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Topological Imitations and Reni-Mecchia-Zimmermann's Conjecture

Dedicating this paper to Professor Yukio Matsumoto for his 60th birthday

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Abstract. M. Reni has shown that there are at most nine mutually inequivalent knots in the 3-sphere whose 2-fold branched covering spaces are mutually homeomorphic, hyperbolic 3-manifolds. By observing that the Z-homology sphere version of M. Reni's result still holds, M. Mecchia and B. Zimmermann showed that there are exactly nine mutually inequivalent, knots in Z-homology 3-spheres whose 2-fold branched covering spaces are mutually homeomorphic, hyperbolic 3-manifolds, and conjectured that there exist exactly nine mutually inequivalent, knots in the true 3-sphere whose 2-fold branched covering spaces are mutually homeomorphic, hyperbolic 3-manifolds. Their proof used an argument of AID imitations published in 1992. The main result of this paper is to solve their conjecture affirmatively by combining their argument with a theory of strongly AID imitations published in 1997.

1. Reni-Mecchia-Zimmermann's conjecture

Let $M(K)$ be the double branched covering space branched along a knot K in the 3-sphere S^3. By an affirmative solution of Thurston's 3-orbifold conjecture (cf. [1]), the following finiteness result became a folk result (see M. Reni and B. Zimmerman [16] for a general survey as well as M. Mecchia and B. Zimmermann [12]):

Finiteness Theorem. *Given any knot K in S^3, there are finitely many knots K' in S^3 with a homeomorphism $M(K') \cong M(K)$.*

Then we may have a question asking how many (unoriented) knots K' in S^3 with a homeomorphism $M(K') \cong M(K)$ exist for any given knot K in S^3. Here

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are some known examples.

Example 1.1. Let K_i $(i = 1, 2, \cdots, n)$ be mutually inequivalent, non-invertible, prime oriented knots in S^3, and $K = K_1 \# K_2 \# \cdots \# K_n$. Then there are 2^{n-1} mutually inequivalent, unoriented knots $K' = K_1 \# \pm K_2 \# \cdots \# \pm K_n$ such that $M(K')$ is homeomorphic to $M(K)$.

Example 1.1 is proved by the uniqueness of prime decompositions of knots. We note that the 3-manifold $M(K)$ has incompressible 2-spheres in this example.

Example 1.2. Let $K = P(p_1, p_2, \cdots, p_n)$ be a pretzel knot (see [8] for example) such that $n > 3$ is odd and p_i $(i = 1, 2, \cdots, n)$ are distinct odd integers > 1. Then there are $\frac{(n-1)!}{2}$ mutually inequivalent, unoriented knots $K' = P(p_1, p'_2, \cdots, p'_n)$ with $(p'_2, \cdots, p'_n)$ the permutations of $(p_2, \cdots, p_n)$ such that $M(K')$ is homeomorphic to $M(K)$.

In Example 1.2, we note that $M(K)$ is a Seifert manifold with incompressible tori, although it has no incompressible 2-sphere. Example 1.2 is proved by classification of pretzel knots and generalized into the Montesinos knots. Examples 1.1 and 1.2 are also regarded as a consequence of the property of a Conway mutation that any two Conway-mutant knots in S^3 have homeomorphic 2-fold branched covering spaces, observed by O. Ja. Viro [18] (see [8]). Using this property, we can show the following example from a technique of topological imitations in [5](see also [6]):

Example 1.3. For every integer $n \geqq 1$, we have a hyperbolic knot K in S^3 such that $M(K)$ is a torus sum of n hyperbolic pieces and there are exactly 3^{n-1} mutually inequivalent, unoriented knots K' with a homeomorphism $M(K') \cong M(K)$.

An idea of showing this example for $n \geqq 2$ is to use the fact that for any given 2-string tangle T in a 3-ball B^3, there are infinitely many tangle imitations (B^3, T^*) of (B^3, T) whose double branched covering spaces are mutually non-homeomorphic, hyperbolic 3-manifolds with isometry group Z_2. We next observe the case that $M(K)$ has no incompressible 2-sphere nor incompressible torus. By using an affirmative solution of Thurston's 3-orbifold conjecture, $M(K)$ is a Seifert fiber space with at most 3 exceptional fibers or a hyperbolic 3-manifold. The following proposition is also known (cf. [16]):

Proposition 1.4. *Let $M(K)$ be a Seifert fiber space with at most 3 exceptional fibers.*

