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Existence of Nonoscillatory Solution of Second Order Nonlinear Neutral Delay Equations

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Abstract. In this paper, we study nonoscillatory solutions of a class of second order nonlinear neutral delay differential equations with positive and negative coefficients. Some sufficient conditions for existence of nonoscillatory solutions are obtained.

1. Introduction

Consider the second order nonlinear neutral delay differential equation with positive and negative coefficients

$$[r(t)(x(t)+p(t)x(t-\tau))']' + Q_1(t)f(x(t-\sigma_1)) - Q_2(t)g(x(t-\sigma_2)) = 0, \qquad (E)$$

where $t \geq t_0$, $\tau \in (0,\infty)$, $\sigma_1, \sigma_2 \in [0,\infty)$, $p, Q_1, Q_2, r \in C([t_0,\infty), R)$, $f, g \in C(R,R)$. Throughout this paper, we assume that

- (c_1) f and g satisfy local Lipschitz Condition, and $xf(x) > 0$, $xg(x) > 0$, for $x \neq 0$.
- (c_2) $r(t) > 0$, $Q_i \geq 0$, $\int^\infty R(t)Q_i(t)dt < \infty$, $(i = 1,2)$, where $R(t) = \int_{t_0}^t \frac{1}{r(s)}ds$.
- (c_3) $aQ_1(t) - Q_2(t)$ s eventually nonnegative for every $a > 0$.

Second order neutral delay differential equations have applications in problems dealing with vibrating masses attaches to an elastic bar and in some variational problems (see Hale [5]).

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Let $u \in C([t_0-\rho,\infty),R)$, where $\rho=\max\{\tau,\sigma_1,\sigma_2\}$, be a given function and let y_0 be a given constant. Using the method of steps, equation (E) has a unique solution $x \in C([t_0-\rho,\infty),R)$, in the sense that both $x(t)+p(t)x(t-\tau)$ and $r(t)(x(t)+p(t)x(t-\tau))'$ are continuously differentiable for $t \geq t_0$, $x(t)$ satisfies equation (E) and

$$x(s)=u(s) \text{ for } s\in[t_0-\rho,t_0], \qquad (x(t)+p(t)x(t-\tau))'|_{t=t_0}=y_0.$$

For further questions concerning existence and uniqueness of solutions of neutral delay differential equations, (see Hale [5]).

A solution of equation (E) is called oscillatory if it has arbitrarily large zeros, and otherwise it is non-oscillatory.

We observe that the oscillatory and asymptotic behavior of solutions for second order neutral and non-neutral delay differential equations has been studied in many papers, e.g. [1]-[4], [6]-[10]. The second order neutral equation (E) received much less attention, which is due mainly to the technical difficulties arising in its analysis. See [1], [2], [4] for reviews of this theory.

This paper was motivated by recent paper [6], where there the authors give a criterion for the existence of non-oscillatory solution of second order linear neutral delay equation

$$\frac{d^2}{dt^2}[x(t)+p(t)x(t-\tau)]+Q_1(t)x(t-\sigma_1)-Q_2(t)x(t-\sigma_2)=0, \qquad (E_1)$$

where $p\in R$, $\tau\in(0,\infty)$, $\sigma_1,\sigma_2\in[0,\infty)$, $Q_1,Q_2\in C([t_0,\infty),\ R^+)$. The purpose of this paper is to present some new criteria for the existence of non-oscillatory solution of (E), which extend results in [6], [7].

2. Main results

Our main results are the following:

