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Negative Definite Functions on Hypercomplex Systems

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Abstract. We present a concept of negative definite functions on a commutative normal hypercomplex system $L_1(Q, m)$ with basis unity. Negative definite functions were studied in [5] and [4] for commutative groups and semigroups respectively. The definition of such functions on Q is a natural generalization of that defined on a commutative hypergroups.

1. Preliminaries

Let Q be a complete separable locally compact metric space of points $p, q, r, \cdots$; $\beta(Q)$ is the σ-algebra of Borel subsets, and $B_0(Q)$ is the subring of $B(Q)$, which consists of sets with compact closure. We shall consider the Borel measures; i.e., positive regular measures on $B(Q)$, finite on compact sets. We denote by $C(Q)$ the space of continuous functions on Q; $C_b(Q), C_\infty(Q)$ and $C_0(Q)$ consists respectively of bounded, tending to zero at infinity and compactly supported functions from $C(Q)$.

A hypercomplex system with the basis Q is defined by its structure measure $c(A, B, r)$ $(A, B \in B(Q); r \in Q)$. A structure measure $c(A, B, r)$ is a Borel measure in A (respectively B) if we fix B, r (respectively A, r) which satisfies the following properties:

(H1) $\forall\, A, B \in \beta_0(Q)$, the function $c(A, B, r) \in C_0(Q)$,

(H2) $\forall\, A, B \in \beta_0(Q)$ and $s, r \in Q$, the following associativity relation holds

$$\int_Q c(A, B, r) d_r c(E_r, C, s) = \int_Q C(B, C, r) d_r C(A, E_r, s), \quad C \in B(Q).$$

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(H3) The structure measure is said to be commutative if

$$c(A,B,r) = c(B,A,r), \quad (A,B \in B_0(Q)).$$

A measure m is said to be a multiplicative measure if

$$\int_Q c(A,B,r)dm(r) = m(A)m(B); \quad A,B \in \beta_0(Q).$$

(H4) We will suppose the existence of a multiplicative measure. Under certain restrictions imposed on the commutative structure measure, multiplicative measure exists. (See [10]).

Consider the space $L_1(Q,m) = L_1$ of functions on Q with respect to the multiplicative measure m.

Theorem 1.1. *For any $f,g \in L_1(Q,m)$, the convolution*

$$
\begin{aligned}
(f*g)(r) &= \int_Q f(p)d_p \int_Q g(q)d_q c(E_p,E_q,r) \\
&= \int_Q \int_Q f(p)g(q)c(p,q,r)dm(p)dm(q) \\
&= \int_Q \int_Q f(p)g(q)dm_r(p,q)
\end{aligned}
\tag{1.1}
$$

is well defined. (*See* [2]).

The space $L_1(Q,m)$ with the convolution (1.1) is a Banach algebra which is commutative if (H3) holds. This Banach algebra is called the hypercomplex system with the basis Q.

It is obvious that $C(A,B,r) = (K_A * K_B)(r)$; $A,B \in \beta_0(Q)$ and K_A is the characteristic function of the set A.

A hypercomplex system may or may not have a unity. If a unity not included in $L_1(Q,m)$, then it is convenient to join it formally to L_1.

A non zero measurable and bounded almost everywhere function $Q \ni r \to \chi(r) \in C$ is said to be a character of the hypercomplex system L_1 if $\forall\, A,B \in \beta_0(Q)$

$$\int_Q c(A,B,r)\chi(r)dm(r) = \chi(A)\chi(B),$$

$$\int \chi(r)dm(r) = \chi(C), \qquad C \in \beta_0(Q).$$

(H5) A hypercomplex system is said to be normal, if there exists an involution homomorphism $Q \ni r \to r^* \in Q$, such that $m(A) = m(A^*)$ and $c(A,B,C) = c(C,B^*,A)$, $c(A,B,C) = c(A^*,C,B)$, $(A,B \in \beta_0(Q))$, where

$$c(A,B,C) = \int_C c(A,B,r)dm(r).$$

(H6) A normal hypercomplex system possesses a basis unity if there exists a point $e \in Q$ such that $e^* = e$ and

