

Centroaffine Minimal Surfaces with Constant Curvature Metric

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To the memory of Professor Haruo Kitahara

ABSTRACT. We classify centroaffine minimal surfaces with constant curvature metric under some natural conditions on the cubic differentials.

0. Introduction

In the early 20th century Tzitzéica [10], [11], [12] made an epoch in affine differential geometry. He showed that the ratio of the Gaussian curvature to the fourth power of the support function from the origin for a surface in the Euclidean 3-space \mathbf{R}^3 is invariant under centroaffine transformations and studied surfaces such that the above quantity is constant. Nowadays such surfaces are known as proper affine spheres [2]. They are one of classes of surfaces which are now called integrable-like surfaces with constant Gaussian or mean curvature. On the other hand in the context of centroaffine geometry Liu-Wang [6], [13] introduced a new class of hypersurfaces called centroaffine minimal hypersurfaces and classified such surfaces in \mathbf{R}^3 with vanishing centroaffine Chebyshev operator. Moreover Schief [8] showed that centroaffine minimal surfaces in \mathbf{R}^3 are also integrable-like Bianchi surfaces or surfaces with harmonic inverse mean curvature in Euclidean geometry [3]. In particular these surfaces possess a one-parameter family of deformations preserving a certain geometric quantity such as Chebyshev angle, the ratio of the principal curvatures or the centroaffine metric.

In this paper, we obtain classification of centroaffine minimal surfaces with constant curvature metric under the conditions that the coefficients of the cubic differentials coincide with each other for indefinite surfaces and the coefficient of the cubic differential is real-valued for definite surfaces. Therefore our classification might be regarded as a generalization of the results about affine spheres with constant curvature metric due to Magid-Ryan [7] and Simon [9] and also an analog

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of the results about Bianchi surfaces with constant Chebyshev angle and Bonnet surfaces with constant curvature due to the author [5] and Colares-Kenmotsu [4] respectively since Bonnet surfaces are dual to isothermic surfaces with harmonic inverse mean curvature.

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1. Preliminaries

First we consider indefinite centroaffine surfaces in \mathbf{R}^3 . Since the Gaussian curvature K of such a surface is negative we can parametrize the surface locally by asymptotic line coordinates (u, v) :

$$F : D \rightarrow \mathbf{R}^3,$$

where $D \subset \mathbf{R}^2$. Let d be the distance from the origin of \mathbf{R}^3 to the tangent plane to F , called the support function. If we put

$$\rho = -\frac{1}{4} \log \left(-\frac{K}{d^4} \right), \quad h = -\det \begin{pmatrix} F_u \\ F_v \\ F_{uv} \end{pmatrix} \bigg/ \det \begin{pmatrix} F \\ F_u \\ F_v \end{pmatrix},$$

$$a = h \det \begin{pmatrix} F \\ F_u \\ F_{uu} \end{pmatrix} \bigg/ \det \begin{pmatrix} F \\ F_u \\ F_v \end{pmatrix}, \quad b = h \det \begin{pmatrix} F \\ F_v \\ F_{vv} \end{pmatrix} \bigg/ \det \begin{pmatrix} F \\ F_u \\ F_v \end{pmatrix}$$

then the Gauss equations have the following form [8]:

$$(1.1) \quad \begin{cases} F_{uu} = \left(\frac{h_u}{h} + \rho_u \right) F_u + \frac{a}{h} F_v, \\ F_{uv} = -hF + \rho_v F_u + \rho_u F_v, \\ F_{vv} = \left(\frac{h_v}{h} + \rho_v \right) F_v + \frac{b}{h} F_u \end{cases}$$

with the compatibility conditions:

$$(1.2) \quad \begin{cases} (\log h)_{uv} = -h - \frac{ab}{h^2} + \rho_u \rho_v, \\ a_v + \rho_u h_u = \rho_{uu} h, \\ b_u + \rho_v h_v = \rho_{vv} h. \end{cases}$$

The quantities $hdudv$, adu^3 and bdv^3 , called the centroaffine metric and the cubic differentials respectively, are globally defined and invariant under centroaffine transformations. The Gaussian curvature κ of $hdudv$ is given by

$$(1.3) \quad \kappa = -\frac{(\log h)_{uv}}{h}.$$

If ρ is constant and $a, b \neq 0$ then we may assume that $a = b = 1$ by changing the coordinates if necessary, which reduces (1.2) to the Tzitzéica equation for indefinite proper affine spheres.