Theorem 1. *Suppose that Conditions* $(c_1)-(c_3)$ *hold and that there exists a constant* p_0 *such that*

$$|p(t)|\leq p_0<\frac{1}{2} \quad \text{eventually.} \tag{1}$$

Then (E) *had a non-oscillatory solution.*

Proof. Choose constants $N_1\geq M_1>0$ such that

$$\frac{1}{1-p_0}<N_1\leq\frac{1-M_1}{p_0}<\frac{1}{p_0}. \tag{2}$$

Let X be the set of all continuous and bounded functions on $[t_0,\infty)$ with the sup norm. Set

$$A_1=\{x\in X: M_1\leq x(t)\leq N_1,\ t\geq t_0\}.$$

Let $L_f(A_1),\ L_g(A_1)$ denote Lipschitz constants of functions $f,\ g$ on the set A_1, respectively, and

$$L_1 = \max\{L_f(A_1), L_g(A_1)\}, \qquad \alpha_1 = \max_{x\in A_1}\{f(x)\}, \quad \beta_1 = \min_{x\in A_1}\{f(x)\},$$
$$\alpha_2 = \max_{x\in A_1}\{g(x)\}, \quad \beta_2 = \min_{x\in A_1}\{g(x)\}.$$

Choose a $t_1 > t_0+\rho, \rho = \max\{\tau,\sigma_1,\sigma_2\}$. Sufficiently large such that

$$aQ_1(t) - Q_2(t) \geq 0 \text{ for } t \geq t_1 \quad \text{and} \quad a > 0.$$

$$|p(t)| \leq p_0 < \frac{1}{2} \text{ for } t \geq t_1.$$

$$\int_{t_1}^{\infty} R(s)[Q_1(s)+Q_2(s)]ds \quad < \quad \frac{1-p_0}{L_1} \tag{3}$$

$$0 \leq \int_{t_1}^{\infty} R(s)[\alpha_1 Q_1(s) - \beta_2 Q_2(s)]ds \quad \leq \quad (1-p_0)N_1 - 1, \text{ and} \tag{4}$$

$$\int_{t_1}^{\infty} R(s)[\beta_1 Q_1(s) - \alpha_2 Q_2(s)]ds \quad \geq \quad 0. \tag{5}$$

Define a mapping $T_1 : A_1 \to X$ as follows

$$(T_1x)(t) = \begin{cases} 1 - p(t)x(t-\tau) \\ +R(t)\int_t^{\infty}[Q_1(s)f(x(s-\sigma_1)) - Q_2(s)g(x(s-\sigma_1))]ds \\ +\int_{t_1}^{t} R(s)[Q_1(s)f(x(s-\sigma_1)) - Q_2(s)g(x(s-\sigma_1))]ds, & t \geq t_1, \\ \\ (T_1x)(t_1), & t_0 \leq t \leq t_1. \end{cases}$$

Clearly, T_1x is continuous. For every $x \in A_1$ and $t \geq t_1$, using (1) and (4) we get

$$(T_1x)(t) \leq 1 + p_0N_1 + \int_{t_1}^{\infty} R(s)[\alpha_1Q_1(s) - \beta_2Q_2(s)]ds \leq N_1, \quad t > t_1.$$

On the other hand, in view of (1), (2) and (5) we have

$$(T_1x)(t) \geq 1 - p_0N_1 \geq M_1, \quad t > t_1.$$

Thus we proved that $T_1A_1 \subset A_1$. Since A_1 is a bounded, closed and convex subset of X we have to prove that T_1 is a contraction mapping on A_1 to apply the contract ion principle.

Now, for $x_1,\ x_2 \in A_1$ and $t \geq t_1$, in view of (3) we have

$$
\begin{aligned}
& |(T_1x_1)(t) - (T_1x_2)(t)| \\
\leq\ & p_0|x_1(t-\tau) - x_2(t-\tau)| + R(t)\int_t^\infty Q_1(s)|f(x_1(s-\sigma_1)) - f(x_2(s-\sigma_1))|ds \\
& + R(t)\int_t^\infty Q_2(s)|g(x_1(s-\sigma_2)) - g(x_2(s-\sigma_2))|ds \\
& + \int_{t_1}^t R(s)Q_1(s)|f(x_1(s-\sigma_1)) - f(x_2(s-\sigma_1))|ds \\
& + \int_{t_1}^t R(s)Q_2(s)|g(x_1(s-\sigma_2)) - g(x_2(s-\sigma_2))|ds \\
\leq\ & p_0\|x_1 - x_2\| \\
& + L_1\|x_1 - x_2\|\{\int_t^\infty R(s)[Q_1(s)+Q_2(s)]ds + \int_{t_1}^t R(s)[Q_1(s)+Q_2(s)]ds\} \\
=\ & \|x_1 - x_2\|\{p_0 + L_1\int_{t_1}^\infty R(s)[Q_1(s)+Q_2(s)]ds\} \\
=\ & q_0\|x_1 - x_2\|,
\end{aligned}
$$

