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Oscillation of Second Order Nonlinear Elliptic Differential Equations

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Abstract. By using general means, some oscillation criteria for second order nonlinear elliptic differential equation with damping

$$\sum_{i,j=1}^{N} D_i[a_{ij}(x)D_jy] + \sum_{i=1}^{N} b_i(x)D_iy + p(x)f(y) = 0$$

are obtained. These criteria are of a high degree of generality and extend the oscillation theorems for second order linear ordinary differential equations due to Kamenev, Philos and Wong.

1. Introduction

In this paper, we consider the oscillatory behavior of second order nonlinear elliptic differential equation with damping

$$\sum_{i,j=1}^{N} D_i[a_{ij}(x)D_jy] + \sum_{i=1}^{N} b_i(x)D_iy + p(x)f(y) = 0 \tag{1.1}$$

in $\Omega(a)$, where $N \geq 2$, $x = (x_1, \cdots, x_N) \in \mathbb{R}^N$, $|x| = [\sum_{i=1}^{N} x_i^2]^{1/2}$, $D_i = \partial/\partial x_i$ for all i, and $\Omega(a) = \{x \in \mathbb{R}^N : |x| \geq a\}$ for some $a > 0$.

Throughout this paper we shall assume that the following conditions hold.

(A_1) $A = (a_{ij})_{N\times N}$ is a real symmetric positive definite matrix function with $a_{ij} \in C^{1+\mu}_{loc}(\Omega(a), \mathbb{R})$ for all i, j, $\mu \in (0, 1)$.

Denote by $\lambda_{\max}(x)$ the largest eigenvalue of the matrix A. There exists a function $\lambda \in C([a, \infty), \mathbb{R}^+)$ such that

$$\lambda(r) \geq \max_{|x|=r} \lambda_{\max}(x) \quad \text{for } r \geq a;$$

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(A_2) $p \in C^{\mu}_{loc}(\Omega(a), \mathbb{R})$ and $f \in C(\mathbb{R}, \mathbb{R}) \cup C^1(\mathbb{R} - \{0\}, \mathbb{R})$, $f'(y) \geq k > 0$, $yf(y) > 0$ for $y \neq 0$;

(A_3) $b_i \in C^{1+\mu}_{loc}(\Omega(a), \mathbb{R})$ for all i.

By a solution of Eq.(1.1) we mean a function $y \in C^{2+\mu}_{loc}(\Omega)$, which satisfies Eq.(1.1) almost everywhere on Ω. Regarding the question of existence of solutions of Eq.(1.1) we refer the reader to monograph [1]. A nontrivial solution $y(x)$ of Eq.(1.1) is called oscillatory if the set $\{\, x \in \Omega(a) : y(x) = 0 \,\}$ is unbounded; otherwise it is said to be nonoscillatory. Eq.(1.1) is oscillatory if all its solutions are oscillatory.

In the last decades, there has been an increasing interest in obtaining sufficient conditions for the oscillation for different classes of second order elliptic differential equations. In the absence of damping, many results are obtained for the particular cases of Eq.(1.1) such as the semilinear elliptic differential equation

$$\sum_{i,j=1}^{N} D_i[a_{ij}(x) D_j y] + p(x) f(y) = 0. \tag{1.2}$$

In 1980, Noussair and Swanson [3] first extended the well-known Wintner Theorem [6] to Eq.(1.2) based on $N-$dimensional vector partial Riccati type transformation

$$w(x) = -\frac{\alpha(|x|)}{f(y(x))}(A\nabla y)(x), \tag{1.3}$$

where $\alpha \in C^2(0, \infty)$ is an arbitrary positive function, ∇y denotes the gradient of y.

The survey paper by Swanson [5] contains a complete bibliography up to 1979. Very recently, Xu [8]-[10] and Zhang et al [11] employed the technique of Noussair and Swanson [3] and obtained several oscillation results for Eq.(1.2). However, their results cannot be applied to the nonlinear damped elliptic differential equation (1.1).

Motivated by the work of [3], [4], [7], in this paper we shall establish, by using a generalized Riccati technique [3] and the general means [4], [7], some oscillation criteria for Eq.(1.1). These criteria are of a high degree of generality and extend the oscillation theorems for second order linear ordinary differential equations due to Kamenev [2], Philos [4] and Wong [7]. Our methodology is somewhat different from that of the previous authors. We believe that our approach is simpler and also provides a more unified account of Kamenev-type oscillation theorems.

