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Weighted L^p Boundedness for the Function of Marcinkiewicz

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Abstract. In this paper, we prove a weighted norm inequality for the Marcinkiewicz integral operator $\mathcal{M}_{\Omega,h}$ when h satisfies a mild regularity condition and Ω belongs to $L(\log L)^{1l2}(\mathbf{S}^{n-1})$, $n \geq 2$. We also prove the weighted L^p boundedness for a class of Marcinkiewicz integral operators $\mathcal{M}^*_{\Omega,h,\lambda}$ and $\mathcal{M}_{\Omega,h,S}$ related to the Littlewood-Paley g^*_λ-function and the area integral S, respectively.

1. Introduction and statement of results

Let $n \geq 2$ and $\mathbf{S}^{n-1}$ be the unit sphere in $\mathbf{R}^n$ equipped with the induced Lebesgue measure $d\sigma = d\sigma(\cdot)$. Let Ω be a homogeneous function of degree 0 satisfying $\Omega \in L^1(\mathbf{S}^{n-1})$ and

$$\int_{\mathbf{S}^{n-1}} \Omega(x')\, d\sigma(x') = 0, \tag{1.1}$$

where $x' = x/|x| \in \mathbf{S}^{n-1}$ for any $x \neq 0$.

For a suitable C^1 function Φ on $\mathbf{R}_+$ and a measurable function $h : \mathbf{R}_+ \longrightarrow \mathbf{C}$, the Marcinkiewicz integral operator $\mathcal{M}_{\Omega,\Phi,h}$ is defined by

$$\mathcal{M}_{\Omega,\Phi,h}f(x) = \left(\int_0^\infty |F_{\Omega,\Phi,h,t}f(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F_{\Omega,\Phi,h,t}f(x) = \int_{|y|\leq t} |y|^{1-n}\, \Omega(y')h(|y|)f(x-\Phi(|y|)y')dy.$$

We are also interested in studying the related Marcinkiewicz integral operators

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$\mathcal{M}_{\Omega,\Phi,h,S}$ and $\mathcal{M}^*_{\Omega,\Phi,h,\lambda}$ which are defined by

$$\mathcal{M}_{\Omega,\Phi,h,S}f(x) = \left(\int_{\Gamma(x)} |F_{\Omega,\Phi,h,t}f(x)|^2 \frac{dydt}{t^{n+3}}\right)^{1/2},$$

$$\mathcal{M}^*_{\Omega,\Phi,h,\lambda}f(x) = \left(\int\int_{\mathbf{R}^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |F_{\Omega,\Phi,h,t}f(x)|^2 \frac{dydt}{t^{n+3}}\right)^{1/2},$$

where $\lambda > 1$ and $\Gamma(x) = \left\{(y,t) \in \mathbf{R}^{n+1}_+ : |x-y| < t\right\}$.

If $\Phi(t) = t$ we denote $\mathcal{M}_{\Omega,\Phi,h}, \mathcal{M}_{\Omega,\Phi,h,S}$ and $\mathcal{M}^*_{\Omega,\Phi,h,\lambda}$ by $\mathcal{M}_{\Omega,h}, \mathcal{M}_{\Omega,h,S}$ and $\mathcal{M}^*_{\Omega,h,\lambda}$, respectively. Also, if $h \equiv 1$ and $\Phi(t) = t$ we denote $\mathcal{M}_{\Omega,\Phi,h}, \mathcal{M}_{\Omega,\Phi,h,S}$ and $\mathcal{M}^*_{\Omega,\Phi,h,\lambda}$ by $\mathcal{M}_{\Omega,h}, \mathcal{M}_{\Omega,S}$ and $\mathcal{M}^*_{\Omega,\lambda}$, respectively.

We point out that $\mathcal{M}_{\Omega,S}$ and $\mathcal{M}^*_{\Omega,\lambda}$ are respectively related to the Lusin area integral S and the Littlewood-Paley g^*_λ-function.

The operator $\mathcal{M}_\Omega$ was introduced by E. M. Stein in [13] as an extension of the notion of Marcinkiewicz function. Stein showed that if $\Omega \in \text{Lip}_\alpha(\mathbf{S}^{n-1})$, $(0 < \alpha \leq 1)$, then $\mathcal{M}_\Omega$ is of type (p,p) for $p \in (1,2]$ and of weak type $(1,1)$ (see [13]). Subsequently, Benedek, Calderón and R. Panzone proved that $\mathcal{M}_\Omega$ is of type (p,p) for $p \in (1,\infty)$ if $\Omega \in C^1\left(\mathbf{S}^{n-1}\right)$ (see [2]). Some years later, T. Walsh [18] showed that if $p \in (1,\infty)$, $r = \min\{p,p'\}$, and $\Omega \in L(\log L)^{1/r}(\log\log L)^{2(1-2/r')}(\mathbf{S}^{n-1})$, then $\mathcal{M}_\Omega$ is bounded on $L^p(\mathbf{R}^n)$. In particular, by Walsh's result we have $\mathcal{M}_\Omega$ is of type $(2,2)$ if $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$. Moreover,Walsh showed the optimality of the condition $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$ for the L^2 boundedness of $\mathcal{M}_\Omega$ in the sense that the exponent $1/2$ in $L(\log L)^{1/2}(\mathbf{S}^{n-1})$ cannot be replaced by any smaller numbers. Very recently, Al-Salman-Al-Qassem-Chen-Pan in [1] were able to prove the L^p boundedness $(1 < p < \infty)$ of $\mathcal{M}_\Omega$ if $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$.

On the other hand, the weighted L^p boundedness of $\mathcal{M}_\Omega$ has also attracted the attention of many authors in the recent years. Indeed, Torchinsky and Wang in [17] proved that if $\Omega \in \text{Lip}_\alpha(\mathbf{S}^{n-1})$, $(0 < \alpha \leq 1)$, then $\mathcal{M}_\Omega$ is bounded on $L^p(\omega)$ for $p \in (1,\infty)$ and $\omega \in A_p$ (The Muckenhoupt's weight class, see [9] for the definition). The result of Torchinsky-Wang was improved by Ding, Fan and Pan in [4] who were able to show that $\mathcal{M}_{\Omega,h}$ is bounded on $L^p(\omega)$ for $p \in (1,\infty)$ provided that $h \in L^\infty(\mathbf{R}_+)$, $\Omega \in L^q(\mathbf{S}^{n-1})$, $q > 1$ and $\omega^{q'} \in A_p(\mathbf{R}^n)$. In a recent paper, Ming-Yi Lee and Chin-Cheng Lin in [11] showed that $\mathcal{M}_{\Omega,h}$ is bounded on $L^p(\omega)$ for $p \in (1,\infty)$ if $h \in L^\infty(\mathbf{R}_+)$, $\Omega \in H^1(\mathbf{S}^{n-1})$ and $\omega \in \tilde{A}^I_p(\mathbf{R}^n)$, where $\tilde{A}^I_p(\mathbf{R}^n)$ is a special class of radial weights introduced by Duoandikoetxea [6] and its definition will recalled below.

