

Some New Hilbert Type Inequalities

SHABAN RASLAN SALEM

Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City, Cairo, Egypt

e-mail : shrsalem@hotmail.com

ABSTRACT. In this paper, we obtain some inequalities similar to Hilbert type. Some new inequalities are also given.

1. Introduction

The Hilbert integral inequality is given as follows [1]. Let $p > 1, q > 1$, $1/p + 1/q = 1$, $f, g > 0$. If $0 < \int_0^\infty f^p(t)dt < \infty$, and $0 < \int_0^\infty g^q(t)dt < \infty$, then we have

$$(1.1) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy \leq \frac{\pi}{\sin(\frac{\pi}{p})} \left(\int_0^\infty f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy \right)^{\frac{1}{q}},$$

where $\pi/\sin(\pi/p)$ is best possible.

The Hilbert's double series inequality is given as follows [1]. Let $p > 1, q > 1$, $1/p + 1/q = 1$, $a_m, b_n > 0$. If $0 < \sum_{m=1}^\infty a_m^p < \infty$, and $0 < \sum_{n=1}^\infty b_n^q < \infty$, then we have

$$(1.2) \quad \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} \leq \frac{\pi}{\sin(\frac{\pi}{p})} \left(\sum_{m=1}^\infty a_m^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty b_n^q \right)^{\frac{1}{q}},$$

where $\pi/\sin(\pi/p)$ is best possible.

Recently, many interesting extensions and generalizations of (1.1) and (1.2) appeared in [2], [3], [4], [5], [6]. The main purpose of the present paper is to establish some new generalized inequalities similar to Hilbert's inequality. We also establish some new inequalities.

2. Integral inequalities

First, we introduce some Lemmas.

Received March 3, 2004, and, in revised form, August 19, 2004.

2000 Mathematics Subject Classification: 26D15.

Key words and phrases: Hilbert's inequality, Hölder inequality, Beta function, weight function, weight coefficient.

Lemma 2.1. For $p > 1$, $q > 1$, $1/p + 1/q = 1$, define the weight function $w_1(x)$ as

$$(2.1) \quad w_1(x) = \int_0^\infty \frac{1}{(Ax^\alpha + By^\beta)^{\frac{p\gamma}{2}}} \left(\frac{x}{y}\right)^{\frac{p\lambda}{2}} dy \\ \left(A, B > 0; \alpha, \beta, \gamma, \lambda \geq 0; y \in [0, \infty) \right).$$

Then we get

$$(2.2) \quad w_1(x) = \frac{A^{-\frac{p\gamma}{2}}}{\beta} \left(\frac{A}{B}\right)^{\frac{2-p\lambda}{2\beta}} x^{\frac{p\lambda}{2} + \frac{\alpha}{2\beta}[2-p(\lambda+\beta\gamma)]} B\left(\frac{2-p\lambda}{2\beta}, \frac{p(\lambda+\beta\gamma)-2}{2\beta}\right),$$

where $B(a, b)$ is the well-known Beta function where $a > 0$ and $b > 0$.

Proof. Setting $u = By^\beta/Ax^\alpha$ in (2.1), we have

$$w_1(x) = \frac{A^{-\frac{p\gamma}{2}}}{\beta} \left(\frac{A}{B}\right)^{\frac{2-p\lambda}{2\beta}} x^{\frac{p\lambda}{2} + \frac{\alpha}{2\beta}[2-p(\lambda+\beta\gamma)]} \int_0^\infty u^{\frac{2-2\beta-p\lambda}{2\beta}} (1+u)^{-\frac{p\gamma}{2}} du.$$

Since $B(a, b) = \int_0^\infty t^{a-1}(1+t)^{-(a+b)} dt$ ($a > 0, b > 0$), then we get (2.2) immediately. \square

Lemma 2.2. For $p > 1$, $q > 1$, $1/p + 1/q = 1$, define the weight function $w_2(y)$ as

$$w_2(y) = \int_0^\infty \frac{1}{(Ax^\alpha + By^\beta)^{\frac{q\gamma}{2}}} \left(\frac{y}{x}\right)^{\frac{q\lambda}{2}} dx \quad \left(x \in [0, \infty) \right),$$

where $A, B, \alpha, \beta, \gamma$, and λ are given as in (2.1). Then, similarly as in getting (2.2), we have

$$(2.3) \quad w_2(y) = \frac{B^{-\frac{q\gamma}{2}}}{\alpha} \left(\frac{B}{A}\right)^{\frac{2-q\lambda}{2\alpha}} y^{\frac{q\lambda}{2} + \frac{\beta}{2\alpha}[2-q(\lambda+\alpha\gamma)]} B\left(\frac{2-q\lambda}{2\alpha}, \frac{q(\lambda+\alpha\gamma)-2}{2\alpha}\right).$$

