \circ B)Q$, we have the following:

$$T(A) = P(\mathbf{a}\mathbf{b}^t \circ \mathbf{c}\mathbf{d}^t)Q = P(\mathbf{a} \circ \mathbf{c})(\mathbf{b} \circ \mathbf{d})^t Q = \big(P(\mathbf{a} \circ \mathbf{c})\big)\big(Q^t(\mathbf{b} \circ \mathbf{d})\big)^t. \qquad (2.1)$$

Thus (2.1) implies that

$$r(T(A)) = r\Big(\big(P(\mathbf{a} \circ \mathbf{c})\big)\big(Q^t(\mathbf{b} \circ \mathbf{d})\big)^t\Big) = 1$$

and

$$P(T(A)) = |P(\mathbf{a} \circ \mathbf{c})| + |Q^t(\mathbf{b} \circ \mathbf{d})| = |\mathbf{a} \circ \mathbf{c}| + |\mathbf{b} \circ \mathbf{d}| = |\mathbf{a}| + |\mathbf{b}| = P(A).$$

For the case $m = n$ and $T(A) = (A^t \circ B)Q$, we can show that $r(T(A)) = 1$ and $P(T(A)) = P(A)$ by the similar method as above. $\square$

Furthermore, the converse of Theorem 2.1 is also true. We will show its proof in Theorem 2.7(below). Also Example 2.3(below) shows that a linear operator preserving ranks of all rank-1 matrices need not be (P, Q, B)-operator.

We note that a matrix has perimeter 2 if and only if it is a cell with nonzero scalar multiplication. We say that $A \in \mathcal{M}_{m,n}(\mathbb{K})$ is a *row* (or *column*) *matrix* if A has nonzero entries only in one row (column, respectively). A *line matrix* is a row matrix or a column matrix. Thus we have the following Lemma:

Lemma 2.2. *Let T be a linear operator on $\mathcal{M}_{m,n}(\mathbb{K})$. If T preserves rank and perimeter 2 of every rank-1 matrix, then the following statements hold*:

(1) *T maps a cell into a cell with nonzero scalar multiplication*;

(2) *T maps a line matrix into a line matrix.*

Proof. (1) follows from the property that T preserves perimeter 2. (2) If not, there exist two distinct cells E and F in same row (or column) such that $T(E)$ and $T(F)$ lie in two different rows and different columns. Then we have $r(E + F) = 1$, while $r(T(E + F)) = r(T(E) + T(F)) = 2$. This contradicts to the fact that T preserves rank 1. □

The following is an example of a linear operator that preserves rank and perimeter 2 of rank-1 matrices, but it does not preserve perimeter $2n (n \geq 2)$ and is not a (P, Q, B)-operator.

Example 2.3. Let T be a linear operator on $\mathcal{M}_{n,n}(\mathbb{K})$ with $n \geq 2$ defined by

$$T(A) = \Big(\sum_{i,j=1}^{n} a_{ij} \Big) E_{kk} = \max\{a_{ij} \,|\, i, j = 1, \cdots, n\} E_{kk}$$

for all $A = [a_{ij}] \in \mathcal{M}_{n,n}(\mathbb{K})$, where k is a fixed integer in $\{1, 2, \cdots, n\}$. Then it is easy to verify that T is a linear operator and preserves rank and perimeter 2 of each rank-1 matrix. But T does not preserve perimeter $2n$: for, if $J \in \mathcal{M}_{n,n}(\mathbb{K})$ is a matrix whose entries are all 1, then J has rank 1 and perimeter $2n$, but $T(J) = E_{kk}$ has rank 1 and perimeter 2. Hence T is not a (P, Q, B)-operator by Theorem 2.1.

Let $\mathcal{R}_i = \{E_{ij} \,|\, 1 \leq j \leq n\}$, $\mathcal{C}_j = \{E_{ij} \,|\, 1 \leq i \leq m\}$, $\mathbb{R} = \{\mathcal{R}_i \,|\, 1 \leq i \leq m\}$ and $\mathbb{C} = \{\mathcal{C}_j \,|\, 1 \leq j \leq n\}$. For a linear operator T on $\mathcal{M}_{m,n}(\mathbb{K})$, define $T^*(A) = [T(A)]^*$ for all A in $\mathcal{M}_{m,n}(\mathbb{K})$. Let $T^*(\mathcal{R}_i) = \{T^*(E_{ij}) \,|\, 1 \leq j \leq n\}$ for each $i = 1, \cdots, m$ and $T^*(\mathcal{C}_j) = \{T^*(E_{ij}) \,|\, 1 \leq i \leq m\}$ for each $j = 1, \cdots, n$.

