Ideal Theory in Commutative A-semirings

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ABSTRACT. In this paper, we investigate and characterize the class of A-semirings. A characterization of the Thierrin radical of a proper ideal of an A-semiring is given. Moreover, when P is a Q-ideal in the semiring R, it is shown that P is primary if and only if R/P is nilpotent.

The concept of semirings was introduced by H. S. Vandiver in 1935 and has since then been studied by many authors (e.g., [1]-[5], [11]-[13]). In several papers from 1956 to 1958, K. Iséki ([7]-[10]) developed a large amount of ideal theory for semirings that are not necessarily commutative under either operation. Many of Iséki's results were topological in nature; however, he gave several characterizations of prime ideals and has defined and studied the Thierrin radical of an ideal. It is the purpose of this paper to present a development of ideal theory for commutative semirings and to connect this theory with the theory developed by Iséki. P. J. Allen ([1]) introduced the notion of a Q-ideal and a construction process was presented by which one can build the quotient structure of a semiring modulo a Q-ideal. Maximal homomorphisms were defined and examples of such homomorphisms were given. Using these notions, the Fundamental Theorem of Homomorphisms for rings was generalized to include a large class of semirings. The results proven in [1] will be used throughout this paper. Since the theory of ideals plays an important role in the theory of quotient semirings, in this paper, we will make an intensive study of the notions of prime, completely prime, and primary ideals in commutative semrings. The notion of an A-semiring will be defined and a characterization of an A-semiring will be presented. With the aid of these notions, further algebraic properties of the radical of an ideal in an A-semiring will be given. It will also be shown that a proper Q-ideal I in the semiring R is primary if and only if every zero divisor in

R/I in nilpotent.

There are many different definitions of a semiring appearing in the literature. Throughout this paper, a semiring will be defined as follows:

Definition 1. A set R together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$, respectively) will be called a *semiring* provided:

  (i) addition is a commutative operation;

 (ii) there exist $0 \in R$ such that $x + 0 = x$ and $x0 = 0x = 0$ for each $x \in R$, and

 (iii) multiplication distributes over addition both from the left and from the right.

Definition 2. A subset I of a semiring R will be called an *ideal* if $a, b \in I$ and $r \in R$ implies $a + b \in I, ra \in I$ and $ar \in I$.

Definition 3. An ideal P in the semiring R is said to be *prime* provided;

  (1) $P \neq R$; and

  (2) if A and B are ideals in R such that $AB \subset P$, then $A \subset P$ or $B \subset P$, where $AB = \{ab \,|\, a \in A \text{ and } b \in B\}$.

Definition 4. An ideal P in the semiring R is said to be *completely prime* provided;

  (1) $P \neq R$; and

  (2) if $a, b \in R$ such that $ab \in P$, then $a \in P$ or $b \in P$.

Prime ideals in commutative semirings can be characterized in the following way:

Theorem 5. *If P is a proper ideal in a commutative semiring R, then the following statements are equivalent:*

  (1) *P is prime*;

  (2) *P is completely prime*;

  (3) *if A and B are ideals in R such that $P \subsetneq A$ and $P \subsetneq B$, then $AB \nsubseteq P$.*

Proof. Iséki ([8]) proved that prime and completely prime are equivalent concepts in a commutative semiring.

$(1) \Rightarrow (3)$: Let P be a prime ideal in the commutative semiring R, and let A and B be ideals in R such that $P \subsetneq A$ and $P \subsetneq B$. Assume $AB \subseteq P$. Since P is prime, it follows that $A \subset P$ or $B \subset P$, a contradiction. Therefore, $AB \nsubseteq P$.