2. Main results

First of all, we introduce the general means [4], [7]. Let

$$D = \{(r, s) : r \geq s \geq a\}, \quad \text{and} \quad D_0 = \{(r, s) : r > s \geq a\}.$$

We say that the function $H \in C(D, \mathbb{R})$ belongs to the class $\Im$ (written $H \in \Im$), if

(H_1) $H(r, r) = 0$ for $r \geq a$, $H(r, s) > 0$ on D_0;

(H_2) H has a continuous and nonpositive partial derivative in D_0 with respect to the second variable;

(H_3) there exists a function $h \in C(D, \mathbb{R})$ such that

$$-\frac{\partial}{\partial s}H(r,s) = h(r,s)H(r,s).$$

Let $\rho \in C([a,\infty), \mathbb{R}^+)$, $\kappa \in C([a,\infty), \mathbb{R})$, we now define an integral operator X^ρ_τ (see [7]) in terms of $H(r,s)$ and $\rho(s)$ as

$$X^\rho_\tau(\kappa; r) = \int_\tau^r H(r,s)\kappa(s)\rho(s)ds, \quad r > \tau \geq a. \tag{2.1}$$

Motivated by the work of Noussair and Swanson [3], we apply a generalized Riccati transformation which is different from (1.3), that is

$$W(x) = \frac{1}{f(y)}(A\nabla y)(x) + \frac{1}{2k}B, \tag{2.2}$$

where $B = (b_1(x), \cdots, b_N(x))^T$.

The key point to note here is that the term $(1/2k)B$ appearing in (2.2) is very important. Without this term, our method can't be applied to Eq.(1.1) (cf [3], [8]-[11]).

The following Lemma is a modified version of Lemma 1 in [3], and will be useful for establishing oscillation criteria for Eq.(1.1).

Lemma 2.1. *Let $y(x)$ be a nonoscillatory of Eq.(1.1), then the N-dimensional vector function $W(x)$ given by (2.2) satisfies the generalized Riccati inequality*

$$\mathrm{div\,W(x)} \leq -\mathrm{kW^TA^{-1}W} - \mathrm{p(x)} + \frac{1}{4\mathrm{k}}\mathrm{B^TA^{-1}B} + \frac{1}{2\mathrm{k}}\sum_{\mathrm{i}=1}^{\mathrm{N}}\mathrm{D_ib_i}. \tag{2.3}$$

Proof. Differentiation of the i-th component of (2.2) with respect to x_i gives

$$D_iW_i(x) = -\frac{f'(y)}{f^2(y)}D_iy\left(\sum_{i=1}^N A_{ij}D_jy\right) + \frac{1}{f(y)}D_i\left(\sum_{j=1}^N A_{ij}D_jy\right) + \frac{1}{2k}D_ib_i,$$

for i. Summation over i and use of Eq.(1.1) and (2.2) lead to

$$\begin{aligned}
\operatorname{div} W(x) &= -\frac{f'(y)}{f^2(y)}(\nabla y)^TA\nabla y - \frac{1}{f(y)}[\,p(x)f(y) + B^T\nabla y\,] + \frac{1}{2k}\sum_{i=1}^N D_ib_i \\
&\leq -k\left[W - \frac{1}{2k}B\right]^T A^{-1}\left[W - \frac{1}{2k}B\right] - p(x) \\
&\quad -B^TA^{-1}\left[W - \frac{1}{2k}B\right] + \frac{1}{2k}\sum_{i=1}^N D_ib_i
\end{aligned}$$

$$= \quad -kW^TA^{-1}W - p(x) + \frac{1}{4k}B^TA^{-1}B + \frac{1}{2k}\sum_{i=1}^{N} D_ib_i.$$

This completes the proof. □

For simplicity, we always assume that the function $H \in \Im$ and the integral operator X_τ^ρ defined by (2.1). In addition, we define the functions g and P as following:

$$g(r) = \frac{kr^{1-N}}{\omega_N \lambda(r)}, \quad P(r) = \int_{S_r} \left[p(x) - \frac{1}{4k}B^TA^{-1}B - \frac{1}{2k}\sum_{i=1}^{N} D_ib_i\right] d\sigma,$$

where $S_r = \{x \in \mathbb{R}^N : |x| = r\}$ for $r > 0$, σ denotes the measure on S_r and ω_N denotes the surface of the unit sphere in $\mathbb{R}^N$, i.e., $\omega_N = 2\pi^{N/2}/\Gamma(N/2)$.