Let F be a centroaffine minimal surface. Then it is not so hard to see that $\rho_{uv} = 0$ (cf. [6], [8], [13]). Hence ρ can be chosen to be $\rho = c_1u + c_2v + c_3$ for some $c_1, c_2, c_3 \in \mathbf{R}$, which reduces (1.1) and (1.2) to the following:

$$(1.4) \quad \begin{cases} F_{uu} = \left(\frac{h_u}{h} + c_1\right) F_u + \frac{a}{h} F_v, \\ F_{uv} = -hF + c_2F_u + c_1F_v, \\ F_{vv} = \left(\frac{h_v}{h} + c_2\right) F_v + \frac{b}{h} F_u, \end{cases}$$

$$(1.5) \quad \begin{cases} (\log h)_{uv} = -h - \frac{ab}{h^2} + c_1c_2, \\ a_v + c_1h_u = 0, \\ b_u + c_2h_v = 0. \end{cases}$$

In particular, indefinite centroaffine minimal surfaces possess a one-parameter family of deformations preserving the centroaffine metric since (1.5) is invariant under the transformations:

$$a \rightarrow \lambda a, \quad b \rightarrow \frac{b}{\lambda}, \quad c_1 \rightarrow \lambda c_1, \quad c_2 \rightarrow \frac{c_2}{\lambda} \quad (\lambda \in \mathbf{R} \setminus \{0\}).$$

Lemma 1.1. *Let F be a centroaffine minimal surface with $a = b$. If κ is constant then $\kappa = 0, 1$. Moreover*

$$\begin{cases} a \in \mathbf{R}, \quad h \in \mathbf{R} \setminus \{0\} & \text{if } \kappa = 0, \\ a = c_1 = c_2 = 0 \text{ or } a \neq 0 & \text{if } \kappa = 1. \end{cases}$$

Proof. From (1.3) and the first equation of (1.5) we have

$$(1.6) \quad (\kappa - 1)h^3 + c_1c_2h^2 - a^2 = 0.$$

First we consider the case $a = 0$. Then h is constant or $\kappa = 1, c_1c_2 = 0$. If h is constant then we have $\kappa = 0$ from (1.3). If $\kappa = 1$ and $c_1c_2 = 0$, then it is not so hard to see that $c_1 = c_2 = 0$ from (1.3) and the second and the third equations of (1.5). Second we consider the case $a \neq 0$. Then using the second and the third equations of (1.5), we have $Lh_v + c_1h_u = 0$ and $Lh_u + c_2h_v = 0$, where

$$L = \pm \frac{3(\kappa - 1)h + 2c_1c_2}{2\sqrt{(\kappa - 1)h + c_1c_2}}.$$

Note that

$$L^2 - c_1c_2 = \frac{9(\kappa - 1)^2h^2 + 8(\kappa - 1)c_1c_2h}{4\{(\kappa - 1)h + c_1c_2\}}.$$

If $L^2 - c_1c_2 \neq 0$ or $L^2 - c_1c_2 = 0$, $\kappa \neq 1$ then h is constant. Hence $\kappa = 0$ and a is constant from (1.3) and (1.6). \square

2. Indefinite case

Let F be an indefinite centroaffine minimal surface with $a = b$, $\kappa = 0$. From Lemma 1.1 and (1.5) we may assume that $a = 0$, $h = c_1 = c_2 = 1$ or $a = 1$. A direct computation leads to the following:

Proposition 2.1(cf. [7]). *If $a = 0$, $h = c_1 = c_2 = 1$ then up to centroaffine congruence*

$$F = (e^u, e^v, e^{u+v}),$$

i.e., the image of F is a piece of the hyperbolic paraboloid.