$(3) \Rightarrow (2)$: Assume P is not completely prime. Thus, there exists $a, b \in R$ such that $ab \in P$, where $a \notin P$ and $b \notin P$. Let N denote the natural numbers and let

$Na = \{na \mid n \in N\}$, where na denotes n sums of a. Let C_a denote the union of the following collection of subsets of R:

$$\{P, Ra, Na, P + Ra, P + Na, Ra + Na, P + Ra + Na\}.$$

Similarly, let C_b denote the union of the collection

$$\{P, Rb, Nb, P + Rb, P + Nb, Rb + Nb, P + Rb + Nb\}.$$

An inspection will show that C_a and C_b are ideals in R. Moreover, $P \subsetneq C_a$ and $P \subsetneq C_b$. Since P satisfies statement (3), it follows that $C_a C_b \nsubseteq P$. Moreover, an inspection of the form of each element in $C_a C_b$ shows that $C_a C_b \subset P$, a contradiction. $\qquad\square$

Definition 6. Let S be a non-empty, multiplicatively closed subset ($s_1, s_2 \in S$ implies $s_1 s_2 \in S$) of a semiring R. An ideal P of R is said to be *maximal with respect to S* provided;

(1) $P \cap S = \emptyset$; and

(2) if A is an ideal of R such that $P \subsetneq A$, then $A \cap S \neq \emptyset$.

Theorem 7. *Let S be a non-empty, multiplicatively closed subset of a semiring R. If A is an ideal in R such that $A \cap S = \emptyset$, then there exists an ideal P of R such that $A \subset P$ and P is maximal with respect to S.*

Proof. Let F denote the collection of all ideals in R which contain A and are disjoint from S. Since $A \in F$, it follows that $F \neq \emptyset$. Define a relation $\leq$ on F by $B_1 \leq B_2 \Leftrightarrow B_1 \subset B_2$. F is a partially ordered set under the relation $\leq$. If $\{B_i\}_{i \in I}$ is a non-empty chain in F, then $\cup_{i \in I} B_i \in F$ and $B_i \leq \cup_{i \in I} B_i$, for every $i \in I$. Thus, every non-empty chain in F has an upper bound in F. Zorn's lemma implies that F has a maximal element. Such a maximal element satisfies the conclusion of the theorem. $\qquad\square$

Theorem 8. *Let S be a non-empty, multiplicatively closed subset of a commutative semiring R and let P be an ideal in R. If P is maximal with respect to S, then P is prime.*

Proof. Let B and C be ideals in R such that $P \subsetneq B$ and $P \subsetneq C$. Since P is maximal with respect to S, it follows that $B \cap S \neq \emptyset$ and $C \cap S \neq \emptyset$. Let $b \in B \cap S$ and $c \in C \cap S$. Since S is closed under multiplication, it follows that $bc \in (BC) \cap S$. Since $P \cap S = \emptyset$, it follows that $BC \nsubseteq P$. Theorem 5 implies P is prime. $\qquad\square$

Definition 9. An ideal M in a semiring R is said to be *maximal* provided;

(1) $M \subsetneq R$; and

(2) if A is an ideal in R such that $M \subsetneq A$, then $A = R$.

For semirings with an identity, Definition 6 and Definition 7 can be connected by the following:

Theorem 10. *Let M be an ideal in the semiring R. If R has an identity 1, then the following statements are equivalent*:

(1) M *is maximal*;

(2) M *is maximal with respect to* $\{1\}$.

Proof. $(1) \Rightarrow (2)$: Since M is maximal, it follows that $M \cap \{1\} = \emptyset$. Theorem 7 implies there exists an ideal P in R such that P is maximal with respect to $\{1\}$ and $M \subset P$. Clearly, $P \cap \{1\} = \emptyset$ implies $P \subsetneq R$. Since M is maximal, it follows that $M = P$. Thus, M is maximal with respect to $\{1\}$.

$(2) \Rightarrow (1)$: Assume M is not maximal. Then there exists an ideal A in R such that $M \subsetneq A \subsetneq R$. Clearly, $A \subsetneq R$ implies $A \cap \{1\} = \emptyset$. Thus M is not maximal with respect to $\{1\}$, a contradiction. $\qquad\square$

The following theorem is an immediate consequence of Theorem 8 and Theorem 10.

Theorem 11. *Let R be a commutative semiring with an identity. If M is a maximal ideal in R, then M is prime.*

The following example shows that a maximal ideal in a commutative semiring without an identity may not be prime.

Example 12. Let R denote the semiring of positive even integers with the usual addition and multiplication. If $M = \{x \in R \mid x > 2\}$, then M is a maximal ideal in R. Since $2 \notin M$ and $2 \cdot 2 = 4 \in M$, it follows that M is not prime.