where we used sup norm. This immediately implies that

$$\|T_1x_1 - T_1x_2\| \leq q_0\|x_1 - x_2\|,$$

where in view of (3), $q_0 < 1$, which proves that T_1 is a contraction mapping. Consequently T_1 has the unique fixed point x, which is obviously a positive solution of (E). This completes the proof of Theorem 1. □

Theorem 2. *Suppose that conditions* $(c_1) - (c_3)$ *hold, and if one of the following two conditions is satisfied*:

$$\text{(i)} \quad p(t) \geq 0 \text{ eventually, and } 0 < p_1 < 1; \tag{6}$$

$$\text{(ii)} \quad p(t) \leq 0 \text{ eventually, and } -1 < p_2 < 0, \tag{7}$$

where $p_1 = \lim_{t\to\infty}\sup P(t)$, $p_2 = \lim_{t\to\infty}\inf P(t)$. *Then* (E) *has a nonoscillatory solution.*

Proof. (i). Suppose (6) hold. Choose constants $N_2 \geq M_2 > 0$ such that

$$1 - p_1 < N_2 \leq \frac{4}{3p_1 + 1}[(1-p_1) - M_2]. \tag{8}$$

Let X be the set as in Theorem 1. Set

$$A_2 = \{x \in X : M_2 \leq x(t) \leq N_2, \quad t \geq t_0\}.$$

Define

$$L_2 = \max\{L_f(A_2), L_g(A_2)\}, \qquad \alpha_1 = \max_{x\in A_2}\{f(x)\}, \quad \beta_1 = \min_{x\in A_2}\{f(x)\},$$
$$\alpha_2 = \max_{x\in A_2}\{g(x)\}, \quad \beta_2 = \min_{x\in A_2}\{g(x)\},$$

where $L_f(A_2)$, $L_g(A_2)$ are Lipschitz constants of functions f, g on the set A_2, respectively.

Choose a $t_2 > t_0 + \rho$ sufficiently large such that

$$0 \le p(t) < \frac{1+3p_1}{4} \qquad \text{for } t \ge t_2. \tag{9}$$

$$\int_{t_2}^{\infty} R(s)[Q_1(s)+Q_2(s)]ds < \frac{3(1-p_1)}{4L_2}, \tag{10}$$

$$0 \le \int_{t_2}^{\infty} R(s)[\alpha_1 Q_1(s) - \beta_2 Q_2(s)]ds \le N_2 + (p_1 - 1), \text{ and} \tag{11}$$

$$\int_{t_2}^{\infty} R(s)[\beta_1 Q_1(s) - \alpha_2 Q_2(s)]ds \ge 0. \tag{12}$$

Define a mapping $T_2 : A_2 \to X$ as follows

$$(T_2x)(t) = \begin{cases} 1 - p_1 - p(t)x(t-\tau) \\ +R(t)\int_t^\infty [Q_1(s)f(x(s-\sigma_1)) - Q_2(s)g(x(s-\sigma_2))]ds \\ +\int_{t_2}^t R(s)[Q_1(s)f(x(s-\sigma_1)) - Q_2(s)g(x(s-\sigma_2))]ds, & t \ge t_2, \\ \\ (T_2x)(t_2), & t_0 \le t \le t_2. \end{cases}$$