Lemma 2.4. *Let T be a linear operator on $\mathcal{M}_{m,n}(\mathbb{K})$. Suppose that T preserves rank and perimeters 2 and $p \,(\geq 3)$ of rank-1 matrices. Then*

(1) *T maps two distinct cells in a row (column) into two distinct cells in a row or in a column with nonzero scalar multiplication*;

(2) *if T maps a row matrix into a row (or column if $m = n$) matrix then T maps every row matrix into a row (or column if $m = n$) matrix, and if T maps a*

column matrix into a column (or row if $m = n$) matrix then T maps every column matrix into a column (or row if $m = n$) matrix.

Proof. (1) Suppose $T(E_{ij}) = \alpha E_{rl}$ and $T(E_{ih}) = \beta E_{rl}$ for some distinct pairs $(i,j),(i,h) \in \Delta_{m,n}$ and some nonzero scalars $\alpha, \beta \in \mathbb{K}$. Then T maps the ith row matrix into rth row or lth column matrix by Lemma 2.2. Thus for any rank-1 matrix A with perimeter $p\,(\geq 3)$ which dominates $E_{ij} + E_{ih}$, we can easily show that $T(A)$ has perimeter at most $p-1$, a contradiction. Thus T maps two distinct cells in a row into two distinct cells in a row or in a column with nonzero scalar multiplication.

(2) If not, then there exist rows $\mathcal{R}_i$ and $\mathcal{R}_j$ such that $T^*(\mathcal{R}_i) \subseteq \mathcal{R}_r$ and $T^*(\mathcal{R}_j) \subseteq \mathcal{C}_s$ for some $(r,s) \in \Delta_{m,n}$. Consider a rank-1 matrix $D = E_{ip} + E_{iq} + E_{jp} + E_{jq}$ with $p \neq q$. Then we have

$$\begin{aligned} T(D) &= T(E_{ip} + E_{iq}) + T(E_{jp} + E_{jq}) \\ &= (\alpha_1 E_{rp'} + \alpha_2 E_{rq'}) + (\beta_1 E_{p''s} + \beta_2 E_{q''s}) \end{aligned}$$

for some $p' \neq q'$ and $p'' \neq q''$ and some nonzero scalars $\alpha_i, \beta_i \in \mathbb{K}$ by (1). Therefore $r\,(T(D)) \neq 1$ and T does not preserve rank 1, a contradiction. Hence T maps each row matrix into a row (or column if $m = n$) matrix. Similarly, T maps each column matrix into a column (or row if $m = n$) matrix. □

For a linear operator T on $\mathcal{M}_{m,n}(\mathbb{K})$ preserving rank and perimeter 2 of rank-1 matrices, we define the corresponding mapping $T' : \Delta_{m,n} \to \Delta_{m,n}$ by $T'(i,j) = (k,l)$ whenever $T(E_{ij}) = b_{ij}E_{kl}$ for some nonzero scalar $b_{ij} \in \mathbb{K}$. Then T' is well-defined by Lemma 2.2-(1).

Lemma 2.5. *Let T be a linear operator preserving both rank and perimeters 2 and $k(k \geq 4, k \neq n+1)$ of rank-1 matrices. Then T' is a bijection on $\Delta_{m,n}$.*

Proof. By Lemma 2.2, we have that for any $E_{ij} \in \mathbb{E}_{m,n}$, there exist $E_{rl} \in \mathbb{E}_{m,n}$ and nonzero $b_{ij} \in \mathbb{K}$ such that $T(E_{ij}) = b_{ij}E_{rl}$. Without loss of generality, we may assume that T maps the ith row of a matrix into the rth row with nonzero scalar multiplication. Suppose that $T'(i,j) = T'(p,q)$ for some distinct pairs $(i,j),(p,q) \in \Delta_{m,n}$. By the definition of T', we have $T(E_{ij}) = b_{ij}E_{rl}$ and $T(E_{pq}) = b_{pq}E_{rl}$ for some nonzero scalars $b_{ij}, b_{pq} \in \mathbb{K}$. Lemma 2.4 implies that $i \neq p$ and $j \neq q$. Furthermore T maps the ith row and the pth row of a matrix into the rth row.

Case 1. $4 \leq k \leq n$: Claim: we can choose a $2 \times (k-2)$ submatrix A from ith and pth row, but $T(A)$ is a $1 \times k$ submatrix in the rth row. If the claim is true, then $P(A) = k$, while $P(T(A)) = k+1$, a contradiction.