$$c(A, B, e) = m(A^* \cap B), \qquad A, B \in \beta(Q).$$

We should remark that, for a normal hypercomplex system, the mapping

$$L_1(Q, m) \ni f(r) \to f^*(r) \in L_1(Q, m)$$

is an involution in the Banach algebra L_1, the multiplicative measure is unique and the characters of such a system are continuous. (See [1]). A character χ of a normal hypercomplex system is said to be Hermitian if

$$\chi(r^*) = \overline{\chi(r)}, \qquad (r \in Q).$$

Denote the set of all bounded Hermitian characters by X_h, i.e.,

$$X_h = \{\chi \in C_b(Q) : \chi \neq 0, \int c(A, B, r)\chi(r)dm(r) = \chi(A)\chi(B), \overline{\chi(r)} = \chi(r^*)\}.$$

Let $L_1(Q, m)$ be a hypercomplex system with compact basis, $\hat{Q}$ be a dual countable basis (collection of all characters $\chi, \phi, \psi, \cdots$), and $\hat{m}$ be a Plancherel measure. The space $L_1(\hat{Q}, \hat{m}) = l_1(\hat{m})$ becomes a hypercomplex system with discrete basis if we define a dual structure measure $\hat{c}$ by the formula

$$\hat{c}(\chi, \phi, \psi) = \hat{m}(\chi)\hat{m}(\phi) \int_Q \chi(r)\phi(r)\overline{\psi(r)}dm(r), \quad (\chi, \phi, \psi \in \hat{Q}) \tag{1.2}$$

and assume that the integral in (1.2) is nonnegative.

This dual hypercomplex system is normal if we set $\chi^* = \overline{\chi}$, and it has a basis unity $\hat{e} \equiv 1$. See [2].

2. Generalized translation operators and hypercomplex system

In a series of works originated as early as in 1938, J. Delsarte [7], [8], and then B. M. Levitan [11], [12] noticed that some facts of classical harmonic analysis can be generalized by replacing exponential functions $e^{i\lambda q}$ $(q, \lambda \in R^1)$ by some family of complex-valued functions $\chi(q, \lambda)$ which inherit the following property of the indicated exponential functions. The exponential functions are connected with the family of ordinary translation operators R_p $(p \in R^1)$ acting upon complex-valued functions $f(q)$ $(q \in R^1)$ according to the rule

$$(R_P f)(q) = f(p + q),$$

i.e.,

$$R_p e^{i\lambda q} = e^{i\lambda p} e^{i\lambda q} \tag{2.1}$$

for any λ.

For functions $\chi(q,\lambda)$, where q varies in some set Q and λ in another set $\hat{Q}$, there should exist a family of linear "generalized translation" operators R_p $(p \in Q)$, acting on the functions of the variable $q \in Q$ such that an equality of (2.1) type is valid.

$$(R_p\chi(.,\lambda))(q) = \chi(p,\lambda)\chi(q,\lambda) \qquad (p,q \in Q,\ \lambda \in \hat{Q}).$$

It is natural that the family of such operators R_p should have some additional properties, similar to a usual shift. It was clear from [8], [12] that it is important to study not only generalized translations but a convolution of functions associated with these translations. So by analogy with the usual convolution

$$(f * g)(q) = \int_R f(p)g(q-p)dp = \int_R f(p)(R_{-p}g)(q)dp, \tag{2.2}$$

it is possible to introduce a generalized convolution $*$ similar to (2.2), associated with the generalized translation operators:

$$(f * g)(q) = \int_Q f(p)(R_{p^*}g)(q)dm(p) \tag{2.3}$$

which is equivalent to the form (1.1).

In (2.3), the involution * in Q is used instead of the inverse in R, and m is the multiplicative measure.

Let $L_1(Q,m)$ be a hypercomplex system with a basis Q and Φ be a space of complex-valued functions on Q. Assume that an operator valued function $Q \ni p \to R_p : \Phi \to \Phi$ is given such that the function $g(p) = (R_pf)(q)$ belongs to Φ for any $f \in \Phi$ and any fixed $q \in Q$. The operators $R_p\,(p \in Q)$ are called generalized translation operators, provided that the following axioms are satisfied:

(T1) Associativity axiom: The equality

$$(R_p^q(R_qf))(r) = (R_q^r(R_pf))(r)$$

holds for any elements $p,q \in Q$.