Theorem 2.1. *Suppose that there exist a function $\rho \in C^1([a,\infty),\mathbb{R}^+)$ such that*

$$\limsup_{r\to\infty} \frac{1}{H(r,a)} X_a^\rho \left(P - \frac{1}{4g}\left[h - \frac{\rho'}{\rho}\right]^2\right) = \infty. \tag{2.4}$$

Then Eq.(1.1) *is oscillatory.*

Proof. Let $y(x)$ be a nonoscillatory solution of Eq.(1.1). Without loss of generality, we may assume that $y(x) > 0$ for all $x \in \Omega(a)$. By Lemma 2.1, the generalized Riccati inequality (2.3) holds in $\Omega(a)$. Since $\lambda^{-1}(x)$ is the smallest eigenvalue of A^{-1}, then

$$(W^TA^{-1}W)(x) \ge \lambda^{-1}(x)|W(x)|^2 \ge \lambda^{-1}(|x|)|W(x)|^2. \tag{2.5}$$

Inequalities (2.3) and (2.5) imply that

$$\operatorname{div} W(x) \le -\frac{k}{\lambda(|x|)}|W(x)|^2 - p(x) + \frac{1}{4k}B^TA^{-1}B + \frac{1}{2k}\sum_{i=1}^{N} D_ib_i. \tag{2.6}$$

Define

$$Z(r) = \int_{S_r} W(x)\cdot\nu(x)d\sigma, \tag{2.7}$$

where $\nu(x) = x/|x|$, $|x| \ne 0$, denotes the outward unit normal to S_r, $r = |x|$. Using Green's formula in (2.7), and in view of (2.6), we get

$$Z'(r) = \int_{S_r} \operatorname{div}W(x)\, d\sigma \le -P(r) - \frac{k}{\lambda(r)}\int_{S_r} |W(x)|^2\, d\sigma. \tag{2.8}$$

By the Schwarz inequality, we have

$$\int_{S_r} |W(x)|^2\, d\sigma \ge \frac{r^{1-N}}{\omega_N}\left[\int_{S_r} W(x)\cdot\upsilon(x)\, d\sigma\right]^2.$$

Thus, by (2.8), we have

$$Z'(r) \le -P(r) - g(r)Z^2(r). \tag{2.9}$$

Applying the integral operator X_τ^ρ $(\tau \ge a)$ to (2.9), we obtain

$$X_\tau^\rho\left(\left[h - \frac{\rho'}{\rho}\right] Z\right) + X_\tau^\rho\left(gZ^2\right) + X_\tau^\rho(P) \le H(r,\tau)\rho(\tau)Z(\tau). \tag{2.10}$$

Completing squares of Z in (2.10) yields

$$\begin{aligned} & X_\tau^\rho\left(g\left[Z + \frac{1}{2g}\left(h - \frac{\rho'}{\rho}\right)\right]^2\right) + X_\tau^\rho\left(P - \frac{1}{4g}\left[h - \frac{\rho'}{\rho}\right]^2\right) \\ \le \quad & H(r,\tau)\rho(\tau)Z(\tau). \end{aligned} \tag{2.11}$$

Set $\tau = a$ and divide (2.11) through by $H(r,a)$. Note that the first term is nonnegative, so

$$\frac{1}{H(r,a)} X_a^\rho\left(P - \frac{1}{4g}\left[h - \frac{\rho'}{\rho}\right]^2\right) \le \rho(a)Z(a). \tag{2.12}$$

Take limsup in (2.12) as $r \to \infty$. Condition (2.4) gives a desired contradiction in (2.12). This completes the proof. □

The following Theorem 2.2 treats the case when it is not possible to verify easily condition (2.4).