Theorem 2.3. Let f, g be real-valued functions defined on $[0, \infty)$ such that $0 < \int_0^\infty x^{\frac{p\lambda}{2} + \frac{\alpha}{2\beta}(2-p(\lambda+\beta\gamma))} f^p(x) dx < \infty$ and $0 < \int_0^\infty y^{\frac{q\lambda}{2} + \frac{\beta}{2\alpha}(2-q(\lambda+\beta\gamma))} g^q(y) dy < \infty$. Then we have

$$(2.4) \quad \begin{aligned} & \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(Ax^\alpha + By^\beta)^\gamma} dx dy \\ & \leq \left\{ \frac{A^{-\frac{p\gamma}{2}}}{\beta} \left(\frac{A}{B}\right)^{\frac{2-p\lambda}{2\beta}} B\left(\frac{2-p\lambda}{2\beta}, \frac{p(\lambda+\beta\gamma)-2}{2\beta}\right) \int_0^\infty x^{\frac{p\lambda}{2} + \frac{\alpha}{2\beta}[2-p(\lambda+\beta\gamma)]} f^p(x) dx \right\}^{\frac{1}{p}} \\ & \quad \times \left\{ \frac{B^{-\frac{q\gamma}{2}}}{\alpha} \left(\frac{B}{A}\right)^{\frac{2-q\lambda}{2\alpha}} B\left(\frac{2-q\lambda}{2\alpha}, \frac{q(\lambda+\alpha\gamma)-2}{2\alpha}\right) \int_0^\infty y^{\frac{q\lambda}{2} + \frac{\beta}{2\alpha}[2-q(\lambda+\alpha\gamma)]} g^q(y) dy \right\}^{\frac{1}{q}}, \end{aligned}$$

where $A, B > 0$, $\alpha, \beta, \gamma, \lambda \geq 0$; $p > 1$, $q > 1$, $1/p + 1/q = 1$.

Proof. By Hölder's inequality, we have

$$\begin{aligned}
& \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(Ax^\alpha + By^\beta)^\gamma} dx dy \\
&= \int_0^\infty \int_0^\infty \left[\frac{f(x)}{(Ax^\alpha + By^\beta)^{\frac{\gamma}{2}}} \left(\frac{x}{y} \right)^{\frac{\lambda}{2}} \right] \left[\frac{g(y)}{(Ax^\alpha + By^\beta)^{\frac{\gamma}{2}}} \left(\frac{y}{x} \right)^{\frac{\lambda}{2}} \right] dx dy \\
&\leq \left\{ \int_0^\infty \int_0^\infty \frac{f^p(x)}{(Ax^\alpha + By^\beta)^{\frac{p\gamma}{2}}} \left(\frac{x}{y} \right)^{\frac{p\lambda}{2}} dx dy \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \int_0^\infty \frac{g^q(y)}{(Ax^\alpha + By^\beta)^{\frac{q\gamma}{2}}} \left(\frac{y}{x} \right)^{\frac{q\lambda}{2}} dx dy \right\}^{\frac{1}{q}}.
\end{aligned}$$

Using (2.2) and (2.3), we get (2.4) directly. \square

Remarks 2.4.

1. Let $A = B = 1$, $\alpha = \beta = 1$, $p = q = 2$, $\gamma = 1$, $\lambda = \frac{1}{2}$ in (2.4). Then we have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy \leq \pi \left(\int_0^\infty f^2(x) dx \right)^{\frac{1}{2}} \left(\int_0^\infty g^2(y) dy \right)^{\frac{1}{2}},$$

which is Hilbert's integral inequality.

2. Let $A = B = 1$, $\alpha = \beta = 1$, $p = q = 2$ in (2.4). Then we get

$$\begin{aligned}
(2.5) \quad & \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\gamma} dx dy \\
&\leq B(1-\lambda, \gamma+\lambda-1) \left(\int_0^\infty x^{1-\gamma} f^2(x) dx \int_0^\infty y^{1-\gamma} g^2(y) dy \right)^{\frac{1}{2}}.
\end{aligned}$$

Let $1-\lambda = \frac{\gamma}{2}$ in (2.5). Then we obtain

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\gamma} dx dy \leq B\left(\frac{\gamma}{2}, \frac{\gamma}{2}\right) \left(\int_0^\infty x^{1-\gamma} f^2(x) dx \int_0^\infty y^{1-\gamma} g^2(y) dy \right)^{\frac{1}{2}},$$

which is the same result in B. Yang [2].