(T2) There exists an element $e \in Q$ such that R_e is the identity in Φ. (See [3]).

3. Positive and negative definite functions on hypercomplex system

Let $L_1(Q,m)$ be a commutative normal hypercomplex system with basis unity.

Definition 3.1. A continuous bounded function $\varphi(r)$ $(r \in Q)$ is called positive definite if the inequality

$$\sum_{i,j=1}^{N} \lambda_i\overline{\lambda_j}(R_{r_i^*}\varphi)(r_j) \geq 0 \tag{3.1}$$

holds for all $r_1, \cdots, r_n \in Q$, and $\lambda_1, \cdots, \lambda_n \in C, \quad (n \in N)$.

If the generalized translation operators R_t extended to L_∞ map $C_b(Q)$ into $C_b(Q \times Q)$, then the inequality (3.1) of positive definiteness is equivalent for the functions $\varphi(r) \in C_b(Q)$ to

$$\int_Q \int_Q (R_t\varphi)(s^*)x(t)\overline{x(s)}dtds \geq 0, \quad x \in L_1(Q, m).$$

By $P(Q)$, we shall denote the set of all continuous positive definite functions on Q.

Theorem 3.2. *Every function $\varphi \in P(Q)$ admits a unique representation in the form of an integral*

$$\varphi(r) = \int_{X_h} \chi(r)d\mu(\chi), \qquad \chi \in X_h, \tag{3.2}$$

where μ is a nonnegative finite regular measure on the space X_h. Conversely, each function of the form (3.2) *belongs to $P(Q)$.*

For the proof, see [1].

Theorem 3.2 is an analog of the Bochner theorem for hypercomplex systems.

Corollary 3.3. *If $\varphi \in P(Q)$; then the following properties holds*:

(i) $\varphi(e) \geq 0$;

(ii) $\varphi(r^*) = \overline{\varphi(r)} \qquad \forall\, r \in Q$;

(iii) $|\varphi(r)| \leq \varphi(e) \qquad \forall\, r \in Q$;

(iv) $|R_s(\varphi)(t)|^2 \leq (R_{s^*}\varphi)(s)(R_{t^*}\varphi)(t)$;

(v) $|\varphi(s) - \varphi(t)|^2 \leq 2\varphi(e)[\varphi(e) - Re(R_s\varphi)(t^*)] \qquad (s, t \in Q)$.

Definition 3.4. A continuous bounded function $\psi : Q \to C$ is called negative definite if for any $r_1, \cdots, r_n \in Q$ and $c_1, \cdots, c_n \in C$

$$\sum_{i,j=1}^{n} [\psi(r_i) + \overline{\psi(r_j)} - (R_{r_j^*}\psi)(r_i)]c_i\overline{c_j} \geq 0. \tag{3.3}$$

For example each constant function, $c \geq 0$ is negative definite. Obviously the following holds for a negative definite function ψ:

$$\begin{aligned} &\psi(e) \geq 0, \quad \overline{\psi(r)} = \psi(r^*), \quad (R_r * \psi)(r) \in R \qquad \text{and} \\ &\psi(r) + \psi(r^*) \geq (R_{r^*}\psi)(r). \end{aligned} \tag{3.4}$$

Let us abbreviate the set of negative definite functions on Q by $N(Q)$.

We note that $\psi = \psi^*$, and $Re\ \psi$ is non negative if $(R_{r^*}\psi)(r) \geq 0$.