We remark that on $\mathbf{S}^{n-1}$, for any $q > 1$ and $0 < \alpha \leq 1$, the following inclusions hold and are proper:

$$C^1(\mathbf{S}^{n-1}) \subseteq \text{Lip}_\alpha(\mathbf{S}^{n-1}) \subseteq L^q(\mathbf{S}^{n-1}) \subseteq L(\log^+ L)(\mathbf{S}^{n-1}) \subseteq H^1(\mathbf{S}^{n-1}). \tag{1.2}$$

Also, it is easy to see that $L(\log L)(\mathbf{S}^{n-1}) \subseteq L(\log L)^{1/2}(\mathbf{S}^{n-1})$. However, it is known that $L(\log L)^{1/2}(\mathbf{S}^{n-1})$ is disjoint from $H^1(\mathbf{S}^{n-1})$ in the sense that

$L(\log L)^{1/2}(\mathbf{S}^{n-1}) \nsubseteq H^1(\mathbf{S}^{n-1})$ and $H^1(\mathbf{S}^{n-1}) \nsubseteq L(\log L)^{1/2}(\mathbf{S}^{n-1})$ (see [3] for more details).

Our chief concern in this paper is studying the weighted $L^p(\omega)$ boundedness of $\mathcal{M}_{\Omega,h}$ under the natural condition $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$ and under a very weak condition on h. To sate our main results, we need to recall some definitions.

We say that $\Omega \in L(\log L)^{\alpha}(\mathbf{S}^{n-1})$ $(\alpha > 0)$ if Ω satisfies

$$\int_{\mathbf{S}^{n-1}} |\Omega(x)| \left(\log(2 + |\Omega(x)|)\right)^{\alpha} d\sigma(x) < \infty.$$

For $\gamma > 1$, we say that $h \in \Delta_{\gamma}(\mathbf{R}_+)$ if h is a measurable function on $\mathbf{R}_+$ satisfying

$$\|h\|_{\Delta_\gamma} = \sup_{R>0} \left(\frac{1}{R} \int_0^R |h(t)|^{\gamma} dt \right)^{1/\gamma} < \infty.$$

Let $L^p(\omega)$ be the weighted L^p spaces associated to the weight $\omega \geq 0$ which is defined by

$$L^p(\mathbf{R}^n, \omega(x)dx) = \left\{ f : \|f\|_{L^p(\omega)} = \left(\int_{\mathbf{R}^n} |f(x)|^p \omega(x) dx \right)^{1/p} < \infty \right\}.$$

Definition 1.1. Let $\omega(t) \geq 0$ and $\omega \in L^1_{loc}(\mathbf{R}_+)$. For $1 < p < \infty$, we say that $\omega \in A_p(\mathbf{R}_+)$ if there is a positive constant C such that for any interval $I \subset \mathbf{R}_+$,

$$\left(|I|^{-1} \int_I \omega(t) dt \right) \left(|I|^{-1} \int_I \omega(t)^{-1/(p-1)} dt \right)^{p-1} \leq C < \infty.$$

We say that $\omega \in A_1(\mathbf{R}_+)$ if there is a positive constant C such that

$$|I|^{-1} \int_I \omega(t) dt \leq C \ . \ ess \ \inf_{t \in I} \omega(t) \ for \ \ any \ \ interval \ \ I \subset \mathbf{R}_+.$$

It is easy to verify that $\omega \in A_1(\mathbf{R}_+)$ if and only if there is a positive constant C such that

$$M_{HL}\omega(t) \leq C\omega(t) \text{ for } a.e. \ t \in \mathbf{R}_+,$$

where $M_{HL}(f)$ is the Hardy-Littlewood maximal function of f.

Definition 1.2. Let $1 \leq p < \infty$. We say that $\omega \in \tilde{A}_p(\mathbf{R}_+)$ if

$$\omega(x) = \nu_1(|x|)\nu_2(|x|)^{1-p},$$

where either $\nu_i \in A_1(\mathbf{R}_+)$ is decreasing or $\nu_i^2 \in A_1(\mathbf{R}_+)$, $i = 1, \ 2$.

Let $A_p^I(\mathbf{R}^n)$ be the weight class defined by exchanging the cubes in the definitions of A_p for all n-dimensional intervals with sides parallel to coordinate axes (see [9]). Let $\tilde{A}_p^I = \tilde{A}_p \cap A_p^I$. If $\omega \in \tilde{A}_p$, it follows from [6] that the standard Hardy-Littlewood maximal function $M_{HL}f$ is bounded on $L^p(\mathbf{R}^n, \omega(|x|)dx)$. Therefore, if $\omega(t) \in \tilde{A}_p(\mathbf{R}_+)$, then $\omega(|x|) \in A_p(\mathbf{R}^n)$.

We shall need the following lemma which can be proved by the same argument as in the proof of the elementary properties of A_p weight class (see for example [9]):

Lemma 1.3. *If $1 \le p < \infty$, then the weight class $\tilde{A}_p^I(\mathbf{R}_+)$ has the following properties*:

(i) $\tilde{A}_{p_1}^I \subset \tilde{A}_{p_1}^I$, *if* $1 \le p_1 < p_2 < \infty$;

(ii) *For any $\omega \in \tilde{A}_p^I$, there exists an $\varepsilon > 0$ such that $\omega^{1+\varepsilon} \in \tilde{A}_p^I$;*

(iii) *For any $\omega \in \tilde{A}_p^I$ and $p > 1$, there exists an $\varepsilon > 0$ such that $p - \varepsilon > 1$ and $\omega \in \tilde{A}_{p-\varepsilon}^I$.*

Our main results are the following:

Theorem 1.1. *Let $h \in \Delta_\gamma(\mathbf{R}_+)$ for some $\gamma > 1$. Let Φ be in $C^2([0,\infty))$, convex, and increasing function with $\Phi(0) = 0$. If $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$ and satisfies* (1.1), *then*

$$\|\mathcal{M}_{\Omega,\Phi,h}(f)\|_{L^p(\mathbf{R}^n)} \le C_p \|f\|_{L^p(\mathbf{R}^n)} \tag{1.3}$$

is bounded on $L^p(\mathbf{R}^n)$ for $|1/p - 1/2| < \min\{1/\gamma', 1/2\}$.

Theorem 1.2. *Let Φ be in $C^2([0,\infty))$, convex, and increasing function with $\Phi(0) = 0$. Suppose $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$ satisfying* (1.1) *and $h \in \Delta_\gamma(\mathbf{R}_+)$ for some $\gamma \ge 2$. If p and ω satisfy one of the following conditions*:

(a) $2 < p < \infty$ *and* $\omega \in \tilde{A}_{p/2}^I(\mathbf{R}_+)$;

(b) $\gamma' < p \le 2$ *and* $\omega \in \tilde{A}_{p/\gamma'}^I(\mathbf{R}_+)$,

then there exists $C_p > 0$, independent of f, such that

$$\|\mathcal{M}_{\Omega,\Phi,h}(f)\|_{L^p(\omega)} \le C_p \|f\|_{L^p(\omega)}. \tag{1.4}$$

Theorem 1.3. *Let $h \in \Delta_\gamma(\mathbf{R}_+)$ for some $\gamma \ge 2$. Let Φ be in $C^2([0,\infty))$, convex, and increasing function with $\Phi(0) = 0$. If $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$, then there exists $C_p > 0$ such that*

$$\|\mathcal{M}_{\Omega,\Phi,h,S}(f)\|_{L^p(\omega)} + \left\|\mathcal{M}^*_{\Omega,\Phi,h,\lambda}(f)\right\|_{L^p(\omega)} \le C_p \|f\|_{L^p(\omega)} \tag{1.5}$$

for $2 \le p < \infty$ and $\omega \in \tilde{A}_{p/2}^I(\mathbf{R}_+)$.