Clearly, T_2x is continuous. For every $x \in A_2$ and $t \ge t_2$, using (c_3) and (11) we get

$$\begin{aligned} & (T_2x)(t) \\ = \ & 1 - p_1 - p(t)x(t-\tau) + R(t)\int_t^\infty [Q_1(s)f(x(s-\sigma_1)) - Q_2(s)g(x(s-\sigma_2))]ds \\ & + \int_{t_2}^t R(s)[Q_1(s)f(x(s-\sigma_1)) - Q_2(s)g(x(s-\sigma_2))]ds \\ \le \ & 1 - p_1 + \int_t^\infty R(s)[\alpha_1 Q_1(s) - \beta_2 Q_2(s)]ds + \int_{t_2}^t R(s)[\alpha_1 Q_1(s) - \beta_2 Q_2(s)]ds\} \\ = \ & 1 - p_1 + \int_{t_2}^\infty R(s)[\alpha_1 Q_1(s) - \beta_2 Q_2(s)]ds \le N_2, \quad t \ge t_2. \end{aligned}$$

Furthermore, in view of (8) and (9) we have

$$\begin{aligned}
&(T_2x)(t)\\
\geq\ & 1-p_1-\frac{1+3p_1}{4}N_2+R(t)\int_t^\infty[\beta_1Q_1(s)-\alpha_2Q_2(s)]ds\\
&+\int_{t_2}^t R(s)[\beta_1Q_1(s)-\alpha_2Q_2(s)]ds\\
\geq\ & 1-p_1-\frac{1+3p_1}{4}\frac{4}{1+3p_1}[(1-p_1)-M_2]=M_2,\quad t\geq t_2.
\end{aligned}$$

Thus we proved that $T_2A_2\subset A_2$. Since A_2 is a bounded, closet and convex subset of X we have to prove that T_2 is a contraction mapping on A_2 to apply the contraction principle.

Now for $x_1,\ x_2\in A_2$ and $t\geq t_2$ we have

$$\begin{aligned}
&|(T_2x_1)(t)-(T_2x_2)(t)|\\
\leq\ & p_1|x_1(t-\tau)-x_2(t-\tau)|+R(t)\int_t^\infty Q_1(s)|f(x_1(s-\sigma_1))-f(x_2(s-\sigma_1))|ds\\
&+R(t)\int_t^\infty Q_2(s)|g(x_1(s-\sigma_2))-g(x_2(s-\sigma_2))|ds\\
&+\int_{t_2}^t R(s)Q_1(s)|f(x_1(s-\sigma_1))-f(x_2(s-\sigma_1))|ds\\
&+\int_{t_2}^t R(s)Q_2(s)|g(x_1(s-\sigma_2))-g(x_2(s-\sigma_2))|ds\\
\leq\ & p_1\|x_1-x_2\|\\
&+L_2\|x_1-x_2\|\{\int_t^\infty R(s)[Q_1(s)+Q_2(s)]ds+\int_{t_2}^t R(s)[Q_1(s)+Q_2(s)]ds\}\\
=\ & \|x_1-x_2\|\{p_1+L_2\int_{t_1}^\infty R(s)[Q_1(s)+Q_2(s)]ds\}\\
=\ & \|x_1-x_2\|\{p_1+L_2\frac{3(1-p_1)}{4L_2}\}\\
=\ & \frac{3+p_1}{4}\|x_1-x_2\|=q_1\|x_1-x_2\|,\quad \text{where we used sup norm.}
\end{aligned}$$

This immediately implies that

$$\|(T_2x_1)(t)-(T_2x_2)(t)\|\leq q_1\|x_1-x_2\|,$$

where in view of (6), $q_1<1$, which proves that T_2 is a contraction mapping, consequently T_2 has the unique fixed point x, which is obviously a positive solution of (E).

(ii). Suppose (7) holds. Choose constants $N_3 \geq M_3 > 0$ such that

$$0 < M_3 < 1 + p_2 \quad \text{and} \quad N_3 > \frac{4}{3}.$$

Set

$$A_3 = \{x \in X : M_3 \leq x(t) \leq N_3, \ t \geq t_0\}.$$

Define $L_3, \ \alpha_1, \ \beta_1, \ \alpha_2, \ \beta_2$ as in Theorem 1 with A_3 instead of A_1. Choose a $t_3 > t_0 + \rho$ sufficiently large such that

$$-1 < \frac{3p_2 - 1}{4} \leq p(t) \leq 0, \quad t \geq t_3 \tag{13}$$

$$\int_{t_3}^{\infty} R(s)[Q_1(s) + Q_2(s)]ds < \frac{3(1 + p_2)}{4L_3}, \tag{14}$$

$$0 \leq \int_{t_3}^{\infty} R(s)[\alpha_1 Q_1(s) - \beta_2 Q_2(s)]ds < (1 + p_2)(\frac{3}{4}N_3 - 1), \quad \text{and} \tag{15}$$

$$\int_{t_3}^{\infty} R(s)[\beta_1 Q_1(s) - \alpha_2 Q_2(s)]ds \geq 0. \tag{16}$$

