KYUNGPOOK Math. J. 46(2006), 11-18

On Direct Sums of Lifting Modules and Internal Exchange Property

Wu Dejun
Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou, Gansu 730050, P. R. China
e-mail : wudj@lut.cn

Abstract. Let R be a ring with identity and let $M = M_1 \oplus M_2$ be an amply supplemented R-module. Then it is proved that M_i has (D_1) and is M_j-*ojective for $i \neq j$, $i = 1, 2$, if and only if for any coclosed submodule X of M, there exist $M_1' \leq M_1$ and $M_2' \leq M_2$ such that $M = X \oplus M_1' \oplus M_2'$.

1. Introduction

Throughout this paper all rings will have an identity and all modules will be unital left R-modules. $N \leq M(N|M)$ will mean N is a submodule(a direct summand) of the module M. For $M = \oplus_{i \in I} M_i$ and $K \subseteq I$, $M(K) = \oplus_{i \in K} M_i$.

A module is extending (or satisfies (C_1)) if every submodule is essential in a direct summand. Dually, a module M is called a lifting module(or satisfies (D_1)), if for any submodule N of M, there exists a direct summand K of M such that $K \leq N$ and $N/K \ll M/K$, equivalently, for any submodule N of M there exist submodules K, K' of M such that $M = K \oplus K', K \leq N$ and $N \cap K' \ll K'$. Lifting modules generalize discrete and quasi-discrete ones; they have been studied extensively(see, for examples, [1], [2], [6], [8], [9]) but many questions remain unresolved.

An open problem is to find sensible necessary and sufficient conditions for the direct sum of lifting modules to be lifting. If M_1 and M_2 are relatively projective, quasi-projective and (D_1)-modules then $M = M_1 \oplus M_2$ is a (D_1)-module([9, Theorem 9]). Let $M = \oplus_{i=1}^{n} M_i$ be a finite direct sum of relatively projective modules M_i. Then M is lifting if and only if M is an amply supplemented and M_i is lifting for all $1 \leq i \leq n$([1, Corollary 2.9]). However, it is not a sufficient condition for a finite direct sum of lifting modules to be a lifting module. Let p be a prime integer and M denote the $\mathbb{Z}-$module, $(\mathbb{Z}/p\mathbb{Z}) \oplus (\mathbb{Z}/p^2\mathbb{Z})$. Then M is a lifting module and $\mathbb{Z}/p\mathbb{Z}$ is not $\mathbb{Z}/p^2\mathbb{Z}$-projective(see [9, Example 4]).

In this paper we consider when the direct sum of two lifting modules is lifting. In [3] the authors claim that for any closed submodule X of $M = M_1 \oplus M_2$, M decomposes as $M = X \oplus M_1' \oplus M_2'$ with $M_i' \leq M_i$, if and only if M_i has (C_1) and

Received November 19, 2003, and, in revised form, October 2, 2004.
2000 Mathematics Subject Classification: 16D10, 16D99.
Key words and phrases: lifting module, M-*cojective module, M-cojective module.

is M_j-ojective for $i \neq j$. Dually, we prove that if $M = M_1 \oplus M_2$ be an amply supplemented R-module. Then it is proved that M_i has (D_1) and is M_j-*cojective for $i \neq j$ if and only if for any coclosed submodule X of M, there exist $M_1' \leq M_1$ and $M_2' \leq M_2$ such that $M = X \oplus M_1' \oplus M_2'$.

2. Preliminaries

Let M be a module and $S \leq M$. S is called small in M(denoted by $S \ll M$) if for any $T \leq M, S + T = M$ implies $T = M$. For $N, L \leq M, N$ is a supplement of L in M if $N + L = M$ with $N \cap L \ll N$. Following [7], a module M is called supplemented if every submodule of M has a supplement in M. On the other hand, the module M is amply supplemented if, for any submodules A, B of M with $M = A + B$ there exists a supplement P of A in M such that $P \leq B$. Following [10], the module M is called a weakly supplemented module if for each submodule A of M there exists a submodule B of M such that $M = A + B$ and $A \cap B \ll M$.