If $a = 1$ then (1.4) and (1.5) become

$$(2.1) \quad \begin{cases} F_{uu} = c_1F_u + \frac{1}{h}F_v, \\ F_{uv} = -hF + c_2F_u + c_1F_v, \\ F_{vv} = c_2F_v + \frac{1}{h}F_u, \end{cases}$$

$$(2.2) \quad h^3 - c_1c_2h^2 + 1 = 0.$$

If we put $u = x + y$, $v = C(x - y)$ with

$$(2.3) \quad C^3 + c_2hC^2 - c_1hC - 1 = 0, \quad C > 0$$

then (2.1) becomes

$$(2.4) \quad \begin{cases} F_{xx} = -2hCF + \frac{C^2 + 2c_2hC + c_1h}{h}F_x, \\ F_{xy} = \frac{-C^2 + c_1h}{h}F_y, \\ F_{yy} = 2hCF + \frac{C^2 - c_1h}{h}F_x - 2(c_2C - c_1)F_y. \end{cases}$$

From (2.2), (2.3) and the first equation of (2.4) we have

$$F = \begin{cases} A_1(y)e^{\alpha x} + A_2(y)e^{\beta x} & \text{if } D_1 \neq 0, \\ A_1(y)xe^{\alpha x} + A_2(y)e^{\alpha x} & \text{if } D_1 = 0, \end{cases}$$

where A_1 and A_2 are \mathbf{R}^3 -valued functions of y and

$$\alpha = \frac{2C(C + c_2h)}{h}, \beta = \frac{-C^2 + c_1h}{h}, D_1 = 3C^2 + 2c_2hC - c_1h.$$

From the second equation of (2.4) A_1 is constant. Hence from the third equation of (2.4) we have the following:

Proposition 2.2(cf. [6]). *Up to centroaffine congruence*

$$F = \begin{cases} \left(e^{\alpha x}, e^{\beta x + (\gamma + \sqrt{D_2})y}, e^{\beta x + (\gamma - \sqrt{D_2})y} \right) & \text{if } D_1 \neq 0, D_2 > 0, \\ \left(e^{\alpha x}, ye^{\beta x + \gamma y}, e^{\beta x + \gamma y} \right) & \text{if } D_1 \neq 0, D_2 = 0, \\ \left(e^{\alpha x}, e^{\beta x + \gamma y} \cos(\sqrt{-D_2}y), e^{\beta x + \gamma y} \sin(\sqrt{-D_2}y) \right) & \text{if } D_1 \neq 0, D_2 < 0, \\ \left(e^{\alpha x}, e^{\alpha x + 2\gamma y}, \left(x + \frac{\alpha}{2\gamma}y\right) e^{\alpha x} \right) & \text{if } D_1 = 0, \gamma \neq 0, \end{cases}$$

where $\gamma = c_1 - c_2C$, $D_2 = \gamma^2 - (\beta^2 + 2hC)$.

Remarks. (1) The surfaces given in Proposition 2.2 are special cases in the classification of centroaffine minimal surfaces with vanishing centroaffine Chebyshev operator due to Liu-Wang [6].

(2) A direct computation shows that if $\gamma = 0$ then $c_1 = c_2$, $C = 1$, $D_1 \neq 0$, $D_2 < 0$. In this case the image of F is a piece of

$$(2.5) \quad \left\{ (X, Y, Z) \in \mathbf{R}^3 \mid (Y^2 + Z^2)^{-\frac{1}{h} + c_1} X^{-\frac{1}{h} - c_1} = 1 \right\}.$$

In particular the surfaces with $a = 1$, $c_1, c_2 \neq 0$ can be obtained by deformations of (2.5) and if $c_1 = c_2 = 0$ then F is a flat proper affine sphere obtained by Magid-Ryan [7].

Let F be an indefinite centroaffine minimal surface with $a = b$, $\kappa = 1$. From Lemma 1.1, (1.3) and (1.5) we may assume that $a = c_1 = c_2 = 0$, $h = -2/(u + v)^2$ or $a = h = -2/(u + v)^2$, $c_1 = c_2 = -1$. A direct computation leads to the following:

Proposition 2.3(cf. [9]). *If $a = c_1 = c_2 = 0$, $h = -2/(u + v)^2$ then up to centroaffine congruence*

$$F = \left(\frac{2}{u + v}, \frac{2uv}{u + v}, \frac{u - v}{u + v} \right),$$

i.e., the image of F is a piece of the hyperboloid of one sheet.