Theorem 3.5. *A function $\psi : Q \to C$ is negative definite if and only if the following conditions are satisfied*:

$$\text{(3.5)} \quad \text{(i)} \quad \psi(e) \geq 0, \quad \psi \textit{ is continuous bounded function}$$

$$\text{(3.6)} \quad \text{(ii)} \quad \overline{\psi(r)} = \psi(r^*) \quad \textit{for each} \quad r \in Q, \quad \textit{and}$$

$$\text{(iii)} \quad \textit{if } r_1, \cdots, r_n \in Q, \textit{ and } c_1, \cdots, c_n \in C \textit{ with } \sum_{i=1}^{n} c_i = 0, \textit{ then}$$

$$\text{(3.7)} \qquad \sum_{i,j=1}^{n} (R_{r_j^*}\psi)(r_i) c_i \overline{c_j} \leq 0$$

holds.

Proof. Suppose first that $\psi \in N(Q)$. It is clear that (i) and (ii) are satisfied. Let $n \in N$; $r_1, \cdots, r_n \in Q$ and $c_1, \cdots, c_n \in C$ be such that

$$\sum_{i=1}^{n} c_i = 0.$$

Then we find

$$\begin{aligned} 0 &\leq \sum_{i,j=1}^{n} \left(\psi(r_i) + \overline{\psi(r_j)} - (R_{r_j^*})(r_i) \right) c_i \overline{c_j} \\ &= \overline{\left(\sum_{j=1}^{n} c_j \right)} \left(\sum_{i=1}^{n} \psi(r_i) c_i \right) + \left(\sum_{i=1}^{n} c_i \right) \overline{\left(\sum_{j=1}^{n} \psi(r_j) c_j \right)} - \sum_{i,j=1}^{n} \left(R_{r_j^*}\psi \right)(r_i) c_i \overline{c_j} \\ &= - \sum_{i,j=1}^{n} \left(R_{r_j^*}\psi \right)(r_i) c_i \overline{c_j}. \end{aligned}$$

Then (iii) is satisfied.

Conversely, suppose that ψ satisfies (i)-(iii), and consider $r_1, \cdots, r_n \in Q$ and $c_1, \cdots, c_n \in C$. For the $(n+1)$-tuples $e, r_1, \cdots, r_n \in Q$ and $c_0, c_1, \cdots, c_n \in C$, where

$$\sum_{i=0}^{n} c_i = 0$$

i.e.,

$$c_0 = - \sum_{i=1}^{n} c_i,$$

we get by (iii)

$$\begin{aligned} 0 &\geq \sum_{i,j=0}^{n} \left(R_{r_j^*}\psi\right)(r_i)c_i\overline{c_j} \\ &= \sum_{i,j=1}^{n} \left(R_{r_j^*}\psi\right)(r_i)c_i\overline{c_j} + \overline{c_0}\sum_{i=1}^{n}\psi(r_i)c_i + c_0\sum_{j=1}^{n}\psi(r_j^*)\overline{c_j} + \psi(e)|c_0|^2 \\ &= \sum_{i,j=1}^{n} \left((R_{r_j^*}\psi)(r_i) - \psi(r_i) - \psi(r_j^*)\right)c_i\overline{c_j} + \psi(e)|c_0|^2 \end{aligned}$$

hence, using (i) and (ii), that

$$\sum_{i,j=1}^{n} \left[\psi(r_i) + \overline{\psi(r_j)} - (R_{r_j^*}\psi)(r_i)\right] c_i\overline{c_j} \geq \psi(e)|c_0|^2 \geq 0. \qquad \square$$

Corollary 3.6. *Let ψ be a function on Q*

(i) *If $\psi \in N(Q)$, then $r \mapsto \psi(r) - \psi(e)$ is negative definite.*

(ii) *If $\varphi \in P(Q)$, then $r \mapsto \varphi(e) - \varphi(r)$ is negative definite.*

Proof. We will use the Theorem 3.5.
(i). Conditions (3.5) and (3.6) are clearly satisfied. For the condition (3.7), let $r_1, \cdots, r_n \in Q$ and $c_1, \cdots, c_n \in C$ be given satisfying

$$\sum_{i=1}^{n} c_i = 0.$$

Then we find

$$\begin{aligned} \sum_{i,j=1}^{n} R_{r_j^*}(\psi(r_j) - \psi(e))c_i\overline{c_j} &= \sum_{i,j=1}^{n} \left(R_{r_j^*}\psi\right)(r_i)c_i\overline{c_j} - \psi(e)\left|\sum_{i=1}^{n} c_i\right|^2 \\ &= \sum_{i,j=1}^{n} \left(R_{r_j^*}\psi\right)(r_i)c_i\overline{c_j} - 0 \\ &= \sum_{i,j=1}^{n} \left(R_{r_j^*}\psi\right)(r_i)c_i\overline{c_j} \\ &\leq 0 \end{aligned}$$

which proves the negative definiteness of $\psi(r) - \psi(e)$.