Define a mapping $T_3 : A_3 \to X$ as follows

$$(T_3 x)(t) = \begin{cases} 1 + p_2 - p(t)x(t - \tau) \\ +R(t) \int_t^{\infty} [Q_1(s) f(x(s - \sigma_1)) - Q_2(s) g(x(s - \sigma_2))]ds \\ + \int_{t_3}^{t} R(s)[Q_1(s) f(x(s - \sigma_1)) - Q_2(s) g(x(s - \sigma_2))]ds, & t \geq t_3, \\ \\ (T_3 x)(t_3) & t_0 \leq t \leq t_3. \end{cases}$$

Clearly, $T_3 x$ is continuous. For every $x \in A_3$ and $t \geq t_3$, using (13) and (15) we get

$$\begin{aligned} & (T_3 x)(t) \\ \leq \; & 1 + p_2 - \frac{3p_2 - 1}{4} N_3 + \int_{t_3}^{\infty} R(s)[\alpha_1 Q_1(s) - \beta_2 Q_2(s)]ds \\ \leq \; & 1 + p_2 - \frac{3p_2 - 1}{4} N_3 + (1 + p_2)(\frac{3}{4} N_3 - 1) \\ = \; & N_3. \end{aligned}$$

Furthermore, in view of (16) we have

$$\begin{aligned} & (T_3 x)(t) \\ \geq \; & 1 + p_2 + R(t) \int_t^{\infty} [\beta_1 Q_1(s) - \alpha_2 Q_2(s)]ds + \int_{t_3}^{t} R(s)[\beta_1 Q_1(s) - \alpha_2 Q_2(s)]ds \\ \geq \; & 1 + p_2 \; > \; M_3. \end{aligned}$$

Thus, we proves that $T_3A_3 \subset A_3$. Since A_3 is a bounded, closed and convex subset of X, we have t_0 prove that T_3 is a contraction mapping on A_3 to apply the contraction principle.

Now, for $x_1,\ x_2 \in A_3$ and $t \geq t_3$, in view of (14) we have

$$\begin{aligned} & |(T_3x_1)(t) - (T_3x_2)(t)| \\ \leq\ & -p_2\|x_1 - x_2\| + L_3\|x_1 - x_2\| \int_{t_3}^{\infty} R(s)[Q_1(s) + Q_2(s)]ds \\ \leq\ & \|x_1 - x_2\|\{-p_2 + \frac{3(1+p_2)}{4}\} \ = \ \frac{3-p_2}{4}\|x_1 - x_2\| \\ =\ & q_2\|x_1 - x_2\|, \quad \text{where we used sup norm.} \end{aligned}$$

This immediately implies

$$\|(T_3x_1)(t) - (T_3x_2)(t)\| \leq q_2\|x_1 - x_2\|,$$

where in view of (7), $q_2 < 1$. This proves that T_3 is a contraction mapping. consequently, T_3 has the unique fixed point x, which is obviously a positive solution of (E). This completes the proof of Theorem 2. □

Theorem 3. *Suppose that conditions* $(c_1) - (c_3)$ *hold and if one of the following two conditions is satisfied*:

(17) $\quad$ (i) $\quad p(t) > 1$ eventually, and $1 < p_2 \leq p_1 < p_2^2 < +\infty;$

(18) $\quad$ (ii) $\quad p(t) < -1$ eventually, and $-\infty < p_2 \leq p_1 < -1,$

where p_1 *and* p_2 *are defined as in theorem 2. Then* (E) *has a non-oscillatory solution.*