It is worth noting that the range of p given in Theorem 1.1 is the full range $(1,\infty)$ whenever $\gamma \geq 2$. Also, the result in Theorem 1.1 extends the result of Al-Salman-Al-Qassem-Chen-Pan in [1] who obtained Theorem 1.1 in the special case $h \equiv 1$ and $\Phi(t) = t$. We remark also that Theorems 1.2 and 1.3 represent an improvement and extension of Theorems 1 and 2 in [4] in the case $\omega \in \tilde{A}_p^I(\mathbf{R}_+)$.

The main tools used in this paper come from [5] and [12]. The paper is divided into three sections. In the second section, we compute certain Fourier transform estimates and prove the weighted $L^p(\omega)$ boundedness of a maximal function. The proofs of Theorems 1.1–1.3 will appear in Section 3, along with a further result.

Throughout the rest of the paper the letter C will stand for a positive constant not necessarily the same one at each occurrence.

2. Some basic lemmas

Definition 2.1. Let h be a measurable function and $\Phi(t)$ be a C^1 function on $\mathbf{R}_+$. For $m \in \mathbf{N} \cup \{0\}$, let $a_m = 2^{(m+1)}$ and Ω_m be a function on $\mathbf{S}^{n-1}$ satisfying the following conditions:

$$\|\Omega_m\|_{L^2(\mathbf{S}^{n-1})} \leq (a_m)^2; \tag{2.1}$$

$$\|\Omega_m\|_1 \leq 1. \tag{2.2}$$

Define the family of measures $\{\sigma_{m,t} : t \in \mathbf{R}_+\}$ and the corresponding maximal operator σ_m^* on $\mathbf{R}^n$ by

$$\begin{aligned} \int_{\mathbf{R}^n} f d\sigma_{m,t} &= \frac{1}{t}\int_{\frac{1}{2}t<|y|\leq t} f(\Phi(|y|)y')h(|y|)\frac{\Omega_m(y')}{|y|^{n-1}}dy, \\ \sigma_m^* f(x) &= \sup_{t\in\mathbf{R}_+} ||\sigma_{m,t}| * f(x)|, \end{aligned}$$

where $|\sigma_{m,t}|$ is defined in the same way as $\sigma_{m,t}$, but with Ω_m replaced by $|\Omega_m|$ and h replaced by $|h|$.

Lemma 2.2. *Let $m \in \mathbf{N}$ and $h \in \Delta_\gamma(\mathbf{R}_+)$ for some γ, $1 < \gamma \leq 2$. Let Ω_m be a function on $\mathbf{S}^{n-1}$satisfying* (2.1)-(2.2) *and* (1.1) *with Ω replaced by Ω_m. Assume that Φ is in $C^2([0,\infty))$, convex, and increasing function with $\Phi(0) = 0$. Then there exist constants C and $0 < \alpha < 1$ such that for all $k \in \mathbf{Z}$ and $\xi \in \mathbf{R}^n$ we have*

$$\|\sigma_{m,t}\| \leq C; \tag{2.3}$$

$$\int_{a_m^k}^{a_m^{k+1}} |\hat{\sigma}_{m,t}(\xi)|^2 \frac{dt}{t} \leq C(m+1)\left|\Phi(a_m^{k-1})\xi\right|^{-\frac{\alpha}{\gamma'(m+1)}}; \tag{2.4}$$

$$\int_{a_m^k}^{a_m^{k+1}} |\hat{\sigma}_{m,t}(\xi)|^2 \frac{dt}{t} \leq C(m+1)\left|\Phi(a_m^{k+1})\xi\right|^{\frac{\alpha}{\gamma'(m+1)}}, \tag{2.5}$$

where $\|\sigma_{m,t}\|$ stands for the total variation of $\sigma_{m,t}$. The constant C is independent of k, m, ξ and $\Phi(\cdot)$.

By (2.2) and the definition of $\sigma_{m,t}$, one can easily see that (2.3) holds with a constant C independent of t and m. Next we prove (2.4). Switching to polar coordinates and then applying Hölder's inequality, we obtain

$$|\hat{\sigma}_{m,t}(\xi)| \leq \left(\frac{1}{t}\int_{\frac{1}{2}t}^{t} |h(s)|^{\gamma} ds\right)^{1/\gamma} \left(\frac{1}{t}\int_{\frac{1}{2}t}^{t} |I_m(s)|^{\gamma'} ds\right)^{1/\gamma'},$$

where

$$I_m(s) = \int_{\mathbf{S}^{n-1}} e^{-i\Phi(s)\xi\cdot x}\Omega_m(x)d\sigma(x).$$

Since $|I_m(s)| \leq 1$ we immediately get

$$\begin{aligned}|\hat{\sigma}_{m,t}(\xi)| &\leq C\left(\frac{1}{t}\int_{\frac{1}{2}t}^{t} |I_m(s)|^{2} ds\right)^{1/\gamma'} \\ &= C\left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} \Omega_m(x)\overline{\Omega_m(y)}Y_{m,t}(\xi,x,y)d\sigma(x)\,d\sigma(y)\right)^{1/\gamma'},\end{aligned}$$

where

$$Y_{m,t}(\xi,x,y) = \int_{1/2}^{1} e^{-i\Phi(ts)\xi\cdot(x-y)}ds.$$

We now show that

$$|Y_{m,t}(\xi,x,y)| \leq C\,|\Phi(t/2)\xi|^{-\alpha}\,|\xi'\cdot(x-y)|^{-\alpha}; \tag{2.6}$$

for some $0 < \alpha < 1/2$.

To this end, we notice that by the assumptions on Φ and the mean value theorem we have

$$\frac{d}{ds}(\Phi(ts)) = t\Phi'(ts) \geq \frac{\Phi(ts)}{s} \geq \Phi(t/2).$$

Thus by van der Corput's lemma, $|Y_{m,t}(\xi,x,y)| \leq |\Phi(t/2)\xi|^{-1}\,|\xi'\cdot(x-y)|^{-1}$. By combining this estimate with the trivial estimate $|Y_{m,t}(\xi,x,y)| \leq 1/2$ and choosing α such that $0 < \alpha < 1/2$ we get (2.6). Applying now Schwarz's inequality and (2.1) we get

$$\begin{aligned}&|\hat{\sigma}_{m,t}(\xi)| \\ \leq\ & C\,|\Phi(t/2)\xi|^{-\alpha/\gamma'}\left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} |\Omega_m(x)\Omega_m(y)|\,|\xi'\cdot(x-y)|^{-\alpha}\,d\sigma(x)\,d\sigma(y)\right)^{1/\gamma'} \\ \leq\ & C\,|\Phi(t/2)\xi|^{-\alpha/\gamma'}\,(a_m)^{4/\gamma'}\left\{\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} |x_1-y_1|^{-2\alpha}\,d\sigma(x)\,d\sigma(y)\right\}^{1/2\gamma'},\end{aligned}$$

where $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$. Since the last integral is finite, we obtain

$$|\hat{\sigma}_{m,t}(\xi)| \leq C\,|\Phi(t/2)\xi|^{-\alpha/\gamma'}\,(a_m)^{4/\gamma'},$$

which easily leads to

$$\begin{aligned}\int_{a_m^k}^{a_m^{k+1}} |\hat{\sigma}_{m,t}(\xi)|^2 \frac{dt}{t} &\leq C(m+1)(a_m)^{8/\gamma'} \left|\Phi(\frac{1}{2}a_m^k)\xi\right|^{-\frac{2\alpha}{\gamma'}} \\ &\leq C(m+1)(a_m)^{8/\gamma'} \left|\Phi(a_m^{k-1})\xi\right|^{-\frac{2\alpha}{\gamma'}}.\end{aligned}$$

By combining the last estimate with the trivial estimate

$$\int_{a_m^k}^{a_m^{k+1}} |\hat{\sigma}_{m,t}(\xi)|^2 \frac{dt}{t} \leq C(m+1)$$

we get (2.4).