Theorem 2.2. *Suppose that there exist functions $\rho \in C^1([a,\infty),\mathbb{R}^+)$, $\varphi_i \in C([a,\infty),\mathbb{R})$, $i = 1,\ 2$, such that for all $\tau \ge a$*

$$\limsup_{r\to\infty} \frac{1}{H(r,\tau)} X_\tau^\rho(P) \ge \varphi_2(\tau) \tag{2.13}$$

and

$$\limsup_{r\to\infty} \frac{1}{H(r,\tau)} X_\tau^\rho\left(\frac{1}{g}\left[h - \frac{\rho'}{\rho}\right]^2\right) \le \varphi_1(\tau), \tag{2.14}$$

where φ_1 and φ_2 satisfy

$$\liminf_{r\to\infty} \frac{1}{H(r,\tau)} X_\tau^\rho\left(\frac{g}{\rho^2}\left[\varphi_2 - \frac{1}{4}\varphi_1\right]_+^2\right) = \infty, \tag{2.15}$$

where $[\varphi(r)]_+ = \max\{\varphi(r), 0\}$. Then Eq.(1.1) is oscillatory.

Proof. Proceeding as in the proof of Theorem 2.1 to get (2.10) and (2.11). Then, by (2.11), we have, for all $r > \tau \geq a$

$$\frac{1}{H(r,\tau)} X_\tau^\rho(P) - \frac{1}{4}\frac{1}{H(r,\tau)} X_\tau^\rho\left(\frac{1}{g}\left[h - \frac{\rho'}{\rho}\right]^2\right) \leq \rho(\tau)Z(\tau).$$

Taking limsup in the above inequality as $r \to \infty$ and applying (2.13) and (2.14), we obtain

$$\varphi_2(\tau) - \frac{1}{4}\varphi_1(\tau) \leq \rho(\tau)Z(\tau),$$

from which it follows that

$$\frac{1}{H(r,\tau)} X_\tau^\rho\left(\frac{g}{\rho^2}\left[\varphi_2 - \frac{1}{4}\varphi_1\right]_+^2\right) \leq \frac{1}{H(r,\tau)} X_\tau^\rho(gZ^2). \tag{2.16}$$

On the other hand, by (2.10)

$$\frac{1}{H(r,\tau)} X_\tau^\rho\left(\left[h - \frac{\rho'}{\rho}\right] Z\right) + \frac{1}{H(r,\tau)} X_\tau^\rho(gZ^2) \leq \rho(\tau)Z(\tau) - \frac{1}{H(r,\tau)} X_\tau^\rho(P).$$

Thus, by (2.13)

$$\begin{aligned} &\liminf_{r\to\infty}\left\{\frac{1}{H(r,\tau)} X_\tau^\rho\left(\left[h - \frac{\rho'}{\rho}\right] Z\right) + \frac{1}{H(r,\tau)} X_\tau^\rho(gZ^2)\right\} \\ \leq\ & \rho(\tau)Z(\tau) - \varphi_2(\tau) \leq C_0, \end{aligned} \tag{2.17}$$

where C_0 is a constant.

Now, we claim that

$$\liminf_{r\to\infty} \frac{1}{H(r,\tau)} X_\tau^\rho(gZ^2) < \infty. \tag{2.18}$$

If (2.18) does not hold, then there exists a sequence $\{r_j\}_{j=1}^\infty \in [a,\infty)$ with $\lim_{j\to\infty} r_j = \infty$ such that

$$\lim_{j\to\infty} \frac{1}{H(r_j,\tau)} X_\tau^\rho(gZ^2) = \infty. \tag{2.19}$$

Hence, by (2.17), for j large enough, we have

$$\frac{1}{H(r_j,\tau)} X_\tau^\rho\left(\left[h - \frac{\rho'}{\rho}\right] Z\right) + \frac{1}{H(r_j,\tau)} X_\tau^\rho(gZ^2) \leq C_0 + 1.$$

So, by (2.19), for j large enough and $\varepsilon > 1$, we have

$$\frac{X_\tau^\rho\left(\left[h - \frac{\rho'}{\rho}\right] Z\right)}{X_\tau^\rho(gZ^2)} < -\frac{1}{\varepsilon},$$

that is

$$X_\tau^\rho(gZ^2) \le \varepsilon \left| X_\tau^\rho \left(\left[h - \frac{\rho'}{\rho} \right] Z \right) \right|. \tag{2.20}$$

By the Schwarz inequality, we have

$$\left| X_\tau^\rho \left(\left[h - \frac{\rho'}{\rho} \right] Z \right) \right|^2 \le X_\tau^\rho(gZ^2) X_\tau^\rho \left(\frac{1}{g} \left[h - \frac{\rho'}{\rho} \right]^2 \right). \tag{2.21}$$