3. Let $\alpha = \beta = 1$, $p = q = 2$ in (2.4). Then we have

$$\begin{aligned}
(2.6) \quad & \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(Ax+By)^\gamma} dx dy \\
&\leq \frac{1}{(AB)^{\frac{\gamma}{2}}} B(1-\lambda, \gamma+\lambda-1) \left(\int_0^\infty x^{1-\gamma} f^2(x) dx \int_0^\infty y^{1-\gamma} g^2(y) dy \right)^{\frac{1}{2}}.
\end{aligned}$$

Let $1-\lambda = \frac{\gamma}{2}$ in (2.6). Then we get

$$\begin{aligned}
& \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(Ax+By)^\gamma} dx dy \\
&\leq \frac{1}{(AB)^{\frac{\gamma}{2}}} B\left(\frac{\gamma}{2}, \frac{\gamma}{2}\right) \left(\int_0^\infty x^{1-\gamma} f^2(x) dx \int_0^\infty y^{1-\gamma} g^2(y) dy \right)^{\frac{1}{2}},
\end{aligned}$$

which is the same result in Y. Bicheng [6].

4. Let $\gamma = 2$ in (2.6). Then we have

$$(2.7) \quad \begin{aligned} & \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(Ax+By)^2} dx dy \\ & \leq \frac{1}{AB} \left(\int_0^\infty \frac{1}{x} f^2(x) dx \int_0^\infty \frac{1}{y} g^2(y) dy \right)^{\frac{1}{2}}. \end{aligned}$$

Let $A = B = 1$ in (2.7). Then we get

$$(2.8) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^2} dx dy \leq \left(\int_0^\infty \frac{1}{x} f^2(x) dx \int_0^\infty \frac{1}{y} g^2(y) dy \right)^{\frac{1}{2}}.$$

5. Let $A = B = 1$, $p = q = 2$, $\gamma = 1$ in (2.4). Then we have

$$(2.9) \quad \begin{aligned} & \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\alpha + y^\beta} dx dy \\ & \leq \frac{1}{\sqrt{\alpha\beta}} \left[B\left(\frac{1-\lambda}{\beta}, \frac{\lambda+\beta-1}{\beta}\right) B\left(\frac{1-\lambda}{\alpha}, \frac{\lambda+\alpha-1}{\alpha}\right) \right. \\ & \quad \left. \int_0^\infty x^{\lambda+\frac{\alpha}{\beta}(1-\lambda-\beta)} f^2(x) dx \int_0^\infty y^{\lambda+\frac{\beta}{\alpha}(1-\lambda-\alpha)} g^2(y) dy \right]^{\frac{1}{2}}. \end{aligned}$$

Let $\alpha = \beta = n$, $\lambda = 0$ in (2.9). Then we get

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^n + y^n} dx dy \leq \frac{\pi}{n \sin \frac{\pi}{n}} \left(\int_0^\infty x^{1-n} f^2(x) dx \int_0^\infty y^{1-n} g^2(y) dy \right)^{\frac{1}{2}}.$$

6. Let $A = B = 1$, $p = q = 2$, $\alpha = \beta = \gamma = n$, $\lambda = 0$ in (2.4). Then we get

$$(2.10) \quad \begin{aligned} & \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x^n + y^n)^n} dx dy \\ & \leq \frac{1}{n} B\left(\frac{1}{n}, \frac{n^2-1}{n}\right) \left(\int_0^\infty x^{1-n^2} f^2(x) dx \int_0^\infty y^{1-n^2} g^2(y) dy \right)^{\frac{1}{2}}. \end{aligned}$$

3. Discrete inequalities

First, we introduce some Lemmas.