We now turn to the proof of (2.5). By using the mean zero property (1.1) of Ω_m we get

$$|\hat{\sigma}_{m,t}(\xi)| \leq \frac{1}{t}\int_{\mathbf{S}^{n-1}}\int_{\frac{1}{2}t}^{t} \left|e^{-i\Phi(s)\xi\cdot x} - 1\right| |h(s)| \, |\Omega_m(x)| \, ds d\sigma(x).$$

Hence by (2.2) and since Φ is increasing we get

$$|\hat{\sigma}_{m,t}(\xi)| \leq C\,|\Phi(t)\xi|\,.$$

By using the same argument as above we get (2.5). The lemma is proved.

By the same argument as in [15, p. 57] we get

Lemma 2.3. *Let φ be a nonnegative, decreasing function on $[0,\infty)$ with $\int_{[0,\infty)}\varphi(t)dt = 1$. Then*

$$\left|\int_{[0,\infty)} f(x-ty')\varphi(t)dt\right| \leq M_{y'}f(x),$$

where

$$M_{y'}f(x) = \sup_{R\in\mathbf{R}} \frac{1}{R}\int_0^R |f(x-sy')|\,ds$$

is the Hardy-Littlewood maximal function of f in the direction of y'.

Lemma 2.4. *Let $m \in \mathbf{N}$, $h \in \Delta_\gamma(\mathbf{R}_+)$ for some $\gamma > 1$, $\gamma' < p < \infty$ and $\omega \in \tilde{A}_{p/\gamma'}(\mathbf{R}_+)$. Let Ω_m be a function on $\mathbf{S}^{n-1}$ satisfying (2.1)-(2.2) and let Φ be in $C^2([0,\infty))$, convex, and increasing function with $\Phi(0) = 0$. Then*

$$\|\sigma_m^*(f)\|_{L^p(\omega)} \leq C_p \|f\|_{L^p(\omega)}\,, \tag{2.7}$$

where C_p is independent of m and f.

Proof. By Hölder's inequality and (2.2), we have

$$\begin{aligned}
&\left|\left|\sigma_{m,t}\right| * f(x)\right| \\
\leq\ & \left(\frac{1}{t}\int_{\frac{1}{2}t}^{t}\left|h(s)\right|^{\gamma} ds\right)^{1/\gamma}\left(\frac{1}{t}\int_{\frac{1}{2}t}^{t}\left|\int_{\mathbf{S}^{n-1}}\Omega_m(y')f(x-\Phi(s)y')d\sigma(y')\right|^{\gamma'} ds\right)^{1/\gamma'} \\
\leq\ & C\left(\frac{1}{t}\int_{\frac{1}{2}t}^{t}\int_{\mathbf{S}^{n-1}}\left|\Omega_m(y')\right|\left|f(x-\Phi(s)y')\right|^{\gamma'} d\sigma(y')ds\right)^{1/\gamma'}.
\end{aligned}$$

Thus

$$\sigma_m^* f(x) \leq C\left(\int_{\mathbf{S}^{n-1}}\left|\Omega_m(y')\right|\mathcal{M}_{\Phi,y'}(|f|^{\gamma'})(x)d\sigma(y')\right)^{1/\gamma'}, \tag{2.8}$$

where

$$\mathcal{M}_{\Phi,y'}f(x) = \sup_{t\in\mathbf{R}_+}\left|\frac{1}{t}\int_0^t f(x-\Phi(s)y')ds\right|.$$

Without loss of generality, we may assume that $\Phi(t) > 0$ for all $t > 0$. By a change of variable we have

$$\mathcal{M}_{\Phi,y'}f(x) \leq \sup_{t\in\mathbf{R}_+}\left(\frac{1}{t}\int_0^{\Phi(t)}\left|f(x-sy')\right|\frac{ds}{\Phi'(\Phi^{-1}(s))}\right).$$

Since the function $\frac{1}{t\Phi'(\Phi^{-1}(s))}$ is nonnegative, decreasing and its integral over $[0,\Phi(t)]$ is equal to 1, by Lemma 2.3 we obtain

$$\mathcal{M}_{\Phi,y'}f(x) \leq M_{y'}f(x). \tag{2.9}$$

By (2.8)-(2.9) and Minkowski's inequality for integrals we get

$$\left\|\sigma_m^* f\right\|_{L^p(\omega)} \leq \left(\int_{\mathbf{S}^{n-1}}\left|\Omega_m(y')\right|\left\|M_{y'}(|f|^{\gamma'})\right\|_{L^{p/\gamma'}(\omega)} d\sigma(y')\right)^{1/\gamma'}. \tag{2.10}$$

By (8) in [6] and since $\omega \in \tilde{A}_{p/\gamma'}(\mathbf{R}_+)$ we have

$$\left\|M_{y'}f\right\|_{L^{p/\gamma'}(\omega)} \leq C\left\|f\right\|_{L^{p/\gamma'}(\omega)} \tag{2.11}$$

with C independent of y'. By (2.2) and (2.10)-(2.11) we get (2.7) which finishes the proof of the lemma. □

3. Proof of theorems

Assume that $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$ and satisfies (1.1). We may assume without loss of generality that σ is normalized so that $\sigma(\mathbf{S}^{n-1}) = 1$. For each $m \in \mathbf{N}$, let $E_m = \left\{y \in \mathbf{S}^{n-1} : 2^{m-1} \le |\Omega(y)| < 2^m\right\}$. Let

$$D_\Omega = \{m \in \mathbf{N} : \ \sigma(E_m) > 2^{-4m}\}, \tag{3.1}$$

and for each $m \in D_\Omega$, let $\theta_m = \|\Omega\|_{L^1(E_m)}$,

$$\Omega_m(x) = \theta_m^{-1}\left(\Omega(x)\chi_{E_m}(x) - \int_{E_m} \Omega(u)d\sigma(u)\right). \tag{3.2}$$

Also, let

$$\Omega_0 = \Omega - \sum_{m \in D_\Omega} \theta_m \Omega_m. \tag{3.3}$$

Let $\theta_0 = 1$. Then the following hold for all $m \in D_\Omega \cup \{0\}$:

$$\int_{\mathbf{S}^{n-1}} \Omega_m(x)d\sigma \ = \ 0; \tag{3.4}$$

$$\|\Omega_m\|_{L^1(\mathbf{S}^{n-1})} \ \le \ C; \tag{3.5}$$

$$\|\Omega_m\|_{L^2(\mathbf{S}^{n-1})} \ \le \ C(a_m)^2; \tag{3.6}$$

$$\sum_{m \in D_\Omega \cup \{0\}} (m+1)^{1/2}\theta_m \ \le \ C\,\|\Omega\|_{L(\log L)^{1/2}(\mathbf{S}^{n-1})}; \tag{3.7}$$

and

$$\Omega = \sum_{m \in D_\Omega \cup \{0\}} \theta_m \Omega_m \tag{3.8}$$

for some positive constant C.