Let M be a module and $B \leq A \leq M$. If $A/B \ll M/B$, then B is called a coessential submodule of A in M. A submodule A of M is called coclosed if A has no proper coessential submodule. Also, we will call B an coclosure(or an s-closure) of A in M, if B is a coessential submodule of A and B is coclosed in M[1].

Let M be a module. Then by [8, Proposition 4.8], M is lifting if and only if M is amply supplemented and every supplement submodule of M is a direct summand.

We list a few lemmas for later use.

Lemma 2.1. *Let M be a module and $N \leq M$. Consider the following conditions*:

(1) *N is a supplement submodule of M;*

(2) *N is coclosed in M;*

(3) *For all $X \leq N$, $X \ll M$ implies $X \ll N$.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) *hold. If M is a weakly supplemented module then* (3) $\Rightarrow$ (1) *holds.*

Proof. [1, Lemma 1.1]. □

Lemma 2.2. *Let $M = M_1 \oplus M_2$ and N, $L \leq M_1$. If N is a supplement of L in M_1, then $N \oplus M_2$ is a supplement of L in M.*

Proof. Let N be a supplement of L in M_1. Then $M_1 = N + L$ and $N \cap L \ll N$. It is easy to see that $M = (N \oplus M_2) + L$ and $(N \oplus M_2) \cap L = N \cap L \ll N$. Thus $N \oplus M_2$ is a supplement of L in M. □

Lemma 2.3. *Let $K \leq L \leq M$. If K is coclosed in M, then K is coclosed in L and the converse is true if L is coclosed in M.*

Proof. [11, Lemma 2.6]. □

Definition 2.4 ([3]). Let $M = \oplus_{i \in I} M_i$ be a direct sum of submodules M_i. Then we say that the decomposition $M = \oplus_{i\in I} M_i$ is exchangeable if for any direct summand N of M we have $M = (\oplus_{i\in I} M_i') \oplus N$ with $M_i' \leq M_i$.

3. *Cojective modules

Let A and B be modules. Following [5], B is called A-ojective if any diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & A \\ {\scriptstyle\varphi}\downarrow & & \\ B & & \end{array}$$

can be embedded in a diagram

$$\begin{array}{cccccc} X & \xrightarrow{i} & A = A_1 & \oplus & A_2 \\ {\scriptstyle\varphi}\downarrow & & {\scriptstyle\varphi_1}\downarrow & & \uparrow{\scriptstyle\varphi_2} \\ B & = & B_1 & \oplus & B_2 \end{array}$$

such that φ_2 is a monomorphism and for $x = a_1 + a_2$ and $\varphi(x) = b_1 + b_2$ one has $b_1 = \varphi_1(a_1)$ and $a_2 = \varphi_2(b_2)$.

Mohamed and Müller named it ojectivity in honor of Oshiro and they characterize it in [3] as follows:

Theorem 3.1. *Let $M = A \oplus B$. Then the following are equivalent*:

(1) *B is A-ojective;*

(2) *For any complemnt C of B, M decomposes as $M = C \oplus A' \oplus B'$ with $A' \leq A$ and $B' \oplus B$.*

According to this characterization, Mohamed and Müller give the following dual definition in [4, Definition 2.3].

Definition 3.2. Let A, B be left R-modules. We say B is A-*cojective if for any supplement C of A in $A \oplus B$, $A \oplus B$ decomposes as $A \oplus B = C \oplus A' \oplus B'$ with $A' \leq A$ and $B' \leq B$. If B is A-*cojective and A is B-*cojective, we say that A and B are mutually *cojective.