Proof of the claim. By Lemma 2.4, T maps distinct cells in each row (or column) to distinct cells with nonzero scalar multiplication. Now, choose E_{ij}, E_{pj} but do not choose E_{iq}, E_{pq}. Since there is a cell E_{ph_1} $(h_1 \neq j, q)$ in the pth row such that $T'(p,h_1) = T'(i,q)$ but $T'(i,h_1) \neq T'(p,j)$, we can choose a $2 \times (4-2)$ submatrix $E_{ij} + E_{ih_1} + E_{pj} + E_{ph_1}$ whose image under T is an 1×4 submatrix in the rth row.

Therefore the claim is satisfied for $k = 4$. Assume that for $k = s$ with $4 \leq s \leq n-1$, the claim is true. Then there is a $2 \times (s-2)$ submatrix

$$X = E_{ij} + \sum_{t=1}^{s-3} E_{ih_t} + E_{pj} + \sum_{t=1}^{s-3} E_{ph_t}$$

such that $T(X)$ is an $1 \times s$ submatrix in the rth row, where $\{j, q, h_1, \cdots, h_{s-3}\}$ is the set of distinct indices. Now, we can choose a cell $E_{ph_{s-2}}$ $(h_{s-2} \neq j, q, h_1, \cdots, h_{s-3})$ such that $T'(i, h_{s-2}) \neq T'(p, j), T'(p, q), T'(p, h_1), \cdots, T'(p, h_{s-3})$. Then we have a $2 \times ((s+1)-2)$ submatrix $A = E_{ij} + \sum\limits_{t=1}^{s-2} E_{ih_t} + E_{pj} + \sum\limits_{t=1}^{s-2} E_{ph_t}$ such that $T(A)$ is an $1 \times (s+1)$ submatrix in the rth row. Thus the claim is satisfied for $k = s+1$. By the mathematical induction, the claim is true.

Case 2. $k = n + \alpha \geq n+2$: Consider a matrix

$$Y = \sum_{s=1}^{n} E_{is} + \sum_{t=1}^{n} E_{p\,t} + \sum_{h=1}^{\alpha-2} \sum_{g=1}^{n} E_{hg}$$

with rank 1 and perimeter k. Then T maps the ith and pth row of Y into the rth row with nonzero scalar multiplication by Lemma 2.4. Thus the perimeter of $T(Y)$ is less than k, a contradiction.

Hence $T'(i, j) \neq T'(p, q)$ for any two distinct pairs $(i, j), (p, q) \in \Delta_{m,n}$. Therefore T' is a bijection on $\Delta_{m,n}$. $\square$

The condition of $k \neq n+1$ in Lemma 2.5 must be necessary. The following example shows that T is a linear operator preserving both rank and perimeters 2 and $n+1$ of rank-1 matrices, but T' is not a bijection on $\Delta_{m,n}$.

Example 2.6. Consider a linear operator T on $\mathcal{M}_{2,3}(\mathbb{K})$ defined by

$$T\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right) = \begin{bmatrix} a+e & b+f & c+d \\ 0 & 0 & 0 \end{bmatrix}.$$

Then we can easily show that T preserves both rank and perimeters 2 and 4 of rank-1 matrices. But T' is not a bijection on $\Delta_{2,3}$.

Theorem 2.7. *Let T be a linear operator on $\mathcal{M}_{m,n}(\mathbb{K})$. Then the following are equivalent:*

(1) *T is a (P, Q, B)-operator;*

(2) *T preserves both rank and perimeter of rank-1 matrices;*

(3) *T preserves both rank and perimeters 2 and k $(k \geq 4, k \neq n+1)$ of rank-1 matrices.*

Proof. (1) $\Rightarrow$ (2): clear by Proposition 2.1. (2) $\Rightarrow$ (3): it is obvious. (3) $\Rightarrow$ (1): Assume (3). Then the corresponding mapping $T : \Delta_{m,n} \to \Delta_{m,n}$ is a bijection by Lemma 2.5. Furthermore, there are two cases: (a) T^* maps $\mathbb{R}$ onto $\mathbb{R}$ and maps $\mathbb{C}$ onto $\mathbb{C}$, or (b) T^* maps $\mathbb{R}$ onto $\mathbb{C}$ and maps $\mathbb{C}$ onto $\mathbb{R}$.