From (2.20) and (2.21), we have

$$\frac{1}{H(r_j,\tau)} X_\tau^\rho(gZ^2) \le \frac{\varepsilon^2}{H(r_j,\tau)} X_\tau^\rho \left(\frac{1}{g} \left[h - \frac{\rho'}{\rho} \right]^2 \right). \tag{2.22}$$

By (2.14), the right-hand side of (2.22) is bounded, which contradicts (2.19). Thus, (2.18) holds. Hence, by (2.16)

$$\liminf_{r\to\infty} \frac{1}{H(r,\tau)} X_\tau^\rho \left(\frac{g}{\rho^2} \left[\varphi_2 - \frac{1}{4}\varphi_1 \right]_+^2 \right) \le \liminf_{r\to\infty} \frac{1}{H(r,\tau)} X_\tau^\rho(gZ^2) < \infty.$$

which contradicts (2.15). □

Remak 2.1. Theorems 2.1 and 2.2 are extensions of Wong's results [7] to Eq.(1.1).

For the case when the substitution (2.7) is modified as

$$U(r) = \phi(r) \left[\int_{S_r} W(x)\cdot \nu(x) d\sigma - \frac{1}{2g(r)} \frac{\phi'(r)}{\phi(r)} \right], \tag{2.23}$$

where $\phi \in C^1([a,\infty),\mathbb{R}^+)$, then we can obtain more sharp oscillation criteria (Theorem 2.3 and Theorem 2.4) for Eq.(1.1).

Theorem 2.3. *Suppose that there exist functions $\phi \in C^2([a,\infty),\mathbb{R}^+)$, $\rho \in C^1([a,\infty),\mathbb{R}^+)$ such that*

$$\limsup_{r\to\infty} \frac{1}{H(r,a)} X_a^\rho \left(\Theta - \frac{1}{4\psi} \left[h - \frac{\rho'}{\rho} \right]^2 \right) = \infty, \tag{2.24}$$

where

$$\psi(r) = \frac{g(r)}{\phi(r)}, \quad \Theta(r) = \phi(r)P(r) + \frac{1}{2}\left(\frac{\phi'(r)}{g(r)} \right)' - \frac{\phi'^2(r)}{4g(r)\phi(r)}.$$

Then Eq.(1.1) is oscillatory.

Proof. Let $y(x)$ be a nonoscillatory solution of Eq.(1.1). Without loss of generality, we may assume that $y(x) > 0$ for all $x \in \Omega(a)$. By Lemma 2.1, the Riccati inequality

(2.3) holds in $\Omega(a)$. Then, using Green's formula in (2.23) and making use of (2.6), we obtain

$$
\begin{aligned}
(2.25)\qquad & U'(r)\\
= \;& \frac{\phi'(r)}{\phi(r)}U(r)+\phi(r)\left\{\int_{S_r}\mathrm{div}W(x)\,d\sigma-\frac{1}{2}\left[\frac{1}{g(r)}\frac{\phi'(r)}{\phi(r)}\right]'\right\}\\
\le \;& \frac{\phi'(r)}{\phi(r)}U(r)-g(r)\phi(r)\left[\int_{S_r}W(x)\cdot\nu(x)d\sigma\right]^2-\phi(r)P(r)\\
& -\frac{\phi(r)}{2}\left[\frac{1}{g(r)}\frac{\phi'(r)}{\phi(r)}\right]'\\
= \;& \frac{\phi'(r)}{\phi(r)}U(r)-\psi(r)\left[U(r)+\frac{\phi'(r)}{2g(r)}\right]^2-\phi(r)P(r)-\frac{\phi(r)}{2}\left[\frac{\phi'(r)}{g(r)\phi(r)}\right]'\\
= \;& -\psi(r)U^2(r)-\Theta(r).
\end{aligned}
$$

Next, proceeding as the proof of Theorem 2.1 we can complete the proof. □

Remark 2.2. Theorem 2.3 improves Theorem 1 for Eq.(1.2) in [10].

Evidently, we can establish the following theorem analogous to Theorem 2.2. The proof is similar to that of Theorem 2.2, and hence omitted.