As supplements need not exist, *cojectivity is not the precise dual of ojectivity. The precise dual of ojectivity as follows (See, [4]):

Let A and B be modules. A is B-cojective if any diagram

$$\begin{array}{ccc} A & & \\ {\scriptstyle\varphi}\downarrow & & \\ X & \xleftarrow{\pi} & B \end{array}$$

can be embedded in a diagram

$$\begin{array}{ccccc} A & = & A_1 & \oplus & A_2 \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi_1} & & \uparrow{\scriptstyle \varphi_2} \\ X & \overset{\pi}{\twoheadleftarrow} & B = B_1 & \oplus & B_2 \end{array}$$

such that φ_2 is onto, $\pi\varphi_1 = \varphi|A_1$ and $\varphi\varphi_2 = \pi|B_2$.

In [4, Theorem 2.8], Mohamed and Müller give the following characterization of cojectivity:

Let $M = A \oplus B$. Then A is B-cojective if and only if whenever $M = N + B$, we have $M = N' \oplus A' \oplus B' = N' + B$ with $N' \leq N, A' \leq A$ and $B' \leq B$. Therefore if A is B-cojective, then A is B-*cojective (See, [4, Proposition 2.9]).

Proposition 3.3. *Let $M = M_1 \oplus M_2$. If M_1 is M_2-projective, then M_1 is M_2-*cojective.*

Proof. Let N be a supplement of M_2 in M. Then $M = N + M_2$ and $N \cap M_2 \ll N$. Since M_1 is M_2-projective, by [1, Lemma 2.5], there exists a submodule N' of N such that $M = N' \oplus M_2$. Clearly $N = N' \oplus (N \cap M_2)$. Hence $N = N'$ and $M = N \oplus M_2$. Thus M_1 is M_2-*cojective. □

Let M be a module. Consider the following condition:

(D_3) For every direct summands K, L of M with $M = K + L, K \cap L$ is a direct summand of M.

Following [8], if the module M is lifting and has (D_3) then it is called a quasi-discrete module.

Let M_1 and M_2 be modules. Following [1], the module M_1 is small M_2-projective if every homomorphism $f : M_1 \to M_2/A$, where A is a submodule of M_2 and $Imf \ll M_2/A$, can be lifted to a homomorphism $\varphi : M_1 \to M_2$.

Proposition 3.4. *Let $M = M_1 \oplus M_2$ be an amply supplemented module with (D_3). If M_1 is M_2-*cojective, then M_1 is small M_2-projective.*

Proof. Let N be a submodule of M such that $(N + M_1)/N \ll M/N$. Then $M = N + M_2$. Since M is amply supplemented there exists a submodule N' of M such that $N' \leq N, M = N' + M_2$ and $N' \cap M_2 \ll N'$, that is, N' is a supplement of M_2 in M. Since M_1 is M_2-*cojective, $M = N' \oplus M_1' \oplus M_2'$ with $M_i' \leq M_i$. By (D_3), $N' \cap M_2$ is a direct summand of M, and so $M = N' \oplus M_2$. By [1, Lemma 2.4], M_1 is small M_2-projective. □

Proposition 3.5. *Let $A_1|A$ and $B_1|B$. If B is A-*cojective, then B_1 is A_1-*cojective.*

Proof. Write $M = A \oplus B,\ A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$.

(1) First we prove that B_1 is A-*cojective. Write $N = A \oplus B_1$, and let X be a supplement of A in N. By Lemma 2.2, $X \oplus B_2$ is a supplement of A in M. As B is A-*cojective, $M = X \oplus B_2 \oplus A' \oplus B'$ with $A' \leq A$ and $B' \leq B$. Hence

$N = X \oplus A' \oplus (N \cap (B_2 \oplus B'))$. The result now follows if we show $N \cap (B_2 \oplus B')) \leq B_1$. Indeed, $N \cap (B_2 \oplus B') = (A \oplus B_1) \cap (B_2 \oplus B') \leq (A \oplus B_1) \cap B = B_1$.