Proof. (i). Suppose that (17) holds. Set $0 < \varepsilon < p_2 - 1$ be sufficiently small such that

$$1 < p_2 - \varepsilon < p_1 + \varepsilon < (p_2 - \varepsilon)^2. \tag{19}$$

Then

$$\frac{1}{p_2 - \varepsilon} < \frac{p_2 - \varepsilon}{p_1 + \varepsilon}. \tag{20}$$

Choose constants $N_4 \geq M_4 > 0$ such that

$$\frac{1}{p_2 - \varepsilon} < N_4 < \frac{p_2 - \varepsilon}{p_1 + \varepsilon}, \text{ and} \tag{21}$$

$$0 < M_4 \leq \frac{1}{p_1 + \varepsilon} - \frac{1}{p_2 - \varepsilon}N_4. \tag{22}$$

Let X be the set as in theorem 1. Set

$$A_4 = \{x \in X : M_4 \leq x(t) \leq N_4, t \geq t_0\}.$$

Choose a $t_4 > t_0 + \rho$ sufficiently large such that

$$p_2 - \varepsilon \le p(t) \le p_1 + \varepsilon \text{ for } t \ge t_4, \tag{23}$$

$$\int_{t_4}^{\infty} R(s)[Q_1(s) + Q_2(s)]ds < \frac{p_1 + p_2}{L_4(p_1 + \varepsilon)}, \tag{24}$$

$$0 \le \int_{t_4}^{\infty} R(s)[\alpha_1 Q_1(s) + \beta_2 Q_2(s)]ds \le (p_2 - \varepsilon)N_4 - 1, \text{ and} \tag{25}$$

$$\int_{t_4}^{\infty} R(s)[\beta_1 Q_1(s) - \alpha_2 Q_2(s)]ds \ge 0, \tag{26}$$

where $\alpha_1,\ \beta_1,\ \alpha_2,\ \beta_2,\ L_4$ are defined as in theorem 1, but with A_4 instead of A_1.

Define a mapping $T_4 : A_4 \to X$ as follows

$$(T_4x)(t) = \begin{cases} \frac{1}{p(t+\tau)} - \frac{1}{p(t+\tau)}x(t+\tau) \\ +\frac{R(t+\tau)}{p(t+\tau)} \int_{t+\tau}^{\infty} [Q_1(s)f(x(s-\sigma_1)) - Q_2(s)g(x(s-\sigma_2))]ds \\ +\frac{1}{p(t+\tau)} \int_{t_4}^{t+\tau} R(s)[Q_1(s)f(x(s-\sigma_1)) \\ -Q_2(s)g(x(s-\sigma_2))]ds, & t \ge t_4, \\ \\ (T_4x)(t_4), & t_0 \le t \le t_4, \end{cases}$$

where $t + \tau \ge t_0 + \max\{\sigma_1, \sigma_2\}$. Clearly, T_4x is continuous. For every $x \in A_4$ and $t \ge t_4$, using (25) we get

$$\begin{aligned} (T_4x)(t) &\le \frac{1}{p_2 - \varepsilon} + \frac{1}{p_2 - \varepsilon} \int_{t_4}^{\infty} R(s)[\alpha_1 Q_1(s) - \beta_2 Q_2(s)]ds \\ &\le \frac{1}{p_2 - \varepsilon} + \frac{1}{p_2 - \varepsilon}[(p_2 - \varepsilon)N_4 - 1] = N_4. \end{aligned}$$

Furthermore, in view of (21)and (26) we have

$$\begin{aligned} (T_4x)(t) &\ge \frac{1}{p_1 + \varepsilon} - \frac{1}{p_2 - \varepsilon}N_4 + \frac{1}{p_1 + \varepsilon}R(t+\tau) \int_{t+\tau}^{\infty} [\beta_1 Q_1(s) - \alpha_2 Q_2(s)]ds \\ &\quad \frac{1}{p_1 + \varepsilon} \int_{t_4}^{t+\tau} R(s)[\beta_1 Q_1(s) - \alpha_2 Q_2(s)]ds \\ &\ge M_4. \end{aligned}$$

Thus, we proved that $T_4 A_4 \subset A_4$. Since A_4 is a bounded, closed and convex subset of X, we have t_0 prove that T_4 is a contraction mapping on A_4 to apply the contraction principle.