(ii). Let $r_1, \cdots, r_n \in Q$ and $c_1, \cdots, c_n \in C$ be given satisfying

$$\sum_{i=1}^{n} c_i = 0.$$

Then we find

$$\begin{aligned}
\sum_{i,j=1}^{n} \left(\varphi(e) - (R_{r_j^*}\varphi)(r_i) \right) c_i \overline{c_j} &= -\sum_{i,j=1}^{n} \left((R_{r_j^*}\varphi)(r_i) - \varphi(e) \right) c_i \overline{c_j} \\
&= -\sum_{i,j=1}^{n} \left(R_{r_j^*}\varphi \right)(r_i) c_i \overline{c_j} + \varphi(e) \left| \sum_{i=1}^{n} c_i \right|^2 \\
&= -\sum_{i,j=1}^{n} \left(R_{r_j^*}\varphi \right)(r_i) c_i \overline{c_j} \\
&\leq 0
\end{aligned}$$

because $\varphi \in P(Q)$, and since the function $\varphi(e) - \varphi(r)$ clearly satisfies (i) and (ii) of Theorem 3.5, it is negative definite. □

Now, we state the definition of negative definiteness in another form.

A continuous bounded function $\psi(r)$ $(r \in Q)$ is called negative definite if the inequality

$$\int_Q \int_Q \left(\psi(r) + \overline{\psi(s)} - (R_{s^*}\psi)(r) \right) x(r) \overline{x(s)} dr ds \geq 0 \tag{3.8}$$

holds for all $x \in L_1$.

If the generalized operators R_t extends to L_∞ map $C_b(Q)$ into $C_b(Q \times Q)$, then the definitions of negative definiteness (3.3) and (3.8) are equivalent for the function $\psi(r) \in C_b(Q)$.

By the condition, we have $(R_t\phi)(s^*) \in C_b(Q \times Q)$, then the last inequality (3.8) clearly implies (3.3). Let us prove the converse assertion. Let Q_n be an increasing sequence of compact sets covering the entire Q. We consider a function $y(r) \in C_0(Q)$ and set $\lambda_i = y(r_i)$ in (3.7). This yields

$$\sum_{i,j=1}^{n} \left(R_{r_i^*}\psi \right)(r_j) y(r_i) \overline{y(r_j)} \leq 0.$$

By integrating this inequality with respect to each $r_1, \cdots, r_n$ over the sets Q_k $(k \in N)$ and collecting similar terms we conclude that

$$nm(Q_k) \int_{Q_k} (R_{r^*}\psi)(r) |y(r)|^2 dr + n(n-1) \int_{Q_k} \int_{Q_k} (R_{r^*}\psi)(s) y(r) \overline{y(s)} dr ds \leq 0.$$

Further, we divide this inequality by n^2 and pass to the limit as $n \to \infty$. We get

$$\int_{Q_k}\int_{Q_k}(R_{r^*}\psi)(s)y(r)\overline{y(s)}drds \le 0$$

for each $k \in N$. By passing to the limit as $k \to \infty$ and applying Lebesgue Theorem [9], we see that the inequality

$$\int_{Q}\int_{Q}(R_{r^*}\psi)(s)y(r)\overline{y(s)}drds \le 0$$

holds for all functions from $C_0(Q)$. Approximating an arbitrary function from L_1 by finite continuous functions we arrive at (3.8) for all $x \in L_1$. Theorem 3.5 completes the conclusion of the desired equivalence.