Remark. All surfaces obtained in Propositions 2.1, 2.2 and 2.3 are also centroaffine Chebyshev surfaces, i.e., they satisfy the condition (cf. [6]):

$$a_v = b_u = 0.$$

Centroaffine Chebyshev surfaces with constant curvature metric are classified by Binder [1].

Computing the remaining case we have the following:

Theorem 2.4. *If $a = h = -2/(u+v)^2$, $c_1 = c_2 = -1$ then up to centroaffine congruence*

$$F = \left(\frac{e^{-(u+v)}}{u+v} \cos(u-v), \frac{e^{-(u+v)}}{u+v} \sin(u-v), 1 - \frac{1}{u+v} \right).$$

3. Definite case

In the following we consider definite centroaffine surfaces in \mathbf{R}^3 . Such a surface can be parametrized by a holomorphic coordinate z such that the second fundamental form is conformal with respect to z . All computations are similar to those of the indefinite case. If we put

$$\rho = -\frac{1}{4} \log \frac{K}{d^4}, \quad h = -\det \begin{pmatrix} F_z \\ F_{\bar{z}} \\ F_{z\bar{z}} \end{pmatrix} \bigg/ \det \begin{pmatrix} F \\ F_z \\ F_{\bar{z}} \end{pmatrix}, \quad a = h \det \begin{pmatrix} F \\ F_z \\ F_{zz} \end{pmatrix} \bigg/ \det \begin{pmatrix} F \\ F_z \\ F_{\bar{z}} \end{pmatrix}$$

then the Gauss equations have the following form:

$$(3.1) \quad \begin{cases} F_{zz} = \left(\frac{h_z}{h} + \rho_z \right) F_z + \frac{a}{h} F_{\bar{z}}, \\ F_{z\bar{z}} = -hF + \rho_{\bar{z}} F_z + \rho_z F_{\bar{z}} \end{cases}$$

with the compatibility conditions:

$$(3.2) \quad \begin{cases} (\log h)_{z\bar{z}} = -h - \frac{|a|^2}{h^2} + |\rho_z|^2, \\ a_{\bar{z}} + \rho_z h_z = \rho_{zz} h. \end{cases}$$

The quantities $hdzd\bar{z}$ and adz^3 are called the centroaffine metric and the cubic differential respectively. Moreover κ is given by

$$(3.3) \quad \kappa = -\frac{(\log h)_{z\bar{z}}}{h}.$$

Let F be a centroaffine minimal surface, i.e., $\rho_{z\bar{z}} = 0$. Then ρ can be chosen to be $\rho = c_1 z + \bar{c}_1 \bar{z} + c_2$ for some $c_1 \in \mathbf{C}$ and $c_2 \in \mathbf{R}$, which reduces (3.1) and (3.2) to the following:

$$(3.4) \quad \begin{cases} F_{zz} = \left(\frac{h_z}{h} + c_1 \right) F_z + \frac{a}{h} F_{\bar{z}}, \\ F_{z\bar{z}} = -hF + \bar{c}_1 F_z + c_1 F_{\bar{z}}, \end{cases}$$

$$(3.5) \quad \begin{cases} (\log h)_{z\bar{z}} = -h - \frac{|a|^2}{h^2} + |c_1|^2, \\ a_{\bar{z}} + c_1 h_z = 0. \end{cases}$$

In particular definite centroaffine surfaces possess a one-parameter family of deformations preserving the centroaffine metric since (3.5) is invariant under the transformations:

$$a \rightarrow \lambda a, \quad c_1 \rightarrow \lambda c_1 \quad (\lambda \in \mathbf{C}, |\lambda| = 1).$$

A direct computation leads to the following:

Lemma 3.1. *Let F be a centroaffine minimal surface with $a = \bar{a}$. If κ is constant then $\kappa = 0, 1$. Moreover*

$$\begin{cases} a \in \mathbf{R}, \quad h \in \mathbf{R} \setminus \{0\} & \text{if } \kappa = 0, \\ a = c_1 = 0 \text{ or } a \neq 0 & \text{if } \kappa = 1. \end{cases}$$