By (3.8) we have

$$\mathcal{M}_{\Omega,\Phi,h}(f) \le \sum_{m \in D_\Omega \cup \{0\}} \theta_m \mathcal{M}_{\Omega_m,\Phi,h}(f)\ . \tag{3.9}$$

Therefore, Theorems 1.1 and 1.2 are proved if we can show that

$$\left\|\mathcal{M}_{\Omega_m,\Phi,h}(f)\right\|_{L^p(\mathbf{R}^n)} \le C_p(m+1)^{1/2}\,\|f\|_{L^p(\mathbf{R}^n)} \tag{3.10}$$

for p satisfying $|1/p - 1/2| < \min\{1/\gamma', 1/2\}$; and

$$\left\|\mathcal{M}_{\Omega_m,\Phi,h}(f)\right\|_{L^p(\omega)} \le C_p(m+1)^{1/2}\,\|f\|_{L^p(\omega)} \tag{3.11}$$

if p and ω satisfy one of the conditions (a) and (b) of Theorem 1.2.

Proof of (3.10). Since $\Delta_{\gamma}(\mathbf{R}_{+}) \subseteq \Delta_{2}(\mathbf{R}_{+})$ for $\gamma \geq 2$, we may assume that $1 < \gamma \leq 2$. Therefore, it suffices to prove (3.10) for p satisfying $|1/p - 1/2| < 1/\gamma'$. For $k \in \mathbf{Z}$ and $m \in \mathbf{N}$ let $\Upsilon_{m,k} = \Phi(a_m^k)$. We notice that $\{\Upsilon_{m,k} : k \in \mathbf{Z}\}$ is a lacunary sequence with $\Upsilon_{m,k+1}/\Upsilon_{m,k} \geq a_m$. Let $\{\Lambda_{k,m}\}_{-\infty}^{\infty}$ be a smooth partition of unity in $(0, \infty)$ adapted to the interval $I_{k,m} = [\Upsilon_{m,k+1}^{-1}, \Upsilon_{m,k-1}^{-1}]$. To be precise, we require the following:

$$\begin{aligned} \Lambda_{k,m} &\in C^{\infty},\ 0 \leq \Lambda_{k,m} \leq 1,\ \sum_{k} \Lambda_{k,m}(t) = 1; \\ \text{supp } \Lambda_{k,m} &\subseteq I_{k,m};\ \left| \frac{d^s \Lambda_{k,m}(t)}{dt^s} \right| \leq \frac{C_s}{t^s}, \end{aligned}$$

where C_s is independent of the lacunary sequence $\{\Upsilon_{m,k} : k \in \mathbf{Z}\}$. Let $\widehat{\Psi_{k,m}}(\xi) = \Lambda_{k,m}(|\xi|)$.

By Minkowski's inequality we have

$$\begin{aligned} &\mathcal{M}_{\Omega_m,\Phi,h} f(x) \\ &= \left(\int_0^{\infty} \left| \sum_{k=0}^{\infty} 2^{-k} \sigma_{m,2^{-k}t} * f(x) \right|^2 \frac{dt}{t} \right)^{1/2} \\ &\leq \sum_{k=0}^{\infty} 2^{-k} \left(\int_0^{\infty} \left| \sigma_{m,2^{-k}t} * f(x) \right|^2 \frac{dt}{t} \right)^{1/2} \\ &= 2 \left(\int_0^{\infty} \left| \sigma_{m,t} * f(x) \right|^2 \frac{dt}{t} \right)^{1/2}. \end{aligned}$$

Decompose

$$f * \sigma_{m,t}(x) = \sum_{j \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} (\Psi_{k+j,m} * \sigma_{m,t} * f)(x) \chi_{[a_m^k, a_m^{k+1})}(t) := \sum_{j \in \mathbf{Z}} S_{j,m}(x,t)$$

and define

$$\mathcal{M}_{j,m} f(x) = \left(\int_0^{\infty} |S_{j,m}(x,t)|^2 \frac{dt}{t} \right)^{1/2}.$$

Then

$$\mathcal{M}_{\Omega_m,\Phi,h}(f) \leq 2 \sum_{j \in \mathbf{Z}} \mathcal{M}_{j,m}(f)$$

holds for $f \in \mathcal{S}(\mathbf{R}^n)$.

We notice that to prove (3.10), it is enough to show that

$$\|\mathcal{M}_{j,m}(f)\|_{L^p(\mathbf{R}^n)} \leq C(m+1)^{1/2} 2^{-\alpha_p |j|} \|f\|_{L^p(\mathbf{R}^n)} \tag{3.12}$$

for some $\alpha_p > 0$ and for p satisfying $|1/p - 1/2| < 1/\gamma'$. This can be achieved by interpolation between a sharp L^2 estimate and a cruder L^p estimate of $\mathcal{M}_{j,m}$.

To this end, as usual a sharp L^2 estimate can be obtained by using Plancherel's theorem. In fact,

$$\begin{aligned}\|\mathcal{M}_{j,m}(f)\|^2_{L^2(\mathbf{R}^n)} &= \sum_{k\in\mathbf{Z}}\int_{\mathbf{R}^n}\int_{a_m^k}^{a_m^{k+1}} |\Psi_{k+j,m}*\sigma_{m,t}*f(x)|^2\,\frac{dt}{t}dx\\ &\le \sum_{k\in\mathbf{Z}}\int_{\Gamma_{k+j,m}}\left(\int_{a_m^k}^{a_m^{k+1}}|\hat{\sigma}_{m,t}(\xi)|^2\,\frac{dt}{t}\right)\left|\hat{f}(\xi)\right|^2 d\xi,\end{aligned}$$

where

$$\Gamma_{k,m}=\{\xi\in\mathbf{R}^n : |\xi|\in I_{k,m}\}.$$

By Lemma 2.2 we have

$$\|\mathcal{M}_{j,m}(f)\|_{L^2(\mathbf{R}^n)} \le C(m+1)^{1/2}\,2^{-\frac{\alpha}{2}|j|}\,\|f\|_{L^2(\mathbf{R}^n)}\,. \tag{3.13}$$

On the other hand, we compute the $L^p(\mathbf{R}^n)$-norm of $\mathcal{M}_{j,m}(f)$. To this end, assume first that $2\le p<\frac{2\gamma}{2-\gamma}$. By duality there exists a nonnegative function g in $L^{(p/2)'}(\mathbf{R}^n)$ with $\|g\|_{(p/2)'}\le 1$ such that

$$\|\mathcal{M}_{j,m}(f)\|^2_{L^p(\mathbf{R}^n)} = \sum_{k\in\mathbf{Z}}\int_{\mathbf{R}^n}\int_{a_m^k}^{a_m^{k+1}} |\sigma_{m,t}*\Psi_{k+j,m}*f(x)|^2\,\frac{dt}{t}g(x)dx.$$