Case (a): We note that $T^*(\mathcal{R}_i) = \mathcal{R}_{\sigma(i)}$ and $T^*(\mathcal{C}_j) = \mathcal{C}_{\tau(j)}$ for all $i = 1, \cdots, m$ and $j = 1, \cdots, n$, where σ and τ are permutations of $\{1, \cdots, m\}$ and $\{1, \cdots, n\}$, respectively. Let P and Q be the permutation matrices corresponding to σ and τ, respectively. Then for any $E_{ij} \in \mathbb{E}_{m,n}$, we can write $T(E_{ij}) = b_{ij}E_{\sigma(i)\tau(j)}$ for some nonzero scalar $b_{ij} \in \mathbb{K}$. Now we claim that $B = [b_{ij}] \in \mathcal{M}_{m,n}(\mathbb{K})$ has rank 1. For, consider an $m \times n$ matrix J, all of whose entries are 1. Then we have

$$T(J) = T\Big(\sum_{i=1}^{m}\sum_{j=1}^{n} E_{ij}\Big) = \sum_{i=1}^{m}\sum_{j=1}^{n} T(E_{ij}) = \sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}E_{\sigma(i)\tau(j)} = PBQ.$$

Since J has rank 1, it follows that $r(T(J)) = 1$, and hence $r(B) = r(PBQ) = r(T(J)) = 1$. Therefore for any $A = [a_{ij}] \in \mathcal{M}_{m,n}(\mathbb{K})$, we have

$$\begin{aligned} T(A) &= T\Big(\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}E_{ij}\Big) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}T(E_{ij}) \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ij}E_{\sigma(i)\tau(j)} = P(A \circ B)Q. \end{aligned}$$

Thus T is a (P, Q, B)-operator.

Case (b): We note that $m = n$ and $T^*(\mathcal{R}_i) = \mathcal{C}_{\sigma(i)}$ and $T^*(\mathcal{C}_j) = \mathcal{R}_{\tau(j)}$ for all i, j, where σ and τ are some permutations of $\{1, \cdots, m\}$. By an argument similar to case (a), we obtain that $T(A)$ is of the form $T(A) = P(A^t \circ B)Q$, and thus T is a (P, Q, B)-operator. $\square$

We say that a linear operator T on $\mathcal{M}_{m,n}(\mathbb{K})$ *strongly preserves* perimeter k of rank-1 matrices if $P(T(A)) = k$ if and only if $P(A) = k$.

Consider a linear operator T on $\mathcal{M}_{n,n}(\mathbb{K})$ with $n \geq 2$ defined by

$$T(A) = \Big(\sum_{i,j=1}^{n} a_{ij}\Big)E_{11}$$

for all $A = [a_{ij}] \in \mathcal{M}_{n,n}(\mathbb{K})$. Then T preserves both rank and perimeter 2 of rank-1 matrices but does not strongly preserve perimeter 2 because $p\Big(\sum_{i,j=1}^{n} E_{ij}\Big) = 2n$, while $p\Big(T\Big(\sum_{i,j=1}^{n} E_{ij}\Big)\Big) = p(E_{11}) = 2$.

Corollary 2.8. *Let T be a linear operator on $\mathcal{M}_{m,n}(\mathbb{K})$. Then T preserves both rank and perimeter of rank-1 matrices if and only if it preserves perimeter 3 and* *strongly preserves perimeter* 2 *of rank-1 matrices.*

Proof. Suppose T preserves perimeter 3 and strongly preserves perimeter 2 of rank-1 matrices. Then T maps each row of a matrix into a row or a column(if $m = n$) with nonzero scalar multiplication. Since T strongly preserves perimeter 2, T maps each cell onto a cell with nonzero scalar multiplication. This means that the corresponding mapping T' is a bijection. Thus T preserves both rank and perimeter of rank-1 matrices by the similar method in the proof of Theorem 2.7.

The converse is immediate. □

Thus we have characterizations of the linear operators that preserve both rank and perimeter of rank-1 matrices over any chain semiring.

References

[1] L. B. Beasley and N. J. Pullman, *Boolean rank-preserving operators and Boolean rank-1 spaces*, Linear Algebra Appl., **59**(1984), 55-77.

[2] L. B. Beasley and N. J. Pullman, *Fuzzy Rank-Preserving Operators*, Linear Algebra Appl., **73**(1986), 197-211.

[3] L. B. Beasley and N. J. Pullman, *Term-rank, permanent and rook-polynomial preservers*, Linear Algebra Appl., **90**(1987), 33-46.

[4] L. B. Beasley, D. A. Gregory and N. J. Pullman, *Nonnegative Rank-Preserving Operators*, Linear Algebra Appl., **65**(1985), 207-223.

[5] S. Z. Song, *Linear operators that preserve column rank of fuzzy matrices*, Fuzzy Sets and Systems, **62**(1994), 311-317.

[6] S. Z. Song, L. B. Beasley, G. S. Cheon and Y. B. Jun, *Rank and perimeter preservers of Boolean rank-*1 *matrices*, J. Korean Math. Soc., **41(2)**(2004), 397-406.