(2) Next we prove that B is A_1-*cojective. Write $L = A_1 \oplus B$, and let Y be a supplement of A_1 in L. By Lemma 2.1, it is easy to see that A_1 is a supplement of Y in L. Then A is a supplement of Y in M by Lemma 2.2. Again by Lemma 2.1, Y is a supplement of A in M. As B is A-*cojective, $M = Y \oplus A^{''} \oplus B^{''}$ with $A^{''} \leq A$ and $B^{''} \leq B$. Hence $L = Y \oplus B^{''} \oplus (L \cap A^{''})$. It remains to show that $L \cap A^{''} \leq A_1$. Let $a^{''} \in L \cap A^{''}$. Then $a^{''} = a_1 + b$ with $a_1 \in A_1$ and $b \in B$. Hence $b = a^{''} - a_1 \in A \cap B = 0$, and so $a^{''} = a_1 \in A_1$.

(3) Our proposition follows from (1) and (2). □

Lemma 3.6. *Let $M = A \oplus B$ where A is B-*cojective and B has (D_1). If X is a coclosed submodule of M with $M = X + B$, then M decomposes as $M = X \oplus A' \oplus B'$ with $A' \leq A$ and $B' \leq B$.*

Proof. Let $M = X + B$. Since B has (D_1), there exists a direct summand B_1 of B such that $B = B_1 \oplus B_2$ and $B_1 \leq X \cap B, X \cap B_2 \ll B_2$. Now $M = A \oplus B_1 \oplus B_2$. Write $N = A \oplus B_2$. Then $X = B_1 \oplus X_1$, where $X_1 = X \cap N$. Hence $M = X + B = X_1 + B_1 + B_2$, and so $N = X_1 + B_2$. Clearly $X_1 \cap B_2 = X \cap B_2 \ll B_2$. Then B_2 is a supplement of X_1 in N. Now X_1 is a coclosed submodule of X, and X is a coclosed submodule of M. It then follows by Lemma 2.3 that X_1 is coclosed in N. It is easy to see that X_1 is a supplement of B_2 in N. Now A is B_2-*cojective, by Proposition 3.5. Now we get $N = X_1 \oplus A' \oplus B_2'$ with $A' \leq A$ and $B_2' \leq B_2$. Hence $M = N \oplus B_1 = X_1 \oplus B_1 \oplus A' \oplus B_2' = X \oplus A' \oplus B_2'$. □

We prove here the dual of the result of [3, Theorem 10].

Theorem 3.7. *Let $M = M_1 \oplus M_2$ be an amply supplemented module. Then M_i has (D_1) and is M_j-*cojective for $i \neq j$ if and only if for any coclosed submodule X of M, we have $M = X \oplus M_1' \oplus M_2'$ with $M_i' \leq M_i$.*

Proof. The sufficiency follows from Definition 3.2 and from the fact that (D_1) is inherited by summands.

Conversely, suppose that M_i has (D_1) and is M_j-*cojective for $i \neq j$. Let X be a coclosed submodule of M. It is easy to see that M/X is amply supplemented, and so $(X + M_1)/X$ has a coclosure in M/X by [1, Proposition 1.5], that is, there exists a coclosed submodule N/X of M/X such that $N/X \leq (X + M_1)/X$ and $(X + M_1)/N \ll M/N$. By [1, Lemma 1.4], N is coclosed in M. As $M = N + M_2$, we get by Lemma 3.6 that $M = N \oplus M_1' \oplus M_2'$ with $M_i' \leq M_i$. Write $N_1 = M_1' \oplus M_2'$. Note that $X = N \cap (X + N_1)$ and $M = N + (X + N_1)$, and so $M/X = N/X \oplus (X + N_1)/X$. Therefore $(X + N_1)/X$ is coclosed in M/X. Again by [1, Lemma 1.4], $X + N_1$ is coclosed in M. Moreover, $M = (X + N_1) + M_1$. Again by Lemma 3.6, $M = (X + N_1) \oplus M_1^{''} \oplus M_2^{''}$ with $M_i^{''} \leq M_i$. Write $N_2 = M_1^{''} \oplus M_2^{''}$. Hence $N_1 = (X + N_1) \cap (N_1 + N_2)$ and $N \cap (X + N_1) \cap (N_1 + N_2) = X \cap (N_1 + N_2) = 0$. So $M = X \oplus (N_1 + N_2) = X \oplus (M_1' + M_2' + M_1^{''} + M_2^{''}) = X \oplus M_1^* \oplus M_2^*$, where $M_1^* = M_1' + M_1^{''}$ and $M_2^* = M_2' + M_2^{''}$. □