Now

$$\begin{aligned}&|\sigma_{m,t}*\Psi_{k+j,m}*f(x)|^2\\ =\;&\left|\frac{1}{t}\int_{\frac{1}{2}t}^{t}\int_{\mathbf{S}^{n-1}}(\Psi_{k+j,m}*f)(x-\Phi(r)y)h(r)\Omega_m(y)d\sigma(y)dr\right|^2\\ \le\;& C\,\|\Omega_m\|_1\left(\frac{1}{t}\int_{\frac{1}{2}t}^{t}\int_{\mathbf{S}^{n-1}}|(\Psi_{k+j,m}*f)(x-\Phi(r)y)|^2\,|\Omega_m(y)|\,|h(r)|^{2-\gamma}\,d\sigma(y)dr\right).\end{aligned}$$

Thus by a change of variable we get

$$\begin{aligned}\|\mathcal{M}_{j,m}(f)\|^2_{L^p(\mathbf{R}^n)} \le\; & C\,\|\Omega_m\|_1\int_{\mathbf{R}^n}\sum_{k\in\mathbf{Z}}\int_{a_m^k}^{a_m^{k+1}}\left(\frac{1}{t}\int_{\frac{1}{2}t}^{t}\int_{\mathbf{S}^{n-1}} g(x+\Phi(r)y)\,|\Omega_m(y)|\times\right.\\ &\left.|h(r)|^{2-\gamma}\,d\sigma(y)dr\right)\frac{dt}{t}\,|\Psi_{k+j,m}*f(x)|^2\,dx.\end{aligned}$$

Therefore, by (2.2) we have

$$\|\mathcal{M}_{j,m}(f)\|^2_{L^p(\mathbf{R}^n)} \le C(m+1)\int_{\mathbf{R}^n}\sum_{k\in\mathbf{Z}}|\Psi_{k+j,m}*f(x)|^2\,M_{|h|^{2-\gamma},m}g(x)dx, \tag{3.14}$$

where

$$M_{|h|^{2-\gamma},m}g(x)=\sup_{t\in\mathbf{R}_+}\left|\frac{1}{t}\int_{\frac{1}{2}t<|y|\le t}g(x+\Phi(|y|)y')\,|h(|y|)|^{2-\gamma}\,\frac{|\Omega_m(y)|}{|y|^{n-1}}dy\right|.$$

By invoking Lemma 2.4 with $\omega=1$ and noticing that $|h(\cdot)|^{2-\gamma}\in\Delta_{\gamma/(2-\gamma)}(\mathbf{R}_+)$ and $(p/2)'>\left(\frac{\gamma}{2-\gamma}\right)'$ we obtain

$$\left\|M_{|h|^{2-\gamma},m}g\right\|_{L^{(p/2)'}(\mathbf{R}^n)}\le C_p\,\|g\|_{L^{(p/2)'}(\mathbf{R}^n)}\le C_p.$$

Therefore, by Hölder's inequality and using Littlewood-Paley theory and Theorem 3 along with the remark that follows its statement in ([14], p.96), we have

$$\begin{aligned}\|\mathcal{M}_{j,m}(f)\|^2_{L^p(\mathbf{R}^n)} &\le C_p(m+1)\left\|\left(\sum_{k\in\mathbf{Z}}|\Psi_{k+j,m}*f|^2\right)^{1/2}\right\|^2_{L^p(\mathbf{R}^n)}\\ &\quad\times\left\|M_{|h|^{2-\gamma},m}g\right\|_{L^{(p/2)'}(\mathbf{R}^n)}\end{aligned}$$

which in turn gives

$$\|\mathcal{M}_{j,m}(f)\|_{L^p(\mathbf{R}^n)}\le C_p(m+1)^{1/2}\,\|f\|_{L^p(\mathbf{R}^n)}\ \text{ for } 2\le p<\frac{2\gamma}{2-\gamma}. \tag{3.15}$$

Now we need to handle the case $\frac{2\gamma}{3\gamma-2}<p<2$. Let $J_{m,k}=[a_m^k,a_m^{k+1})$. By a duality argument, there exist functions $h=h_k(x,t)$ defined on $\mathbf{R}^n\times\mathbf{R}_+$ with $\left\|\left\|\|h_k\|_{L^2(J_{m,k},dt/t)}\right\|_{l^2}\right\|_{L^{p'}}\le 1$ such that

$$\|\mathcal{M}_{j,m}(f)\|_p=\int_{\mathbf{R}^n}\sum_{k\in\mathbf{Z}}\int_{J_{m,k}}(\Psi_{k+j,m}*\sigma_{m,t}*f(x))\,h_k(x,t)\frac{dt}{t}dx.$$

By a change of variable, Hölder's inequality and using Littlewood-Paley theory we have

$$\begin{aligned}\|\mathcal{M}_{j,m}(f)\|_p &\le C_p\,(m+1)^{1/2}\left\|\left(\sum_{k\in\mathbf{Z}}|\Psi_{k+j,m}*f|^2\right)^{\frac{1}{2}}\right\|_p\left\|(T(h))^{1/2}\right\|_{p'}\\ &\le C_p\,(m+1)^{1/2}\,\|f\|_p\,\|T(h)\|^{1/2}_{p'/2},\end{aligned}\tag{3.16}$$

where

$$T(h)(x)=\sum_{k\in\mathbf{Z}}\int_{J_{m,k}}|\sigma_{m,t}*h_k(x,t)|^2\,\frac{dt}{t}.$$

Now, since $p' > 2$, there exists a function $q \in L^{(p'/2)'}(\mathbf{R}^n)$ such that

$$\|T(h)\|_{p'/2} = \sum_{k\in\mathbf{Z}} \int_{\mathbf{R}^n} \int_{J_{m,k}} |h_k(x,t) * \sigma_{m,t}|^2 \frac{dt}{t} q(x)dx.$$

By the same argument as above, we have

$$\begin{aligned} \|T(h)\|_{p'/2} &\leq \int_{\mathbf{R}^n} M_{|h|^{2-\gamma},m} q(x) \left(\sum_{k\in\mathbf{Z}} \int_{J_{m,k}} |h_k(x,t)|^2 \frac{dt}{t} \right) dx \\ &\leq \left\| \left(\sum_{k\in\mathbf{Z}} \int_{J_{m,k}} |h_k(\cdot,t)|^2 \frac{dt}{t} \right) \right\|_{p'/2} \left\| M_{|h|^{2-\gamma},m} q \right\|_{(p'/2)'}. \end{aligned}$$

By invoking Lemma 2.4 with $\omega = 1$ we obtain

$$\left\| M_{|h|^{2-\gamma},m}(q) \right\|_{(p'/2)'} \leq C_p \|q\|_{(p'/2)'} \leq C_p.$$

Thus by our choice of $h_k(x,t)$ we have

$$\|T(h)\|_{p'/2} \leq C_p \left\| \left(\sum_{k\in\mathbf{Z}} \int_{J_{m,k}} |h_k(\cdot,t)|^2 \frac{dt}{t} \right) \right\|_{p'/2} \leq C_p$$

which in turn along with (3.16) leads to the conclusion that

$$\text{(3.17)} \qquad \|\mathcal{M}_{j,m}(f)\|_p \leq C\,(m+1)^{1/2} \|f\|_p \ \text{ for } \frac{2\gamma}{3\gamma-2} < p < 2.$$

By combining (3.15) and (3.17) we get

$$\text{(3.18)} \qquad \|\mathcal{M}_{j,m}(f)\|_p \leq C\,(m+1)^{1/2} \|f\|_p \ \text{ for } p \text{ satisfying } |1/p - 1/2| < 1/\gamma'.$$

Now by interpolation between (3.13) and (3.18) we get (3.12). This completes the proof of (3.10) which in turn concludes the proof of Theorem 1.1.