(1) *Assume that $\pi_1(M(K))$ is a finite group. Then K is uniquely determined.*

(2) *Assume that $\pi_1(M(K))$ is an infinite group. Then $M(K)$ has 3 exceptional fibers and there are at most two knots K' with a homeomorphism $M(K') \cong M(K)$.*

More concretely, we can always have a hyperbolic Montesinos knot with exactly three rational tangles as K'. If there is another knot as K', then $M(K)$ must be a

Brieskorn manifold of a type $(2,p,q)$ *for coprime positive integers* p,q *with* $\frac{1}{p}+\frac{1}{q}<\frac{1}{2}$ *and in this case we have the torus knot* $T(p,q)$ *of the type* (p,q) *as* K'.

Proof. (1) is obtained from F. Waldhausen [19] if $M(K)=S^3$, C. D. Hodgson and J. H. Rubinstein [2] if $M(K)$ is a lens space, and P. Kim [10] and J. M. Montesinos [13] if $M(K)$ is in the other Seifert manifolds with finite fundamental groups.

(2) is proved as follows: Since $\pi_1(M(K))$ is infinite, the Seifert structure on $M(K)$ is unique and the non-trivial covering involution t on $M(K)$ preserves fiber-circles of the Seifert structure setwise. If t reverses the orientation of a fiber-circle, then we have a Montesinos knot K_M with exactly three rational tangles as K' (cf. [13]). If t preserves the orientation of a fiber-circle, then we have a torus knot as K', for K' must be a fiber-circle of the t-orbit fibered space which is S^3, meaning that K' is a torus knot $T(p,q)$ of some type (p,q). Then $M(K)$ must be a Brieskorn manifold $\Sigma(2,p,q)$ (see W. D. Neumann and F. Raymond [14] for its Seifert invariant). The condition that $\pi_1(M(K))$ is infinite is equivalent to the condition that $\frac{1}{p}+\frac{1}{q}\leqq\frac{1}{2}$, which is equivalent to $\frac{1}{p}+\frac{1}{q}<\frac{1}{2}$ since p and q are coprime positive integers. To see that K_M is a hyperbolic knot, we first note that the non-trivial covering involution t on $M(K)=M(K_M)$ sends a generator g of the center $z(\pi_1(M(K)))$ (which is an infinite cyclic group because $\pi_1(M(K))$ is infinite) to the inverse g^{-1} of g. By the equivariant torus theorem, K_M is a simple knot meaning a hyperbolic or torus knot. If K_M is a torus knot, then t also sends g to g, so that $g^2=1$ in $z(\pi_1(M(K)))$, which is a contradiction. Hence K_M must be hyperbolic. □

When $M(K)$ is a hyperbolic 3-manifold, M. Reni proved the following result in [15]:

Theorem (M. Reni). *For a hyperbolic 3-manifold* $M(K)$*, there are at most nine mutually inequivalent, unoriented knots* K' *in* S^3 *with a homeomorphism* $M(K')\cong M(K)$.

In Reni's theorem, one comes to a question asking whether nine mutually inequivalent, unoriented knots K' exist. Observing that Reni's theorem still holds for knots in Z-homology 3-spheres, M. Mecchia and B. Zimmermann solved a homological version of this problem in [12]:

Z-Homology Version Theorem. There are nine mutually inequivalent, unoriented knots K_i in Z-homology 3-spheres S_i $(i=1,2,\cdots,9)$ whose 2-fold branched covering spaces are mutually homeomorphic, hyperbolic 3-manifolds.

Their proof uses an argument of AID imitations in [5]. By a similar method, they also showed (in [12]) that there are six mutually inequivalent, unoriented knots in the true 3-sphere S^3 whose 2-fold branched covering spaces are mutually homeomorphic, hyperbolic 3-manifolds, and conjectured that there exist nine mutually inequivalent, unoriented knots in S^3 whose 2-fold branched covering spaces are mutually homeomorphic, hyperbolic 3-manifolds. The purpose of this paper is to show that their conjecture is true by combining their argument with an argument of strongly AID imitations in [7]. The result is stated as follows:

Theorem 1.5. *There are nine mutually inequivalent, unoriented knots K_i ($i = 1, 2, \cdots, 9$) in S^3 whose 2-fold branched covering spaces are mutually homeomorphic, hyperbolic 3-manifolds.*