Theorem 2.4. *Suppose that there exist functions $\phi \in C^2([a,\infty),\mathbb{R}^+)$, $\rho \in C^1([a,\infty),\mathbb{R}^+)$, and $\varphi_i \in C([a,\infty),\mathbb{R})$, $i = 1,\ 2$, such that for all $\tau \ge a$*

$$
\limsup_{r\to\infty}\frac{1}{H(r,\tau)}X_\tau^\rho(\Theta)\ge\varphi_2(\tau) \tag{2.26}
$$

and

$$
\limsup_{r\to\infty}\frac{1}{H(r,\tau)}X_\tau^\rho\left(\frac{1}{\psi}\left[h-\frac{\rho'}{\rho}\right]^2\right)\le\varphi_1(\tau) \tag{2.27}
$$

where φ_1 and φ_2 satisfy

$$
\liminf_{r\to\infty}\frac{1}{H(r,\tau)}X_\tau^\rho\left(\frac{\psi}{\rho^2}\left[\varphi_2-\frac{1}{4}\varphi_1\right]_+^2\right)=\infty, \tag{2.28}
$$

where ψ and Θ are defined as Theorem 2.3. Then Eq.(1.1) *is oscillatory.*

3. Corollaries and examples

In this section, we would like to establish some oscillation criteria for Eq.(1.1) by properly choosing the weighting functions $\rho(s)$ and $\phi(s)$, and also present some examples that illustrate the obtained results. These examples are new and not covered by any of the results in [3], [8]-[11].

Corollary 3.1. *Let $\alpha > 1$, suppose that*

$$\limsup_{r\to\infty} G^{-\alpha}(r)\int_a^r [G(r)-G(s)]^\alpha P(s)ds = \infty, \tag{3.1}$$

where $G(r) = \int_a^r \frac{s^{1-N}}{\lambda(s)}ds$ for $r \geq a$. Then Eq.(1.1) is oscillatory.

Proof. Let

$$H(r,s) = [G(r)-G(s)]^\alpha, \quad \rho(r) = 1 \quad \text{for } r \geq s \geq a.$$

Then

$$h(r,s) = \alpha[G(r)-G(s)]^{-1}\frac{s^{1-N}}{\lambda(s)},$$

$$\begin{aligned}\int_a^r \frac{H(r,s)}{g(s)}h^2(r,s)\,ds &= \frac{\omega_N\alpha^2}{k}\int_a^r [G(r)-G(s)]^{\alpha-2}\frac{s^{1-N}}{\lambda(s)}\,ds \\ &= \frac{\omega_N}{k}\frac{\alpha^2}{\alpha-1}[G(r)]^{\alpha-1}.\end{aligned}$$

This implies

$$\begin{aligned}&\limsup_{r\to\infty}\frac{1}{H(r,a)}X_a^1\left(P-\frac{1}{4g}\left[h-\frac{\rho'}{\rho}\right]^2\right)\\ &=\limsup_{r\to\infty}\left\{G^{-\alpha}(r)\int_a^r [G(r)-G(s)]^\alpha P(s)\,ds - \frac{\alpha^2\omega_N}{4k(\alpha-1)G(r)}\right\} = \infty.\end{aligned}$$

It follows from Theorem 2.1 that Eq.(1.1) is oscillatory. □

Remark 3.1. Corollary 3.1 extends Kamenev Theorem [2] to Eq.(1.1).

Corollary 3.2. *Let $\alpha > 1$, suppose that there exists a function $\rho \in C^1([a,\infty),\mathbb{R}^+)$ such that*

$$\int_a^\infty \frac{\rho'^2(s)}{\rho(s)}ds < \infty \tag{3.2}$$

and

$$\limsup_{r\to\infty}\frac{1}{r^\alpha}\int_a^r (r-s)^\alpha \rho(s)P(s)\,ds = \infty. \tag{3.3}$$

Then Eq.(1.1) is oscillatory.