Proof of (3.11). As above it is enough to show that

$$\text{(3.19)} \qquad \|\mathcal{M}_{j,m}(f)\|_{L^p(\omega)} \leq C(m+1)^{1/2} 2^{-\alpha_p|j|} \|f\|_{L^p(\omega)}$$

if p and ω satisfy one of the conditions (a) and (b) of Theorem 1.2. The proof of this inequality follows immediately once we prove the estimate

$$\text{(3.20)} \qquad \|\mathcal{M}_{j,m}(f)\|_{L^p(\omega)} \leq C(m+1)^{1/2} \|f\|_{L^p(\omega)}$$

if p and ω satisfy one of the conditions (a) and (b) of Theorem 1.2. In fact, by interpolating between (3.13) and (3.20) with $\omega = 1$ we get

$$\text{(3.21)} \qquad \|\mathcal{M}_{j,m}(f)\|_p \leq C_p(m+1)^{1/2} 2^{-\alpha_p|j|} \|f\|_p \ \text{ for } \gamma' < p < \infty.$$

By Lemma 1.3, for any $\omega \in \tilde{A}_p^I(\mathbf{R}_+)$, there is an $\varepsilon > 0$ such that $\omega^{1+\varepsilon} \in \tilde{A}_p^I(\mathbf{R}_+)$, we get

$$\|\mathcal{M}_{j,m}(f)\|_{L^p(\omega^{1+\varepsilon})} \leq C_p(m+1)^{1/2}\|f\|_{L^p(\omega^{1+\varepsilon})} \tag{3.22}$$

if p and ω satisfy one of the conditions (a) and (b) of Theorem 1.2. By Lemma 1.3 and using Stein and Weiss' interpolation theorem with change of measures [16], we may interpolate between (3.21) and (3.22) to get (3.11) as asserted. So let us turn to the proof of (3.20). □

The key step in the proof of (3.20) will rely on the following lemma.

Lemma 3.1. *Let $m \in \mathbf{N}$, $h \in \Delta_\gamma(\mathbf{R}_+)$ for some $\gamma \geq 2$ and $\omega \in \tilde{A}_{p/\gamma'}(\mathbf{R}_+)$. Let Ω_m be a function on $\mathbf{S}^{n-1}$ satisfying* (2.1)-(2.2) *and let Φ be in $C^2([0,\infty))$, convex, and increasing function with $\Phi(0) = 0$. Then, for arbitrary functions $\{g_k(\cdot)\}_{k\in\mathbf{Z}}$ on $\mathbf{R}^n$, the following vector valued inequality holds*

$$\begin{aligned}&\left\|\left(\sum_{k\in\mathbf{Z}}\int_{a_m^k}^{a_m^{k+1}}|\sigma_{m,t}*g_k|^2\frac{dt}{t}\right)^{1/2}\right\|_{L^p(\omega)} \\ &\leq C_p(m+1)^{1/2}\left\|(\sum_{k\in\mathbf{Z}}|g_k|^2)^{1/2}\right\|_{L^p(\omega)}\end{aligned} \tag{3.23}$$

if p and ω satisfy one of the conditions $(a)$and (b) of Theorem 1.2, where C_p is a positive constant which is independent of m.

Before presenting a proof of this lemma, let us prove (3.20) by applying Lemma 3.1. Let p be as in Lemma 3.1.

$$\begin{aligned}\|\mathcal{M}_{j,m}(f)\|_{L^p(\omega)} &= \left\|\left(\sum_{k\in\mathbf{Z}}\int_{a_m^k}^{a_m^{k+1}}|\sigma_{m,t}*\Psi_{k+j,m}*f|^2\frac{dt}{t}\right)^{1/2}\right\|_{L^p(\omega)} \\ &\leq C_p(m+1)^{1/2}\left\|\left(\sum_{k\in\mathbf{Z}}|\Psi_{k+j,m}*f|^2\right)^{\frac{1}{2}}\right\|_{L^p(\omega)} \\ &\leq C_p(m+1)^{1/2}\|f\|_{L^p(\omega)},\end{aligned} \tag{3.24}$$

where the first inequality follows by Lemma 3.1 and the last inequality follows from a well-known weighted Littlewood-Paley inequality since $\tilde{A}_{p/2}^I(\mathbf{R}_+) \subset \tilde{A}_{p/\gamma'}^I(\mathbf{R}_+) \subset \tilde{A}_{p/\gamma'}(\mathbf{R}_+) \subset A_p(\mathbf{R}_+)$.

Proof of Lemma 3.1. To prove (3.23) we need to consider two cases. We shall use frequently the arguments employed in the proof of (3.14).

Case 1: $\gamma = 2, \gamma' < p < \infty$ and $\omega \in \tilde{A}_{p/2}(\mathbf{R}_+)$. In this case $2 < p < \infty$. By duality, there is a function $u(x) \in L^{(p/2)'}(\omega^{1-(p/2)'})$ satisfying $\|u\|_{L^{(p/2)'}(\omega^{1-(p/2)'})} \leq$

1 such that

$$\left\| \left(\sum_{k\in\mathbf{Z}} \int_{a_m^k}^{a_m^{k+1}} |\sigma_{m,t} * g_k|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(\omega)}^2 = \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} \int_{a_m^k}^{a_m^{k+1}} |\sigma_{m,t} * g_k(x)|^2 \frac{dt}{t} u(x)dx. \tag{3.25}$$

By the same argument as in the proof of (3.14) we get

$$\left\| \left(\sum_{k\in\mathbf{Z}} \int_{a_m^k}^{a_m^{k+1}} |\sigma_{m,t} * g_k|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(\omega)}^2 \le C(m+1) \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} |g_k(x)|^2 M_{1,m}u(x)dx.$$

By Hölder's inequality

$$\left\| \left(\sum_{k\in\mathbf{Z}} \int_{a_m^k}^{a_m^{k+1}} |\sigma_{m,t} * g_k|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(\omega)}^2 \le C(m+1) \left\| \left(\sum_{k\in\mathbf{Z}} |g_k|^2 \right)^{1/2} \right\|_{L^p(\omega)}^2 \|M_{1,m}u\|_{L^{(p/2)'}(\omega^{1-(p/2)'})} . \tag{3.26}$$

It is easy to verify that $\omega \in \tilde{A}_{p/2}(\mathbf{R}^+)$ if and only if $\omega^{1-(p/2)'} \in \tilde{A}_{(p/2)'}(\mathbf{R}^+)$. By the same proof as that of Lemma 2.4 we get

$$\|M_{1,m}u\|_{L^{(p/2)'}(\omega^{1-(p/2)'})} \le C_p \|u\|_{L^{(p/2)'}(\omega^{1-(p/2)'})} \le 1$$

which when combined with (3.26) easily leads to (3.23) for $\gamma = 2$.