Now, for $x_1,\ x_2 \in A_4$ and $t \geq t_4$, in view of (24) we have

$$\begin{aligned} & |(T_4x_1)(t) - (T_4x_2)(t)| \\ \leq\ & -\frac{1}{p_1+\varepsilon}\|x_1 - x_2\| + \frac{L_4}{p_2-\varepsilon}\|x_1-x_2\|.\int_{t_4}^{\infty} R(s)[Q_1(s)+Q_2(s)]ds \\ \leq\ & \|x_1-x_2\|\{-\frac{1}{p_1+\varepsilon} + \frac{1}{p_2-\varepsilon}(1+\frac{p_2-\varepsilon}{p_1+\varepsilon})\} \\ =\ & \frac{1}{p_2-\varepsilon}\|x_1-x_2\| = q_3\|x_1-x_2\|, \end{aligned}$$

where we used sup norm. This immediately implies that

$$\|(T_4x_1)(t) - (T_4x_2)(t)\| \leq q_3\|x_1 - x_2\|.$$

In view of (20), $q_3 < 1$ which proves that T_4 is a contraction mapping. consequently, T_4 has the unique fixed point x, which is obviously a positive solution of (E).

(ii) Suppose that (18) holds, set $0 < \delta < -(1+p_2)$ be sufficiently small such that

$$p_2 - \delta < p_1 + \delta < -1. \tag{27}$$

Choose constant $N_5 \geq M_5 > 0$ such that

$$M_5 < \frac{-1}{1+p_2-\delta} < \frac{-1}{1+p_1+\delta} < N_5. \tag{28}$$

Let X be the set as in theorem 1 set

$$A_4 = \{x \in X : M_4 \leq x(t) \leq M_4,\quad t \geq t_0\}$$

Choose a $t_5 > t_0 + \rho$ sufficiently large such that (c_3) holds and

$$p_2 - \delta < p(t) < p_1 + \delta \quad \text{for} \quad t \geq t_5 \tag{29}$$

$$\int_{t_5}^{\infty} R(s)[Q_1(s)+Q_2(s)]ds < -\frac{1+p_1+\delta}{L_5}, \tag{30}$$

$$0 \leq \int_{t_5}^{\infty} R(s)[\alpha_1 Q_1 - \beta_2 Q_2]ds \leq \frac{p_1+\delta}{p_2-\delta}[1+M_5(1+p_2-\delta)], \tag{31}$$

$$\int_{t_5}^{\infty} R(s)[\beta_1 Q_1(s) - \alpha_2 Q_2(s)]ds \geq 0, \tag{32}$$

where $\alpha_1, \beta_1, \alpha_2, \beta_2, L_5$ are defined as in theorem 1 with A_5 instead of A_1.

Define a mapping $T_5 \to X$ as follows

$$(T_5X)(t) = \begin{cases} \frac{-1}{p(t+\tau)} - \frac{x(t+\tau)}{p(t+\tau)} \\ +\frac{R(t+\tau)}{p(t+\tau)}\int_{t+\tau}^{\infty}[Q_1(s)f(x(s-\sigma_1)) - Q_2(s)g(x(s-\sigma_2))]ds \\ +\frac{1}{p(t+\tau)}\int_{t_5}^{t+\tau} R(s)[Q_1(s)f(x(s-\sigma_1)) - Q_2(s)g(x(s-\sigma_2))]ds, & t \geq t_5, \\ (T_5x)(t), & t_0 \leq t \leq t_5, \end{cases}$$

where $t+\tau \geq t_0+\max\{\sigma_1,\sigma_2\}$. Clearly, T_5x is continuous, for every $x \in A_5$ and $t \geq t_5$, using (c_3) and (32) we get

$$\begin{aligned}(T_5X)(t) \quad &\leq \quad \frac{-1}{p_1+\delta}+\frac{1}{p_1+\delta}N_5+\frac{R(t+\tau)}{p_2-\delta}\int_{t+\tau}^{\infty}[\beta_1Q_1(s)-\alpha_2Q_2(s)]ds \\ &\quad +\frac{1}{p_2-\delta}\int_{t_5}^{t+\tau}[\beta_1Q_1(s)-\alpha_2Q_2(s)]ds \\ &\leq \quad \frac{-1}{p_1+\delta}+\frac{-1}{p_1+\delta}N_5<N_5.\end{aligned}$$