A disconnected link version of Reni's theorem and Theorem 1.5 has been already settled by M. Mecchia, M. Reni and B. Zimmermann. Some points of their results are stated as follows:

- M. Mecchia and M. Reni showed in [11] the 2-component link version of Reni's theorem, meaning that there are at most 9 mutually inequivalent, unoriented 2-component links in S^3 whose 2-fold covering spaces are mutually homeomorphic, hyperbolic 3-manifolds. Then M. Mecchia and B. Zimmermann showed in [12] (by an argument of AID imitations in [5]) that there are 9 mutually inequivalent, unoriented 2-component links in S^3 whose 2-fold branched covering spaces are mutually homeomorphic, hyperbolic 3-manifolds.
- M. Mecchia and B. Zimmermann showed in [12] that for every $r (\geqq 3)$, there are at most three mutually inequivalent, unoriented r-component links in S^3 whose 2-fold branched covering spaces are mutually homeomorphic, hyperbolic 3-manifolds, and (by an argument of AID imitations in [5]) that for every $r (\geqq 3)$, there are three mutually inequivalent, unoriented r-component links in S^3 whose 2-fold branched covering spaces are mutually homeomorphic, hyperbolic 3-manifolds.
- M. Mecchia and B. Zimmermann observed in [12] that any link version of Reni's theorem for links with non-fixed numbers of components splits into the link versions for links with fixed numbers of components stated above, because $H_1(M(L); Z_2) \cong Z_2^{r-1}$ for every r-component link L by a result of M. Sakuma [17], where $M(L)$ denotes the 2-fold branched covering space along L.

In §2, we explain AID and strongly AID imitations. In §3, we give the proof of Main Theorem (Theorem 1.5) after an explanation of M. Mecchia-B. Zimmermann's proof of the Z-homology version theorem.

2. AID and strongly AID imitations

Let M be a closed connected oriented 3-manifold. Let Γ be a graph (without degree one vertices) in M. Possibly, $\Gamma = \emptyset$. Since we treat graphs as topological objects, we do not regard any degree-two vertex as a vertex. Thus, an edge e of Γ is a loop without vertices or a compact arc joining one or two vertices of degree $\geqq 3$ whose interior edge int$e = e - \partial e$ does not contain any vertex of degree ($\geqq 3$). Let $I = [-1, 1]$. The concept of topological imitations comes from non-trivial reflections in the 4-dimensional object $(M, \Gamma) \times I = (M \times I, \Gamma \times I)$.

Definition 2.1. A *reflection in* $(M, \Gamma) \times I$ is a smooth involution α on $(M, \Gamma) \times I$ such that

(1) $\alpha((M,\Gamma)\times 1)=(M,\Gamma)\times(-1)$, and

(2) the fixed point set $\mathrm{Fix}(\alpha,(M,\Gamma)\times I)$ is a pair of a 3-manifold and a graph in it.

By using this reflection, we define an imitation as follows:

Definition 2.2. An *imitation* $q:(M^*,\Gamma^*)\to(M,\Gamma)$ is the composite:

$$(M^*,\Gamma^*)\xrightarrow{\phi}\mathrm{Fix}(\alpha,(M,\Gamma)\times I)\subset(M,\Gamma)\times I\xrightarrow{p}(M,\Gamma),$$

where ϕ is a homeomorphism and p is the projection to the first factor.

See [3] for general properties of imitations. We need some notions of reflections as follows:

Definition 2.3. A reflection α in $(M,\Gamma)\times I$ is:

(1) *standard* if $\alpha(x,t)=(x,-t)$ for all $(x,t)\in M\times I$,

(2) *normal* if $\alpha(x,t)=(x,-t)$ for all $(x,t)\in\partial(M\times I)\cup N\times I$, where N is a tubular neighborhood of Γ in M,

(3) *isotopically standard* if $f^{-1}\alpha f$ is standard for a diffeomorphism f of $M\times I$ with $f|_{\partial(M\times I)\cup N\times I}=1$ such that $[f]=1\in\pi_0 Diff(M\times I,\mathrm{rel}\partial(M\times I)\cup N\times I)$,

(4) *isotopically almost standard* if α defines an isotopically standard reflection in $(M,\Gamma-\mathrm{int}e)\times I$ for any edge e of Γ.

The imitation $q:(M^*,\Gamma^*)\to(M,\Gamma)$ is *normal* or *AID* (= *almost identical*), respectively, if α is normal or isotopically almost standard. By definition, we have AID imitation $\Rightarrow$ normal imitation. Further, if q is normal, then q defines a homeomorphism $\Gamma^*\to\Gamma$. If q is AID and $\Gamma\neq\emptyset$, then q defines a homeomorphism $M^*\to M$.