Case 2: $\gamma > 2$, $\gamma' < p < \infty$ and ω is given as in Theorem 1.2. In this case we need to consider two subcases.

Case 2(i): $2 < p < \infty$. In this case $\omega \in \tilde{A}_{p/2}(\mathbf{R}_+)$.

We argue as in the proof in Case 1. By duality, there is a function $u(x) \in L^{(p/2)'}(\omega^{1-(p/2)'})$ satisfying $\|u\|_{L^{(p/2)'}(\omega^{1-(p/2)'})} \le 1$ such that

$$\left\| \left(\sum_{k\in\mathbf{Z}} \int_{a_m^k}^{a_m^{k+1}} |\sigma_{m,t} * g_k|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(\omega)}^2 = \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} \int_{a_m^k}^{a_m^{k+1}} |\sigma_{m,t} * g_k(x)|^2 \frac{dt}{t} u(x)dx. \tag{3.27}$$

By Schwarz's inequality and (2.2) we have

$$|\sigma_{m,t} * g_k(x)|^2 \le C \left(\frac{1}{t} \int_{\frac{1}{2}t}^{t} |h(r)|^2 dr \right) \left(\frac{1}{t} \int_{\frac{1}{2}t}^{t} \int_{\mathbf{S}^{n-1}} |g(x+\Phi(r)y)|^2 |\Omega_m(y)| d\sigma(y)dr \right).$$

Since $\gamma > 2$, by Hölder's inequality we have

$$\left(\frac{1}{t}\int_{\frac{1}{2}t}^{t}|h(r)|^2\,dr\right) \le C\left(\frac{1}{t}\int_{\frac{1}{2}t}^{t}|h(r)|^{\gamma}\,dr\right)^{2/\gamma} \le C.$$

Thus, we have

$$\left\|\left(\sum_{k\in\mathbf{Z}}\int_{a_m^k}^{a_m^{k+1}}|\sigma_{m,t}*g_k|^2\,\frac{dt}{t}\right)^{1/2}\right\|_{L^p(\omega)}^2 \tag{3.28}$$
$$\le\ C(m+1)\left\|\left(\sum_{k\in\mathbf{Z}}|g_k|^2\right)^{1/2}\right\|_{L^p(\omega)}^2 \|M_{1,m}u\|_{L^{(p/2)'}(\omega^{1-(p/2)'})}\,.$$

Reasoning as above we get (3.23) for the case $\gamma > 2$ and $2 < p < \infty$.

Case 2(ii): $\gamma' < p \le 2$. In this case $\tilde{A}_{p/\gamma'}(\mathbf{R}_+)$.

Let us first consider the case $\gamma' < p < 2$. Since

$$\int_{a_m^k}^{a_m^{k+1}}|\sigma_{m,t}*g_k|^2\,\frac{dt}{t} \le C(m+1)\,|\sigma_m^*(|g_k|)|^2\,,$$

we notice that to prove (3.23) for $\gamma' < p < 2$ and $\omega \in \tilde{A}_{p/\gamma'}(\mathbf{R}_+)$ it suffices to show that

$$\left\|\left(\sum_{k\in\mathbf{Z}}|\sigma_m^*(|g_k|)|^2\right)^{1/2}\right\|_{L^p(\omega)} \le C_p\left\|\left(\sum_{k\in\mathbf{Z}}|g_k|^2\right)^{1/2}\right\|_{L^p(\omega)} \tag{3.29}$$

for $\gamma' < p < 2$ and $\omega \in \tilde{A}_{p/\gamma'}(\mathbf{R}_+)$. The proof of this inequality is easy. In fact, since σ_m^* is a positive operator, $\sup_{k\in\mathbf{Z}}|\sigma_m^*(|g_k|)| \le |\sigma_m^*(\sup_{k\in\mathbf{Z}}|g_k|)|$ and since σ_m^* is bounded on $L^p(\omega)$ for $\omega \in \tilde{A}_{p/\gamma'}(\mathbf{R}_+)$ (by Lemma 2.4), we get

$$\left\|\sup_{k\in\mathbf{Z}}|\sigma_m^*(|g_k|)|\right\|_{L^p(\omega)} \le \left\|\left|\sup_{k\in\mathbf{Z}}|g_k|\right|\right\|_{L^p(\omega)}. \tag{3.30}$$

Moreover, the boundedness of σ_m^* on $L^p(\omega)$ for $\omega \in \tilde{A}_{p/\gamma'}(\mathbf{R}_+)$ implies that

$$\begin{aligned}\left\|\left(\sum_{k\in\mathbf{Z}}|\sigma_m^*\,|g_k|)|^p\right)^{1/p}\right\|_{L^p(\omega)} &= \left(\sum_{k\in\mathbf{Z}}\|\sigma_m^*(|g_k|)\|_{L^p(\omega)}^p\right)^{1/p}\\ &\le C_p\left(\sum_{k\in\mathbf{Z}}\|g_k\|_{L^p(\omega)}^p\right)^{1/p}\\ &= C_p\left\|\left(\sum_{k\in\mathbf{Z}}|g_k|^p\right)^{1/p}\right\|_{L^p(\omega)}.\end{aligned} \tag{3.31}$$

Since $p < 2$, (3.23) easily follows by (3.30), (3.31) and the Riesz-Thorin interpolation theorem ([9, page 481]).

Now it remains to verify (3.23) for $p = 2$ and $\gamma > 2$. However, the proof of this inequality follows by (3.31). This concludes the proof of (3.23) for Case 2 and hence the proof of Lemma 3.1 is complete. □

Proof of Theorem 1.3. A proof of Theorem 1.3 can be obtained by Theorem 1.2 and following a similar argument as in [11]. We omit the details. □

We end this section with the following result concerning power weights $|x|^{\alpha}$.

One of the important special classes of radial weights is the power weights $|x|^{\alpha}$, $\alpha \in \mathbf{R}$. It is know that $|x|^{\alpha} \in A_p(\mathbf{R}^n)$ if and only if $-n < \alpha < n(p-1)$.

Our result regarding this class of weights is the following:

Theorem 3.2. *Let $h \in \Delta_{\gamma}(\mathbf{R}^+)$ with $\gamma \geq 2$. Let Φ be in $C^2([0,\infty))$, convex, and increasing function with $\Phi(0) = 0$. If $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$, then*

$$\|\mathcal{M}_{\Omega,\Phi,h}(f)\|_{L^p(|x|^{\alpha})} \leq C_p \|f\|_{L^p(|x|^{\alpha})}$$

if p and α satisfy one of the following conditions:

(a) $2 < p < \infty$ *and* $\alpha \in (-1,\ p/2 - 1)$;

(b) $\gamma' < p \leq 2$ *and* $\alpha \in (-1,\ p/\gamma' - 1)$.

A proof of this theorem can be obtained by Theorems 1.2 and noticing that $|x|^{\alpha} \in \tilde{A}_p^I(\mathbf{R}_+)$ for $\alpha \in (-1, p-1)$.

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