Lemma 3.1. *For $p > 1$, $q > 1$, $1/p + 1/q = 1$, define the weight coefficient $w_1(m)$ as*

$$(3.1) \quad w_1(m) = \sum_{n=1}^{\infty} \frac{1}{(Am^\alpha + Bn^\beta)^{\frac{p\gamma}{2}}} \left(\frac{m}{n}\right)^{\frac{p\lambda}{2}}, \quad (A, B > 0, \alpha, \beta, \gamma, \lambda \geq 0).$$

Then

$$(3.2) \quad w_1(m) \leq \frac{A^{-\frac{p\gamma}{2}}}{\beta} \left(\frac{A}{B} \right)^{\frac{2-p\lambda}{2p}} m^{\frac{p\lambda}{2} + \frac{\alpha}{2\beta}[2-p(\lambda+\beta\gamma)]} B\left(\frac{2-p\lambda}{2\beta}, \frac{p(\lambda+\beta\gamma)-2}{2\beta}\right).$$

Proof. From (3.1), we have

$$\begin{aligned} w_1(m) &= \sum_{n=1}^{\infty} \frac{1}{B^{\frac{p\gamma}{2}} (\frac{A}{B} m^{\alpha} + n^{\beta})^{\frac{p\gamma}{2}}} \left(\frac{m}{n} \right)^{\frac{p\lambda}{2}} \\ &< \frac{1}{B^{\frac{p\gamma}{2}}} \int_0^{\infty} \frac{1}{(\frac{A}{B} m^{\alpha} + y^{\beta})^{\frac{p\gamma}{2}}} \left(\frac{m}{y} \right)^{\frac{p\lambda}{2}} dy. \end{aligned}$$

In a similar way as in the proof of Lemma 2.1, we get (3.2). \square

Lemma 3.2. For $p > 1$, $q > 1$, $1/p + 1/q = 1$, define the weight coefficient $w_2(n)$ as

$$w_2(n) = \sum_{m=1}^{\infty} \frac{1}{(Am^{\alpha} + Bn^{\beta})^{\frac{q\gamma}{2}}} \left(\frac{n}{m} \right)^{\frac{q\lambda}{2}}.$$

Then

$$(3.3) \quad w_2(n) \leq \frac{B^{-\frac{q\gamma}{2}}}{\alpha} \left(\frac{B}{A} \right)^{\frac{2-q\lambda}{2\alpha}} n^{\frac{q\lambda}{2} + \frac{\beta}{2\alpha}[2-q(\lambda+\alpha\gamma)]} B\left(\frac{2-q\lambda}{2\alpha}, \frac{q(\lambda+\alpha\gamma)-2}{2\alpha}\right).$$

From Lemmas 3.1 and 3.2, we can prove the following theorem.

Theorem 3.3. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers such that $0 < \sum_{m=1}^{\infty} m^{\frac{p\lambda}{2} + \frac{\alpha}{2\beta}[2-p(\lambda+\beta\gamma)]} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} n^{\frac{q\lambda}{2} + \frac{\beta}{2\alpha}[2-q(\lambda+\alpha\gamma)]} b_n^q < \infty$, if $A, B > 0$, $\alpha, \beta, \gamma, \lambda \geq 0$; $p > 1$, $q > 1$, $1/p + 1/q = 1$. Then

$$\begin{aligned} (3.4) \quad &\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(Am^{\alpha} + Bn^{\beta})^{\gamma}} \\ &\leq \left\{ \frac{A^{-\frac{p\gamma}{2}}}{\beta} \left(\frac{A}{B} \right)^{\frac{2-p\lambda}{2p}} B\left(\frac{2-p\lambda}{2\beta}, \frac{p(\lambda+\beta\gamma)-2}{2\beta}\right) \sum_{m=1}^{\infty} m^{\frac{p\lambda}{2} + \frac{\alpha}{2\beta}[2-p(\lambda+\beta\gamma)]} a_m^p \right\}^{\frac{1}{p}} \\ &\quad \times \left\{ \frac{B^{-\frac{q\gamma}{2}}}{\alpha} \left(\frac{B}{A} \right)^{\frac{2-q\lambda}{2\alpha}} B\left(\frac{2-q\lambda}{2\alpha}, \frac{q(\lambda+\alpha\gamma)-2}{2\alpha}\right) \sum_{n=1}^{\infty} n^{\frac{q\lambda}{2} + \frac{\beta}{2\alpha}[2-q(\lambda+\alpha\gamma)]} b_n^q \right\}^{\frac{1}{q}}. \end{aligned}$$

Remarks 3.4.

- Let $A = B = 1$, $\alpha = \beta = 1$, $p = q = 2$ and $\lambda = \frac{1}{2}$ in (3.4). Then we have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} \leq \pi \left\{ \sum_{m=1}^{\infty} a_m^2 \cdot \sum_{n=1}^{\infty} b_n^2 \right\}^{\frac{1}{2}},$$

which is Hilbert's double series inequality.