We assume that a finite group G acts on M faithfully and orientation-preservingly. For a subgroup H of G, let $\mathrm{F}(H,M)$ be the set of fixed points in M of every element of H (except the identity $1\in H$). We note that the set $\mathrm{F}(H,M)$ is a graph in M unless it is empty. We denote the H-orbit 3-manifold of M by M_H and the H-orbit graph of $\mathrm{F}(G,M)$ by $\mathrm{F}(G,M)_H$. Let $p^H_G:M_H\to M_G$ be the canonical (branched or unbranched) covering projection. We say that an edge e of $\mathrm{F}(G,M)_H$ is *unbranched* if the number of connected components of $(p^H_G)^{-1}(p^H_G(\mathrm{int}e))$ is equal to the index $(G:H)$ of the subgroup H. The *unbranched subgraph* of $\mathrm{F}(G,M)_H$ is the subgraph $\mathrm{F}(G,M)^u_H\subset\mathrm{F}(G,M)_H$ consisting of all the unbranched edges e of $\mathrm{F}(G,M)_H$. By definition, we have $\mathrm{F}(G,M)^u_G=\mathrm{F}(G,M)_G$. We denote the H-orbit normal imitation of a G-equivareiant normal imitation $q:M^*\to M$ by $q_H:M^*_H\to M_H$. We note that the normal imitation q_H sends the unbranched subgraph $\mathrm{F}(G,M^*)^u_H$ onto the unbranched subgraph $\mathrm{F}(G,M)^u_H$ homeomorphically.

The existence theorem of an AID imitation is given as follows (see [4], [5] for a more general statement and the proof):

AID Imitation Theorem. *Assume that* $\mathrm{F}(G,M) \neq \emptyset$. *Then there is a G-equivariant normal imitation* $q : M^* \to M$ *such that*

(1) *the G-orbit normal imitation map* $q_G : (M^*_G, \mathrm{F}(G,M^*)_G) \to (M_G, \mathrm{F}(G,M)_G)$ *is an AID imitation, and*

(2) *the 3-manifold* M^* *is a hyperbolic 3-manifold with isometry group* $\mathrm{Isom}M^* = G$.

The existence theorem of a strongly AID imitation is given as follows (see [7] for a more general statement and the proof):

Strongly AID Imitation Theorem. *Assume that* $\mathrm{F}(G,M) \neq \emptyset$. *Then there is a G-equivariant normal imitation* $q : M^* \to M$ *such that*

(1) *the H-orbit normal imitation map* $q_H : (M^*_H, \mathrm{F}(G,M^*)^u_H) \to (M_H, \mathrm{F}(G,M)^u_H)$ *is an AID imitation for every normal subgroup* $H \subset G$ *with* $\mathrm{F}(G,M)^u_H \neq \emptyset$, *and*

(2) *the 3-manifold* M^* *is a hyperbolic 3-manifold with isometry group* $\mathrm{Isom}M^* = G$.

A result given in [9] contains the result that the strongly AID imitation theorem still holds for every (not necessarily normal) subgroup $H \subset G$.