Proof. By Theorem 2.1, it is sufficient to show that

$$\lim_{r\to\infty}\frac{1}{H(r,a)}X_a^\rho\left(\left[h-\frac{\rho'}{\rho}\right]^2\right) < \infty,$$

where $H(r,s) = (r-s)^\alpha$, and $h(r,s) = \alpha(r-s)^{-1}$. Note that

$$\left(h - \frac{\rho'}{\rho}\right)^2 \le 2\left(h^2 + \frac{\rho'^2}{\rho^2}\right).$$

Because $\lim_{r\to\infty} r^{-\alpha}\int_a^r (r-s)^{\alpha-2}\,ds = 0$. In view of (3.2), and using Lemma in [7], p 247, property (14), we have

$$\lim_{r\to\infty} r^{-\alpha}\int_a^r (r-s)^\alpha \frac{\rho'^2(s)}{\rho(s)}\,ds = 0.$$

Thus Corollary 3.2 follows from Theorem 2.1. □

Similar to the proof of Corollary 3.1, by Theorem 3.3, we have

Corollary 3.3. *Let $\alpha > 1$, suppose that*

$$\limsup_{r\to\infty} Q^{-\alpha}(r)\int_a^r [Q(r) - Q(s)]^\alpha \Theta(s)\,ds = \infty, \tag{3.4}$$

where $Q(r) = \int_a^r \psi(s)\,ds$ for $r \ge a$. Then Eq.(1.1) is oscillatory.

Remark 3.2. Corollary 3.3 improves Theorem 4 in [3] for Eq.(1.2).

Corollary 3.4. *Suppose*

$$\liminf_{r\to\infty} \phi(r)\int_r^\infty P(s)\,ds > \frac{1}{4}, \tag{3.5}$$

where $\phi(r) = \int_a^r g(s)\,ds$. Then Eq.(1.1) is oscillatory.

Proof. By (3.5), there exist two numbers $b \ge a$ and $c > \frac{1}{4}$ such that

$$\phi(r)\int_r^\infty P(s)\,ds > c, \quad \text{for } r \ge b.$$

Let

$$H(r,s) = [\phi(r) - \phi(s)]^2 \quad \text{and} \quad \rho(s) = 1,$$

then

$$h(r,s) = 2[\phi(r) - \phi(s)]^{-1} g(s).$$

Thus

$$\begin{aligned} &H(r,s)\left[\Theta(s) - \tfrac{1}{4\psi(s)} h^2(r,s)\right] \\ &= [\phi(r) - \phi(s)]^2 \phi(s)\left[P(s) - \tfrac{\phi'(s)}{4\phi^2(s)}\right] - \phi(s)\phi'(s). \end{aligned}$$

Define

$$\Phi(r) = \int_r^\infty P(s)\,ds.$$

Hence, for all $r > b \geq a$

$$\begin{aligned}
&\int_b^r H(r,s)\left[\Theta(s) - \frac{1}{4\psi(s)}h^2(r,s)\right] ds \\
= \ & \int_b^r [\phi(r)-\phi(s)]^2 \phi(s) d\left(-\Phi(s) + \frac{1}{4\phi(s)}\right) - \int_b^r \phi(s)\phi'(s)\, ds \\
= \ & [\phi(r)-\phi(b)]^2\phi(b)\left[\Phi(b) - \frac{1}{4\phi(b)}\right] - \frac{1}{2}[\phi^2(r) - \phi^2(b)] \\
& + \int_b^r \left[\Phi(s)\phi(s) - \frac{1}{4}\right]\left(-4\phi(r) + 3\phi(s) + \frac{\phi^2(r)}{\phi(s)}\right)\phi'(s)\, ds \\
\geq \ & \left(c - \frac{1}{4}\right)\left[(-\frac{5}{2} - \ln\phi(b))\phi^2(r) + \phi^2(r)\ln\phi(r)\right] - \frac{1}{2}\phi^2(r).
\end{aligned}$$

It follows from Theorem 2.3 that Eq.(1.1) is oscillatory. □

Remark 3.3. Corollary 3.4 improves Theorem 2.3 in [11] for Eq.(1.2).

Remark 3.4. In Theorems 3.1-3.4, let $H(r,s) = (r-s)^\alpha$ and $\alpha > 1$, we can establish the Kamenev-type criterion for Eq.(1.1), here we omit the details.