2. Let $A = B = 1$, $\alpha = \beta = 1$, $p = q = 2$ and $1 - \lambda = \frac{\gamma}{2}$ in (3.4). Then we get

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(m+n)^{\gamma}} \leq B\left(\frac{\gamma}{2}, \frac{\gamma}{2}\right) \left\{ \sum_{m=1}^{\infty} m^{1-\gamma} a_m^2 \sum_{n=1}^{\infty} n^{1-\gamma} b_n^2 \right\}^{\frac{1}{2}}.$$

which is a discrete analogue of B. Yang's integral inequality [2].

3. Let $\alpha = \beta = 1$, $p = q = 2$ and $1 - \lambda = \frac{\gamma}{2}$ in (3.4). Then we have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(Am+Bn)^{\gamma}} \leq \frac{1}{(AB)^{\frac{\gamma}{2}}} B\left(\frac{\gamma}{2}, \frac{\gamma}{2}\right) \left\{ \sum_{m=1}^{\infty} m^{1-\gamma} a_m^2 \sum_{n=1}^{\infty} n^{1-\gamma} b_n^2 \right\}^{\frac{1}{2}},$$

which is the same result in Y. Bicheng [6].

4. Let $A = B = 1$, $\alpha = 1$, $p = q = 2$ in (3.4). Then we get

$$(3.5) \quad \begin{aligned} & \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m^{\alpha} + n^{\beta}} \\ & \leq \frac{1}{\sqrt{\alpha\beta}} \left\{ B\left(\frac{1-\lambda}{\beta}, \frac{\lambda+\beta-1}{\beta}\right) B\left(\frac{1-\lambda}{\alpha}, \frac{\lambda+\alpha-1}{\alpha}\right) \right. \\ & \quad \times \left. \sum_{m=1}^{\infty} m^{\lambda+\frac{\alpha}{\beta}(1-\lambda-\beta)} a_m^2 \sum_{n=1}^{\infty} n^{\lambda+\frac{\beta}{\alpha}(1-\lambda-\alpha)} b_n^2 \right\}^{\frac{1}{2}} \quad (0 \leq \lambda < 1). \end{aligned}$$

Let $\lambda = 0$ and $\alpha = \beta$ in (3.5). Then we have

$$(3.6) \quad \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m^{\alpha} + n^{\alpha}} \leq \frac{\pi}{\alpha \sin \frac{\pi}{\alpha}} \left(\sum_{m=1}^{\infty} m^{1-\alpha} a_m^2 \sum_{n=1}^{\infty} n^{1-\alpha} b_n^2 \right)^{\frac{1}{2}}.$$

5. Let $A = B = 1$, $p = q = 2$, $\alpha = \beta = \gamma$ and $\lambda = 0$ in (3.4). Then we get

$$(3.7) \quad \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(m^{\alpha} + n^{\alpha})^{\alpha}} \leq \frac{1}{\alpha} B\left(\alpha, \frac{\alpha^2 - 1}{\alpha}\right) \left(\sum_{m=1}^{\infty} m^{1-\alpha^2} a_m^2 \sum_{n=1}^{\infty} n^{1-\alpha^2} b_n^2 \right)^{\frac{1}{2}}.$$

4. New integral inequalities

Theorem 4.1. *Let f , g be real-valued functions defined on $[0, \infty)$ such that $0 < \int_0^{\infty} x^{2\lambda p-1} f^p(x) dx < \infty$, $0 < \int_0^{\infty} y^{2\lambda q-1} g^q(y) dy < \infty$; $p > 1$, $q > 1$,*

$1/p + 1/q = 1$. Then

$$(4.1) \quad \begin{aligned} & \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{1+xy} dx dy \\ & \leq \left\{ B\left(1 - \lambda p, \frac{2\lambda p + p - 2}{2}\right) \int_0^\infty x^{2\lambda p - 1} f^p(x) dx \right\}^{\frac{1}{p}} \\ & \quad \times \left\{ B\left(1 - \lambda q, \frac{2\lambda q + q - 2}{2}\right) \int_0^\infty y^{2\lambda q - 1} g^q(y) dy \right\}^{\frac{1}{q}}. \end{aligned}$$

Proof. Using Hölder's inequality, we have

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{1+xy} dx dy \\ & \leq \left\{ \int_0^\infty \int_0^\infty \frac{f^p(x)}{(1+xy)^{\frac{p}{2}}} \left(\frac{x}{y}\right)^{\lambda p} dx dy \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \int_0^\infty \frac{g^q(y)}{(1+xy)^{\frac{q}{2}}} \left(\frac{y}{x}\right)^{\lambda q} dx dy \right\}^{\frac{1}{q}}. \end{aligned}$$