Since the first inequality of (28). Furthermore, in view of (28) and (31) we have

$$\begin{aligned}(T_5X)(t) \quad &\geq \quad \frac{-1}{p_2-\delta}+\frac{-1}{p_2-\delta}M_5+\frac{1}{p_1+\delta}\int_{t_5}^{\infty}R(s)[\alpha_1Q_1(s)-\beta_2Q_2(s)]ds \\ &\geq \quad \frac{-1}{p_2-\delta}+\frac{-1}{p_2-\delta}M_5+\frac{1}{p_1+\delta}\cdot\frac{p_1+\delta}{p_2-\delta}[1+M_5(1+p_2-\delta)]=M_5.\end{aligned}$$

Thus, we proved that $T_5A_5 \subset A_5$. Since A_5 is a bounded, closed and convex subset of X, we have t_0 prove that T_5 is a contraction mapping on A_5 to apply the contraction principle.

Now, for $x_1,\ x_2 \in A_5$ and $t \geq t_5$, in view of (30) we get

$$\begin{aligned}&|(T_5x_1)(t)-(T_5x_2)(t)| \\ \leq\quad & -\frac{1}{p_1+\delta}|x_1(t+\tau)-x_2(t+\tau)| \\ &+\frac{R(t+\tau)}{p(t+\tau)}\int_{t+\tau}^{\infty}Q_1(s)[f(x_1(s-\sigma_1))-f(x_2(s-\sigma_1))]ds \\ &+\frac{R(t+\tau)}{p(t+\tau)}\int_{t+\tau}^{\infty}Q_2(s)[g(x_1(s-\sigma_2))-g(x_2(s-\sigma_2))]ds \\ &+\frac{1}{p(t+\tau)}\int_{t_5}^{t+\tau}R(s)Q_1(s)[f(x_1(s-\sigma_1))-f(x_2(s-\sigma_1))]ds \\ &+\frac{1}{p(t+\tau)}\int_{t_5}^{t+\tau}R(s)Q_2(s)[g(x_1(s-\sigma_2))-g(x_2(s-\sigma_2))]ds \\ \leq\quad & -\frac{1}{p_1+\delta}\|x_1-x_2\|-\frac{L_5}{p_2-\delta}\|x_1-x_2\| \\ &\times\Big\{\int_{t+\tau}^{\infty}R(s)[Q_1(s)+Q_2(s)]ds+\int_{t_5}^{t+\tau}R(s)[Q_1(s)+Q_2(s)]ds\Big\} \\ \leq\quad & \|x_1-x_2\|\cdot\{-\frac{1}{p_1+\delta}-\frac{L_5}{p_2-\delta}\int_{t_5}^{\infty}R(s)[Q_1(s)+Q_2(s)]ds\} \\ <\quad & \|x_1-x_2\|\cdot\{-\frac{1}{p_1+\delta}+\frac{1+p_1+\delta}{p_2-\delta}\} \\ =\quad & q_4\|x_1-x_2\|,\end{aligned}$$

where we used sup norm. This immediately implies that

$$\|(T_5x_1)(t) - (T_5x_2)(t)\| \le q_4\|x_1 - x_2\|,$$

where in view of (27), $q_4 < 1$ which proved that T_5 is a contraction mapping. Consequently, T_5 has the unique fixed point x, which is obviously a positive solution of (E). This completes the proof of theorem 3. □

Remark. If $f(x(t)) = g(x(t)) = x(t), r(t) = 1$ and $p(t) = p = const.$, then theorem 2 and 3 improve the theorem of Kulenovic and Hadziomerspahic ([6]).

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