3. Proof of Reni-Mecchia-Zimmermann's conjecture

Let $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group. M. Mecchia and B. Zimmermann consider an order 32 subgroup $G = Q \times Q/\{(1,1),(-1,-1)\}$ of $SO(4)$ acting orthogonally on S^3 with $S^3/G = S^3$. For $g_i \in Q$ $(i = 1,2)$, the element of G represented by $(g_1, g_2) \in Q \times Q$ is denoted by $g_1 \cdot g_2$. Then the center $z(G)$ of G is the set $\{1 \cdot 1, (-1) \cdot 1 = 1 \cdot (-1)\} (\cong Z_2)$. When we denote by $g_1 * g_2$ the element of $G/z(G)$ represented by $g_1 \cdot g_2 \in G$, the quotient group $G/z(G) (\cong Z_2^4)$ consists of the 16 elements $1*1$, $x*1$, $1*y$, $x*y \in G/z(G)$ $(x, y = i, j, k)$. To understand the action of G on S^3, the Kuratowsky graph in S^3 is introduced by them. Topologically, it is stated as follows: Let C be the unit circle in the plane $R^2 \times 0 \subset R^3$. Let a_i and $-a_i$ $(i = 1,2,3)$ be three mutually distinct pairs of antipodal points in C. Let e_1 be a proper trivial arc in the upper half space $R^3_+ = \{(x_1,x_2,x_3) \in R^3 | x_3 \geqq 0\}$ with $\partial e_1 = \{a_1, -a_1\}$. Let e_2 be the interval $[a_2, -a_2] \subset R^2 \times 0$. Let e_3 be a proper trivial arc in the lower half space $R^3_- = \{(x_1,x_2,x_3) \in R^3 | x_3 \leqq 0\}$ with $\partial e_3 = \{a_3, -a_3\}$. The graph $\Lambda = C \cup e_1 \cup e_2 \cup e_3$ in $R^3 \cup \{\infty\} = S^3$ is called the *Kuratowsky graph* in S^3. The Kuratowsky graph Λ has exactly six vertices (of degree 3) and nine edges, so that we have $H_1(\Lambda; Z_2) = Z_2^3$, and by Alexander duality, $H_1(S^3 - \Lambda; Z_2) = Z_2^4 \cong G/z(G)$. Then we see that there is a $G/z(G)$-regular branched covering $p' : P^3 \to S^3$ branched along Λ. Then the nine elements

$x*y \in G/z(G)$ $(x, y = i, j, k)$ are understood as the monodromies of the nine edges of Λ among which we have a local relation $(x_1 * y_1)(x_2 * y_2)(x_3 * y_3) = 1 * 1$ around every vertex. We color the edges of Λ by $x*y$ $(x, y = i, j, k)$ so as to satisfy this local relation around every vertex. Let $p : S^3 \to S^3$ be the composite branched covering of the $G/z(G)$-regular branched covering $p' : P^3 \to S^3$ and the double unbranched covering $p'' : S^3 \to P^3$. This branched covering is a desired G-regular branched covering branched along the Kratowsky graph Λ, giving the action of G on S^3. Let Γ be the lift of Λ under the branched covering p. Let $H_{x*y}(\cong Z_2^2)$ be the normal subgroup of G which is the preimage of the subgroup $\{1*1, x*y\}(\cong Z_2)$ of $G/z(G)$ under the natural epimorphism $G \to G/z(G)$. For the $x*y$-colored edge e of Λ, we have a G/H_{x*y}-regular branched covering $p'_1 : S^3 \to S^3$ branched along the subgraph $\Lambda - \mathrm{int}e \subset \Lambda$ such that $(p'_1)^{-1}(e) = L_{x*y}$ is a Hopf link in S^3 and the $G/z(G)$-regular branched covering $p' : P^3 \to S^3$ is the composite of the G/H_{x*y}-regular branched covering $p'_1 : S^3 \to S^3$ and the double branched covering $p''_1 : P^3 \to S^3$ branched along the Hopf link L_{x*y}. [*Note:* It suffices to check this result for a particular $x*y$, because we can easily see that for any two edges e, e' of Λ, there is an orientation-preserving homeomorphism $h : S^3 \to S^3$ with $h(\Lambda, e) = (\Lambda, e')$.] The branched covering $p : S^3 \to S^3$ is equal to the composite of the G/H_{x*y}-regular branched covering $p'_1 : S^3 \to S^3$ and the natural H_{x*y}-regular branched covering $p''_1 : S^3 \to S^3_{H_{x*y}} = S^3$ branched along L_{x*y}. Further, the branched covering $p''_1 : S^3 \to S^3$ is the composite of the double branched covering $p''_{11} : S^3 \to S^3$ branched along a trivial component O_{x*y} of L_{x*y} and the double branched covering $p''_{12} : S^3 \to S^3$ branched along the lift $O''_{x*y} = (p''_{11})^{-1}(L_{x*y} - O_{x*y})$ which is a trivial knot. We use the property that the nine elements $x \cdot y$ $(x, y = i, j, k)$ are mutually non-conjugate in G (although $x \cdot y$ and $(-x) \cdot y$ are conjugate). To prove Reni-Mecchia-Zimmermann's conjecture, we first explain M. Mecchia-B. Zimmermann's proof of the Z-homology version theorem.