Setting $u = xy$, we get

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{1+xy} dx dy \\ & \leq \left\{ \int_0^\infty x^{2\lambda p - 1} f^p(x) \int_0^\infty \frac{u^{-\lambda p}}{(1+u)^{\frac{p}{2}}} du dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty y^{2\lambda q - 1} g^q(y) \int_0^\infty \frac{u^{-\lambda q}}{(1+u)^{\frac{q}{2}}} du dy \right\}^{\frac{1}{q}}. \end{aligned}$$

Hence the proof of Theorem 4.1 is complete. \square

We introduce the following Lemmas

Lemma 4.2. For $p > 1, q > 1, 1/p + 1/q = 1$, define the weight function $w_1(x)$ as

$$(4.2) \quad w_1(x) = \int_0^\infty \frac{1}{(x^a y^b + x^c y^d)^{\frac{p\alpha}{2}}} dy \quad \left(d > b, c > a \text{ and } \frac{2}{d} < p\alpha < \frac{2}{b} \right).$$

Then we get

$$(4.3) \quad w_1(x) \leq \frac{x^{\frac{1}{2(d-b)}[-ap\alpha(d-b)+(c-a)(bp\alpha-2)]}}{d-b} B\left(\frac{2-bp\alpha}{2(d-b)}, \frac{dp\alpha-2}{2(d-b)}\right).$$

Proof. Setting $u = x^{c-a} y^{d-b}$ in (4.2), as in the proof of Lemma 2.1, we get (4.3). \square

Lemma 4.3. For $p > 1, q > 1, 1/p + 1/q = 1$, define the weight function $w_2(x)$ as

$$w_2(x) = \int_0^\infty \frac{1}{(x^a y^b + x^c y^d)^{\frac{q\alpha}{2}}} dx \quad \left(d > b, c > a \text{ and } \frac{2}{c} < q\alpha < \frac{2}{a} \right).$$

Then we have

$$(4.4) \quad w_2(x) \leq \frac{y^{\frac{1}{2(c-a)}[-bq\alpha(c-a)+(d-b)(aq\alpha-2)]}}{c-a} B\left(\frac{2-aq\alpha}{2(c-a)}, \frac{cq\alpha-2}{2(c-a)}\right).$$

Theorem 4.4. Let f, g be real-valued functions defined on $[0, \infty)$ such that $0 < \int_0^\infty x^{\frac{1}{2(d-b)}[-ap\alpha(d-b)+(c-a)(bp\alpha-2)]} f^p(x) dx < \infty$ and $0 < \int_0^\infty y^{\frac{1}{2(c-a)}[-bq\alpha(c-a)+(d-b)(aq\alpha-2)]} g^q(y) dy < \infty$. For $p > 1, q > 1$, $1/p + 1/q = 1$, we have

$$(4.5) \quad \begin{aligned} & \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x^a y^b + x^c y^d)^\alpha} dx dy \\ & \leq \left\{ \frac{1}{d-b} B\left(\frac{2-bp\alpha}{2(d-b)}, \frac{dp\alpha-2}{2(d-b)}\right) \int_0^\infty x^{\frac{1}{2(d-b)}[-ap\alpha(d-b)+(c-a)(bp\alpha-2)]} f^p(x) dx \right\}^{\frac{1}{p}} \\ & \quad \times \left\{ \frac{1}{c-a} B\left(\frac{2-aq\alpha}{2(c-a)}, \frac{cq\alpha-2}{2(c-a)}\right) \int_0^\infty y^{\frac{1}{2(c-a)}[-bq\alpha(c-a)+(d-b)(aq\alpha-2)]} g^q(y) dy \right\}^{\frac{1}{q}} \\ & \quad \left(d > b, c > a, \frac{2}{d} < p\alpha < \frac{2}{c}, \frac{2}{c} < q\alpha < \frac{2}{a} \right). \end{aligned}$$

Proof. Using (4.3), (4.4), we get (4.5) directly. \square

Remarks 4.5.