Let F be a centroaffine minimal surface with $a = \bar{a}$, $\kappa = 0$. From Lemma 3.1 and (3.5) we may assume that $a = 0$, $h = c_1 = 1$ or $a = 1$. A direct computation leads to the following:

Proposition 3.2(cf. [7]). *If $a = 0$, $h = c_1 = 1$ then up to centroaffine congruence*

$$F = \left(\frac{e^z + e^{\bar{z}}}{2}, \frac{e^z - e^{\bar{z}}}{2\sqrt{-1}}, e^{z+\bar{z}} \right),$$

i.e., the image of F is a piece of the elliptic paraboloid.

If $a = 1$ then (3.4) and (3.5) become

$$\begin{cases} F_{zz} = c_1 F_z + \frac{1}{h} F_{\bar{z}}, \\ F_{z\bar{z}} = -hF + \bar{c}_1 F_z + c_1 F_{\bar{z}}, \end{cases} \quad h^3 - |c_1|^2 h^2 + 1 = 0.$$

We can carry out a computation as in the indefinite case, which gives special cases in the classification of centroaffine minimal surfaces with vanishing centroaffine Chebyshev operator due to Liu-Wang [6]. Moreover if $c_1 \in \mathbf{R}$ then F can be expressed explicitly, which gives a flat proper affine sphere obtained by Magid-Ryan [7] for $c_1 = 0$:

Proposition 3.3(cf. [6]). *If $c_1 \in \mathbf{R}$ then up to centroaffine congruence*

$$F = \left(e^{(c_1 - \frac{1}{h})(z+\bar{z})}, e^{(c_1 + \frac{1}{h})\frac{z+\bar{z}}{2} + \sqrt{(c_1 + \frac{1}{h})^2 - 2h} \frac{z-\bar{z}}{2\sqrt{-1}}}, e^{(c_1 + \frac{1}{h})\frac{z+\bar{z}}{2} - \sqrt{(c_1 + \frac{1}{h})^2 - 2h} \frac{z-\bar{z}}{2\sqrt{-1}}} \right),$$

i.e., the image of F is a piece of

$$\left\{ (X, Y, Z) \in \mathbf{R}^3 \mid (YZ)^{-\frac{1}{h} + c_1} X^{-\frac{1}{h} - c_1} = 1 \right\}.$$

In particular, the surfaces with $a = 1$ can be obtained by deformations of the above one.

Let F be a centroaffine minimal surface with $a = \bar{a}$, $\kappa = 1$. From Lemma 3.1, (3.3) and (3.5) we may assume that $a = c_1 = 0$, $h = -2/(z + \bar{z})^2$ or $a = h$, $|c_1| = 1$. A direct computation leads to the following:

Proposition 3.4(cf. [9]). *If $a = c_1 = 0$, $h = -2/(z + \bar{z})^2$ then up to centroaffine congruence*

$$F = \left(\frac{2}{z + \bar{z}}, \frac{2|z|^2}{z + \bar{z}}, \frac{\sqrt{-1}(z - \bar{z})}{z + \bar{z}} \right),$$

i.e., the image of F is a piece of the hyperboloid of two sheets.

Remark. All surfaces obtained in Propositions 3.2, 3.3 and 3.4 are also centroaffine Chebyshev surfaces, i.e., they satisfy the condition (cf. [6]):

$$a_{\bar{z}} = 0.$$

See also [1].

Computing the remaining case we have the following:

Theorem 3.5. *If $c_1 = -1$ then up to centroaffine congruence*

$$F = \left(\frac{e^{-(z+\bar{z})+\sqrt{-1}(z-\bar{z})}}{z + \bar{z}}, \frac{e^{-(z+\bar{z})-\sqrt{-1}(z-\bar{z})}}{z + \bar{z}}, 1 - \frac{1}{z + \bar{z}} \right).$$

In particular the surfaces with $a = h$, $|c_1| = 1$ can be obtained by deformations of the above one.

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