Theorem 3.7. *Let $\psi \in N(Q)$ with $Re\psi \ge 0$. Then*

$$\sqrt{(R_r(\psi))(s)} \le \sqrt{|\psi(r)|} + \sqrt{|\psi(s)|}\ ; \qquad r, s \in Q.$$

Proof. Let $\psi \in N(Q)$, then the $n \times n$ matrix $\Bigg(\psi(r_i) + \overline{\psi(r_j)} - (R_{r_j^*}\psi)(r_i)\Bigg)$ is positive Hermitian for any $i, j = 1, \cdots, n$.

Take $n = 2$, and $r, s \in Q$. Since the matrix

$$\begin{pmatrix} \psi(r) + \overline{\psi(r)} - (R_{r^*}\psi)(r) & \psi(r) + \overline{\psi(s)} - (R_{s^*}\psi)(r) \\ \psi(s) + \overline{\psi(r)} - (R_{r^*}\psi)(s) & \psi(s) + \overline{\psi(s)} - (R_{s^*}\psi)(s) \end{pmatrix}$$

has non-negative determinant, we find, using $(R_{r^*}\psi)(s) = \overline{(R_{s^*}\psi)(r)}$, and properties (3.4).

We get

$$\Big|\psi(r) + \overline{\psi(s)} - (R_{s^*}\psi)(r)\Big|^2 \le \Bigg(2Re\psi(r) - (R_{r^*}\psi)(r)\Bigg).$$

$$\begin{aligned} \Bigg(2Re\psi(s) - (R_{s^*}\psi)(s)\Bigg) &\le 4Re\psi(r)Re\psi(s) \\ &\le 4|\psi(r)||\psi(s)|. \end{aligned}$$

Then

$$\Big|(R_{s^*}\psi)(r) - \psi(r) - \overline{\psi(s)}\Big| \le 2\sqrt{|\psi(r)|}\sqrt{|\psi(s)|}$$

and

$$|(R_{s^*}\psi)(r)| \leq \left(\sqrt{|\psi(r)|} + \sqrt{|\psi(s)|}\right)^2. \qquad \square$$

Theorem 3.8. *Let $\psi : Q \to C$ be a function on Q. Assume that*

(i) *ψ is continuous bounded and $\psi(e) \geq 0$.*

(ii) *$\varphi_t : r \to \exp(-t\psi(r))$ are positive definite for each $t > 0$.*

Then ψ is negative definite.

Proof. By (i) the functions φ_t are continuous and $\varphi_t(e) \leq 1$. Therefore Corollary 3.6 (ii) implies that $r \mapsto \frac{1}{t}(1 - \varphi_t(r))$ is negative definite for any $t > 0$. Since

$$\left|\psi(r) - \frac{1}{t}(1 - \varphi_t(r))\right| \leq t \exp|\psi(t)|, \qquad \text{for } 0 < t < 1.$$

We obtain that

$$\lim_{t \to 0} \frac{1}{t}(1 - \varphi_t) = \psi$$

uniformly on compact subsets of Q. Then it is easy to prove that ψ satisfy (3.3).$\square$

We do not know whether the inverse assertion of Theorem 3.8 does hold in general.

4. Negative definite functions on hypergroups

Let K be a commutative hypergroup (see [1], [2]). We define the action of generalized invariant operators $R_r(r \in Q)$ upon arbitrary Borel functions f on K by the formula

$$(R_r f)(s) = (\delta_s * \delta_r)(f),$$

where the convolution

$$K \times K \ni (r, s) \mapsto \delta_r * \delta_s \in M(K)$$

is continuous. $M(k)$ is equipped with weak topology, and δ_r is the Dirac measure.

A continuous function $\psi : K \to C$ is negative definite on K if for $x_1, \cdots, x_n \in K$, $c_1, \cdots, c_n \in C$, the inequality

$$\sum_{i,j=1}^{n} \left(\psi(x_i + \overline{\psi(x_j)}) - \delta_{x_i} * \delta_{\overline{x}_j}(\psi)\right) c_i \overline{c_j} \geq 0$$

holds.

It is fairly easy to observe that all our studied properties and theorems of negative definite functions on hypercomplex system are easily established for the above case (see [6]).

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