1. Let $p = q = 2$ in (4.1). Then we have, for $0 < \lambda < 1/2$,

$$(4.6) \quad \begin{aligned} & \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{1+xy} dx dy \\ & \leq B(1-2\lambda, 2\lambda) \left(\int_0^\infty x^{4\lambda-1} f^2(x) dx \int_0^\infty y^{4\lambda-1} g^2(y) dy \right)^{\frac{1}{2}}. \end{aligned}$$

Let $\lambda = 1/4$ in (4.6). Then we get

$$(4.7) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{1+xy} dx dy \leq \pi \left(\int_0^\infty f^2(x) dx \int_0^\infty g^2(y) dy \right)^{\frac{1}{2}}.$$

2. Let $a = b = 0, c = d = 1$ and $p = q = 2$ in (4.5). Then we have, for $\alpha > 1$,

$$(4.8) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(1+xy)^\alpha} dx dy \leq \frac{1}{\alpha-1} \left(\int_0^\infty \frac{f^2(x)}{x} dx \int_0^\infty \frac{g^2(y)}{y} dy \right)^{\frac{1}{2}}.$$

3. Let $a = b = 0$ in (4.5). Then we get

$$(4.9) \quad \begin{aligned} \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(1+x^c y^d)^\alpha} dx dy & \leq \left(\frac{1}{d} B\left(\frac{1}{d}, \frac{dp\alpha-2}{2d}\right) \int_0^\infty x^{-\frac{c}{d}} f^p(x) dx \right)^{\frac{1}{p}} \\ & \quad \times \left(\frac{1}{c} B\left(\frac{1}{c}, \frac{cq\alpha-2}{2c}\right) \int_0^\infty y^{-\frac{d}{c}} g^q(y) dy \right)^{\frac{1}{q}}, \end{aligned}$$

which, upon setting $\alpha = 1$ and $p = q = 2$, yields

$$(4.10) \quad \begin{aligned} & \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{1+x^cy^d} dx dy \\ & \leq \left(\frac{\pi}{d \sin(\frac{\pi}{d})} \int_0^\infty x^{-\frac{c}{d}} f^2(x) dx \frac{\pi}{c \sin(\frac{\pi}{c})} \int_0^\infty y^{-\frac{d}{c}} g^2(y) dy \right)^{\frac{1}{2}}. \end{aligned}$$

Let $d = c = 2$ in (4.10). Then we get

$$(4.11) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{1+x^2y^2} \leq \frac{\pi}{2} \left(\int_0^\infty \frac{f^2(x)}{x} dx \int_0^\infty \frac{g^2(y)}{y} dy \right)^{\frac{1}{2}}.$$

Let $d = c = n$ in (4.10). Then we have

$$(4.12) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{1+(xy)^n} dx dy \leq \frac{\pi}{n \sin(\frac{\pi}{n})} \left(\int_0^\infty x^{-1} f^2(x) dx \int_0^\infty y^{-1} g^2(y) dy \right)^{\frac{1}{2}}.$$

Theorem 4.6. *Let f, g be positive real-valued functions defined on $[0, \infty)$. Then we have*

$$(4.13) \quad \int_0^1 \int_0^1 \frac{f(x)g(y)}{1+x^ay^b} dx dy < \int_0^1 \int_0^1 \frac{f(x)g(y)}{x^a+y^b} dx dy$$

and

$$(4.14) \quad \int_1^\infty \int_1^\infty \frac{f(x)g(y)}{1+x^ay^b} dx dy < \int_1^\infty \int_1^\infty \frac{f(x)g(y)}{x^a+y^b} dx dy,$$

where $a, b > 0$.

Proof. Since $(1-x^a)(1-y^b) = 1+x^ay^b - (x^a+y^b)$, then $(1-x^a)(1-y^b) > 0$ for $0 < x, y < 1$, also $(1-x^a)(1-y^b) > 0$ for $1 < x, y < \infty$, and hence $1+x^ay^b > x^a+y^b$, from which $\frac{1}{x^a+y^b} > \frac{1}{1+x^ay^b}$.

Since $f(x), g(y)$ are positive valued functions, we have

$$\frac{f(x)g(y)}{1+x^ay^b} < \frac{f(x)g(y)}{x^a+y^b},$$

which, upon integrating, yields (4.13) and (4.14). \square

In a similar way as in the proof of Theorem 4.4, we proved the following theorem

Theorem 4.7. *Let f, g be real-valued functions defined on $[0, \infty)$ such that $0 < \int_0^\infty x^{-\frac{a\alpha p}{2} + \frac{a-c}{d}} f^p(x) dx < \infty$ and $0 < \int_0^\infty y^{-\frac{d\alpha q}{2(a-c)} + \frac{d}{a-c}} g^q(y) dy < \infty$. Then we have, for $a, d \neq 0$,*

$$\begin{aligned}
(4.15) \quad & \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x^a + x^c y^d)^\alpha} dx dy \\
& \leq \left[\frac{1}{d} B\left(\frac{1}{d}, \frac{\alpha p d - 2}{2d}\right) \int_0^\infty x^{-\frac{a\alpha p}{2} + \frac{a-c}{d}} f^p(x) dx \right]^{\frac{1}{p}} \\
& \quad \times \left[\frac{1}{a-c} B\left(\frac{2-c\alpha q}{2(a-c)}, \frac{a\alpha q - 2}{2(a-c)}\right) \int_0^\infty y^{-\frac{d a \alpha q}{2(a-c)} + \frac{d}{a-c}} g^q(y) dy \right]^{\frac{1}{q}}.
\end{aligned}$$

Remarks 4.8.