3.1 M. Mecchia-B. Zimmermann's proof.

We apply the AID imitation theorem to S^3 with G-action, induced from the G-regular branched covering $S^3 \to S^3$ branched along the Kratowsky graph Λ. Then we have a G-equivariant normal imitation $q : M \to S^3$ such that M is a hyperbolic 3-manifold with $\mathrm{Isom}M = G$ and the G-orbit map $q_G : (M_G, \mathrm{F}(G, M)_G) \to (S^3_G, \mathrm{F}(G, S^3)_G) = (S^3, \Lambda)$ is an AID imitation. By a property of an imitation, we see that M is a Z-homology 3-sphere. By Mostow rigidity, the nine mutually non-conjugate elements $x \cdot y \in G$ $(x, y = i, j, k)$ are considered as non-free involutive isometries on M. By $q_{x\cdot y} : M_{x\cdot y} \to S^3_{x\cdot y}$, we denote the $\{(1,1),(x,y)\}$-orbit normal imitation of $q : M \to S^3$. By construction, $S^3_{x\cdot y}$ is homeomorphic to S^3 and the canonical map $S^3 \to S^3_{x\cdot y}$ is the double branched covering branched along the trivial knot O''_{x*y}. Since $q_{x\cdot y}$ is a normal imitation, it follows from some properties of an imitation that $M_{x\cdot y}$ is a Z-homology 3-sphere and the canonical map $M \to M_{x\cdot y}$ is the 2-fold branched covering branched along a knot $K_{x\cdot y}$ which is the lift of the double branched covering $S^3 \to S^3_{x\cdot y}$ branched along the trivial knot O''_{x*y} by the normal imitation $q_{x\cdot y}$. Assume that there is a homeomorphism $h : M_{x\cdot y} \to M_{x'\cdot y'}$ such that

$h(K_{x\cdot y}) = K_{x'\cdot y'}$ for some $x', y' = i, j, k$. Then for the isometries $x \cdot y, x' \cdot y' \in G$, there is an isometry $\omega \in G$ of M induced from h such that $\omega(x\cdot y)\omega^{-1} = x'\cdot y'$. Since $x \cdot y \in G$ $(x, y = i, j, k)$ are mutually non-conjugate, we have $x \cdot y = x' \cdot y'$. Hence the knots $K_{x\cdot y}$ in $M_{x\cdot y}$ $(x, y = i, j, k)$ must be mutually distinct. This completes the proof of the Z-homology version theorem by M. Mecchia-B. Zimmermann. □

We are in a position to prove Theorem 1.5 (Reni-Mecchia-Zimmermann's conjecture). The method is analogous to M. Mecchia-B. Zimmermann's proof of the Z-homology version theorem if we use the strongly AID imitation theorem instead of the AID imitation theorem.

3.2 Proof of Theorem 1.5.

We apply the strongly AID imitation theorem to S^3 with the G-action. Then we have a G-equivariant normal imitation $q : M \to S^3$ such that M is a hyperbolic 3-manifold with Isom$M = G$ and for every normal subgroup $H \subset G$ with $\mathrm{F}(G, S^3)^u_H \neq \emptyset$, the H-orbit normal imitation map $q_H : (M_H, \mathrm{F}(G, M)^u_H) \to (S^3_H, \mathrm{F}(G, S^3)^u_H)$ is an AID imitation. By the same argument as in 3.2, we have mutually distinct knots $K_{x\cdot y}$ in the Z-homology 3-spheres $M_{x\cdot y}$ $(x, y = i, j, k)$ whose double branched covering spaces are the same homology 3-sphere M. In our case, we shall show that $M_{x\cdot y} = S^3$ for all x, y. Since $\mathrm{F}(G, S^3)^u_{H_{x*y}} = L_{x*y}$, a Hopf link in $S^3_{H_{x*y}} = S^3$, we see from a property of a strongly AID imitation that the H_{x*y}-orbit normal imitation map $q_{H_{x*y}} : (M_{H_{x*y}}, \mathrm{F}(G, M)^u_{H_{x*y}}) \to (S^3, L_{x*y})$ is an AID imitation. Thus, we see that $M_{H_{x*y}} = S^3$ and the link $\mathrm{F}(G, M)^u_{H_{x*y}}$ consists of two trivial components. Since $M_{x\cdot y}$ is the double branched covering of $M_{H_{x*y}} = S^3$ branched along a trivial knot which is a component of the link $\mathrm{F}(G, M)^u_{H_{x*y}}$, we have $M_{x\cdot y} = S^3$ for every $x, y = i, j, k$. This completing the proof of Theorem 1.5, namely the Reni-Mecchia-Zimmermann's conjecture. □

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