In view of Definition 2.4, Theorem 3.7 may be reformulated as follows:

Theorem 3.8. *Let $M = M_1 \oplus M_2$ be an amply supplemented module. Then M has (D_1) and the decomposition is exchangeable if and only if, for $i = 1,\ 2$, M_i has (D_1) and is M_j-*cojective for $i \neq j$.*

By analogy with the proof of [3, Theorem 11], we have

Theorem 3.9. *Let $n \geq 2$ be an integer and let $M = \oplus_{i=1}^{n} M_i$ be an amply supplemented module. Then the following are equivalent:*

(1) *M has (D_1) and the decomposition is exchangeable;*

(2) *The M_i have (D_1), and $M_1 \oplus \cdots \oplus M_{i-1}$ and M_i are mutually *cojective, for $2 \leq i \leq n$;*

(3) *The M_i have (D_1), and $M(I)$ is $M(J)$-*cojective for any disjoint nonempty subset I and J of $\{1, 2, \cdots n\}$.*

4. Semi-discrete modules

Definition 4.1 ([3]). Let μ be a cardinal number. A module M is said to have the μ -internal exchange property if any decomposition $M = \oplus_{i \in I} M_i$ with $|I| \leq \mu$, is exchangeable.

Definition 4.2. We call a module M semi-discrete if M has (D_1) and the 2-internal exchange property.

Thus, if M is an amply supplemented module, then M is semi-discrete if and only if for any coclosed submodule C of M and any decomposition $M = A \oplus B$, we have $M = C \oplus A' \oplus B'$ with $A' \leq A$ and $B' \leq B$.

It is well known that a discrete module has the exchange (hence the internal exchange) property, and so is semi-discrete module. However, it is not known whether a quasi-discrete module has the internal exchange property. Let M be a quasi-discrete module. In [8, Corollary 4.19], it is proved that if every hollow summand of M has a local endomorphism ring, then M has exchange property, and so these modules are semi-discrete.

In [8, Lemma 4.23], it is noticed that if M is a quasi-discrete module, then for every decomposition $M = A \oplus B$, A and B are mutually projective. The following is analogue for semi-discrete modules.

Proposition 4.3. *Let M be any module. M is semi-discrete if and only if M has (D_1) and for every decomposition $M = A \oplus B$, A and B are mutually *cojective.*

Proof. The result follows from Theorem 3.7 and from the fact that (D_1) is inherited by summands. □

A module $M = M_1 \oplus \cdots \oplus M_n$ is quasi-discrete if and only if M_i is quasi-discrete and M_j-projective for all $i \neq j$([1, Theorem 2.13]). For $n = 2$ the following is an analogous result for semi-discrete modules.