1. Let $c = 0$ in (4.15). Then we get

$$\begin{aligned}
(4.16) \quad & \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x^a + y^d)^\alpha} dx dy \\
& \leq \left[\frac{1}{d} B\left(\frac{1}{d}, \frac{\alpha p d - 2}{2d}\right) \int_0^\infty x^{-\frac{a\alpha p}{2} + \frac{a}{d}} f^p(x) dx \right]^{\frac{1}{p}} \\
& \quad \times \left[\frac{1}{a} B\left(\frac{1}{a}, \frac{\alpha q a - 2}{2a}\right) \int_0^\infty y^{-\frac{d\alpha q}{2} + \frac{d}{a}} g^q(y) dy \right]^{\frac{1}{q}}.
\end{aligned}$$

2. Let $\alpha = 1$, $a = d = 2$ in (4.16). Then we have

$$\begin{aligned}
(4.17) \quad & \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x^2 + y^2)} dx dy \\
& \leq \frac{1}{2} \left[B\left(\frac{1}{2}, \frac{p-1}{2}\right) \int_0^\infty x^{1-p} f^p(x) dx \right]^{\frac{1}{p}} \times \left[B\left(\frac{1}{2}, \frac{q-2}{2}\right) \int_0^\infty y^{1-q} g^q(y) dy \right]^{\frac{1}{q}}.
\end{aligned}$$

3. Let $\alpha = 2$, $a = d = 2$ in (4.16). Then we get

$$\begin{aligned}
(4.18) \quad & \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x^2 + y^2)^2} dx dy \\
& \leq \frac{1}{2} \left[B\left(\frac{1}{2}, \frac{2p-1}{2}\right) \int_0^\infty x^{1-2p} f^p(x) dx \right]^{\frac{1}{p}} \times \left[B\left(\frac{1}{2}, \frac{2q-1}{2}\right) \int_0^\infty y^{1-2q} g^q(y) dy \right]^{\frac{1}{q}}.
\end{aligned}$$

4. From (1.1), (4.7), (4.13) and (4.14). Then we have

$$\begin{aligned}
(4.19) \quad & \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{1+xy} dx dy < \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy \\
& \leq \pi \left(\int_0^\infty f^2(x) dx \int_0^\infty g^2(y) dy \right)^{\frac{1}{2}}
\end{aligned}$$

5. Let $a = d = \alpha = n$ in (4.16). Then we have

$$(4.20) \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x^n + y^n)^n} dx dy \leq \frac{1}{n} \left[B\left(\frac{1}{n}, \frac{n^2 p - 2}{2n}\right) \int_0^\infty x^{1-\frac{n^2 p}{2}} f^p(x) dx \right]^{\frac{1}{p}} \\ \times \left[B\left(\frac{1}{n}, \frac{n^2 q - 2}{2n}\right) \int_0^\infty y^{1-\frac{n^2 q}{2}} g^q(y) dy \right]^{\frac{1}{q}}.$$

6. A discrete analogue of the inequality (4.14) is

$$(4.21) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m a_n}{1 + m^\alpha n^\beta} < \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m a_n}{m^\alpha + n^\beta}.$$

References

- [1] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1952.
- [2] B. Yang, *On Hilbert's integral inequality*, J. Math. Anal. Appl., **220**(1998), 778-785.
- [3] Z. Lü, *Some new inequalities similar to Hilbert-Pachpatte type inequalities*, J. Inequal. Pure Appl. Math. 3(5) Art., **75**(2002).
- [4] L. He, M. Gao and W. Jia, *On Hardy-Hilbert's integral inequality with parameters*, J. Inequal. Pure Appl. Math. 4(5) Art., **94**(2003).
- [5] Y. Bicheng, *On a new generalization of Hardy-Hilbert's inequality and its applications*, J. Math. Anal. Appl., **233**(1999), 484-597.
- [6] Y. Bicheng, *On new generalization of Hilbert's inequality*, J. Math., Anal. Appl., **248**(2000), 29-40.