Example 3.1. Consider the equation

$$\Delta y + \frac{\cos r}{r}\frac{\partial y}{\partial x_1} + \frac{\sin r}{r}\frac{\partial y}{\partial x_2} + \frac{1}{r^\delta}(\varepsilon + \sin r)(y + y^3) = 0, \quad r \geq 1, \tag{3.6}$$

where $r = \sqrt{x_1^2 + x_2^2}$, $N = 2$, $A = I$, $f(y) = y + y^3$, $\delta \leq 3$, and

$$B = \left(\frac{\cos r}{r}, \frac{\sin r}{r}\right), \quad p(x) = \frac{1}{r^\delta}(\varepsilon + \sin r), \quad g(r) = \frac{2\pi}{r}.$$

By direct calculation we get

$$\begin{aligned}
P(x) &= \int_{S_r}\left[p(x) - \frac{1}{4k}B^T A^{-1} B - \frac{1}{2}\sum_{i=1}^N D_i b_i\right] d\sigma \\
&= \frac{2\pi}{r^{\delta-1}}(\varepsilon + \sin r) - \frac{\pi}{2r}(1 + 2\sin r + 2\cos r).
\end{aligned}$$

For Corollary 3.2, let $\rho(r) = r^{-1}$, if $\delta \leq 3$,

$$\begin{aligned}
&\lim_{r\to\infty}\int_a^r P(s)\rho(s)\, ds \\
= \ & 2\pi \lim_{r\to\infty}\int_a^r \left[\frac{1}{s^{\delta-2}}(\varepsilon + \sin s) - \frac{1}{4s^2}(1 + 2\cos s + 2\sin s)\right] ds = \infty.
\end{aligned} \tag{3.7}$$

Clearly, (3.7) implies (3.3) for any $\alpha > 1$. Thus Eq.(3.6) is oscillatory for $\varepsilon > 0$ and $\delta \le 3$.

Example 3.2. Consider the damping elliptic equation

$$\frac{\partial}{\partial x_1}\left(\frac{1}{r}\frac{\partial y}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\frac{1}{r}\frac{\partial y}{\partial x_2}\right)+\frac{1}{r}\frac{\partial y}{\partial x_1}+\frac{1}{r^2}\left(y+y^3\right)=0, \quad r>1, \tag{3.8}$$

where $r=\sqrt{x_1^2+x_2^2}$, $N=2$, $A(x)=\operatorname{diag}(r^{-1},r^{-1})$, $B=(r^{-1},0)$, $p(x)=r^{-2}$. Since

$$\lambda(r)=\frac{1}{r}, \quad g(r)=\frac{1}{2\pi}, \quad P(x)=\frac{2\pi}{r}-\frac{\pi}{2r^2}.$$

Now, for Theorem 2.4, let $\phi(r)=r^{-1}$, $\rho(r)=1$ and $H(r,s)=(r-s)^2$ for $r\ge s\ge 1$, we have

$$h(r,s)=2(r-s)^{-1}, \quad \psi(r)=\frac{r}{2\pi}, \quad \Theta(r)=\frac{\pi}{r^2}\left(2-\frac{1}{2r}\right).$$

Thus

$$\limsup_{r\to\infty}\frac{1}{H(r,s)}X_\tau^\rho(\Theta)=\lim_{r\to\infty}\frac{1}{r^2}\int_\tau^r (r-s)^2\frac{\pi}{s^2}\left(2-\frac{1}{2s}\right)ds=\frac{2\pi}{\tau}, \quad \tau\ge 1,$$

and

$$\lim_{r\to\infty}\frac{1}{H(r,a)}X_\tau^\rho\left(\frac{1}{\psi}\left[h-\frac{\rho'}{\rho}\right]^2\right)=\lim_{r\to\infty}\frac{8\pi}{r^2}\int_\tau^r\frac{1}{s}\,ds=0.$$

Choosing $\varphi_1(r)=0$ and $\varphi_2(r)=2\pi r^{-1}$, then

$$\lim_{t\to\infty}\frac{1}{H(r,a)}X_\tau^\rho\left(\frac{\psi}{\rho^2}\left[\varphi_2-\frac{1}{4}\varphi_1\right]_+^2\right)=\lim_{r\to\infty}\frac{2\pi}{r^2}\int_a^r (r-s)^2\frac{1}{s}\,ds=\infty.$$

Hence, Theorem 3.4 implies that Eq.(3.8) is oscillatory.

Remark 3.5. The results of this paper are presented in the form of a high degree of generality and thus they give possibilities of deriving a number of oscillation criteria with a choice of $H(r,s)$ different from that discussed in the paper. For example, we may consider

$$H(r,s)=\left[\int_s^r\frac{du}{\theta(u)}\right]^{n-1}, \quad (r,s)\in D.$$

where $n>2$ is a constant and θ is a positive continuous function on $[a,\infty)$.

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