Theorem 4.4. *Let $M = M_1 \oplus M_2$ be an amply supplemented module. Then M is semi-discrete if and only if M_i is semi-discrete and M_j-*cojective for $i \neq j$.*

Proof. The necessity follows from by [3, Proposition 15] and Proposition 4.3. The sufficiency is analogous with the proof of [3, Theorem 19]. □

Corollary 4.5. *Let $n \geq 2$ be an integer and let $M = \oplus_{i=1}^{n} M_i$ be an amply supplemented module. Then the following are equivalent*:

(1) *M is semi-discrete;*

(2) *The M_i are semi-discrete, and $M_1 \oplus \cdots \oplus M_{i-1}$ and M_i are mutually *cojective, for $2 \leq i \leq n$;*

(3) *Every M_i is semi-discrete, and $M(I)$ is $M(J)$- *cojective for any disjoint nonempty subset I and J of $\{1, 2, \cdots n\}$.*

Proof. Theorem 4.4 and induction. □

Proposition 4.6. *Let M be a quasi-projective module. Then the following are equivalent*:

(1) *M is supplemented;*

(2) *M is amply supplemented;*

(3) *M is lifting;*

(4) *M is quasi-discrete;*

(5) *M is discrete;*

(6) *M is semi-discrete.*

Proof. (5) ⇒ (6) ⇒ (3) are clear. Now the result follows by [12, Proposition 2.2]. □

Theorem 4.7. *For any ring R the following are equivalent*:

(1) *R is a left perfect ring;*

(2) *Every quasi-projective left R-module is semi-discrete.*

Proof. This is clear by Proposition 4.6 and [12, Theorem 2.6]. □

Theorem 4.8. *For any ring R the following are equivalent*:

(1) *R is a left perfect ring;*

(2) *Every projective left R-module is semi-discrete.*

Proof. This is clear by Proposition 4.6 and [12, Theorem 2.7]. □

Corollary 4.9. *Let M be a quasi-projective module such that $M = \oplus_{i=1}^{n} M_i$ is a finite direct sum of submodules M_i, $(1 \leq i \leq n)$. Then M is semi-discrete if and only if M_i, $(1 \leq i \leq n)$, is semi-discrete.*

Proof. Necessity is clear. Conversely, suppose that each M_i is semi-discrete. Then, by [12, Proposition 2.8], M is lifting. Hence M is semi-discrete by Proposition 4.6. □

Acknowledgements. I would like to express my gratefulness to Professor Liu Zhongkui and the referee for their invaluable suggestions and comments, which certainly improved the presentation of the paper.

References

[1] D. Keskin, *On lifting modules*, Comm. Algebra, **28(7)**(2000), 3427-3440.

[2] D. Keskin, *Diserete and quasi-discrete modules*, Comm. Algebra, **30(11)**(2002), 5273-5282.

[3] S. H. Mohamed and B. J. Müller, *Ojective modules*, Comm. Algebra, **30(4)**(2002), 1817-1827.

[4] S. H. Mohamed and B. J. Müller, *Cojective modules*, prepint.

[5] H. Hanada, J. Kado and K. Oshiro, On direct sums of extending modules and internal exchange property, Proc. 2^{nd} Japan-China International Symposium on Ring Theory(1995), Okayama(1996), 41-44.

[6] Y. Talebi and N. Vanaja, *The torsion theory cogenerated by M-small modules*, Comm. Algebra, **30(3)**(2002), 1449-1460.

[7] R. Wisbauer, Foundations of Moudule and Ring Theory, Gordon and Breach: Philadelphia, 1991.

[8] S. H. Mohamed and B. J. Müller, Continuous and Discrete Modules, London Math. Soc.; LNS147, Cambridge Univ. Press: Cambridge, 1990.

[9] D. Keskin, *Finite direct sums of (D1)-modules*, Tr. J. of Mathematics, **22**(1998), 85-91.

[10] H. Zöxchinger, *Minimax-Moduln*, J. Algebra, **102**(1986), 1-32.

[11] L. Ganesan and N. Vanaja, *Modules for which every submodule has a unique coclosure*, Comm. Algebra, **30(5)**(2002), 2355-2377.

[12] D. Keskin, *Characterizations of right perfect rings by ⊕-supplemented modules*, Contemp. Math., **259**(2000), 313-318.