In Iséki's development of the Thierrin radical of an ideal, it was necessary to know that an ideal was always contained in a completely prime ideal. Consequently, Iséki did not demand that a completely prime ideal or a prime ideal be proper. Thus, any semiring was a prime and a completely prime ideal. Under the proper circumstances, the Thierrin radical of an ideal could be the entire semiring.

When defining the radical of an ideal in this paper, it will be necessary to know that a proper ideal is always contained in a prime ideal. Since Definition 3 demands that a prime ideal be a proper ideal, we cannot attack this problem in Iséki's manner.

Definition 13. A semiring R is said to be an *A-semiring* provided;

(1) R is commutative; and

(2) every proper ideal in R is contained in a prime ideal of R.

Theorem 14. *A commutative semiring R is an A-semiring if and only if the complement of every proper ideal contains a non-empty multiplicatively closed set.*

Proof. If R is an A-semiring , it is clear that the complement of every proper ideal contains a non-empty multiplicatively closed set. Let B be a proper ideal in R and let S be a non-empty multiplicatively closed subset of $R - B$. Theorem 7 implies there exists an ideal P in R such that $B \subset P$ and P is maximal with respect to S. Theorem 8 implies P is prime. □

Corollary 15. *If R is a commutative semiring with an identity 1, then R is an A-semiring .*

Proof. If B is a proper ideal in R, then $\{1\} \subset R - B$. □

The following examples will show there exist A-semirings that do not have an identity.

Example 16. Let R be the semiring of non-negative integers where $a + b = \max\{a, b\}$ and $ab = \min\{a, b\}$. Then R does not have an identity, and every proper ideal in R is prime.

Example 17. Let J_4 denote the ring of integers modulo 4 and let R be the semiring in Example 16. Let $J_4 \oplus R = \{(a, b) \,|\, a \in J_4 \text{ and } b \in R\}$ denote the direct sum of the semirings J_4 and R. Then $J_4 \oplus R$ is a commutative semiring. Clearly, $J_4 \oplus R$ does not have an identity. If $m \in R$, an inspection will show that $I_m = \{(0, n) \in J_4 \oplus R \,|\, n \leq m\}$ is a proper ideal in $J_4 \oplus R$. Moreover, I_m is not prime, for any $m \in R$. The following argument will show that $J_4 \oplus R$ is an A-semiring . Let M be a proper ideal in $J_4 \oplus R$. Assume $(1, m)$ belongs to M, for each $m \in R$. Consequently, $(2, m) = (1, m) + (1, m) \in M$, $(3, m) = (1, m) + (2, m) \in M$, and $(0, m) = (2, m) + (2, m) \in M$ for each $m \in R$. Therefore, $M = J_4 \oplus R$, a contradiction. Thus, there exists an $m_0 \in R$ such that $(1, m_0) \notin M$, and it is clear that $\{(1, m_0)\}$ is a non-empty multiplicatively closed subset of $(J_4 \oplus R) - M$. Theorem 14 implies $J_4 \oplus R$ is an A-semiring .

Definition 18. Let B be a proper ideal in an A-semiring R. The *radical* of B is denoted by $\sqrt{B}$ and is defined to be the intersection of all of the prime ideals in R that contain B.

The following theorem is an immediate consequence of Definition 18.

Theorem 19. *If B is a proper ideal in an A-semiring R, then $\sqrt{B}$ is an ideal in R and $B \subset \sqrt{B}$.*

Definition 20. A proper ideal B in an A-semiring R is said to be *semi-prime* if $B = \sqrt{B}$.

Theorem 21. *If B is a prime ideal in an A-semiring R, then B is semi-prime.*

Proof. Theorem 19 implies $B \subset \sqrt{B}$. Let $\{P_i\}_{i \in I}$ be the collection of all prime ideals in R that contain B. Clearly, $B \in \{P_i\}_{i \in I}$ and $\sqrt{B} = \cap_{i \in I} P_i \subset B$. Thus, $B = \sqrt{B}$. □

The following example will show that there exist semi-prime ideals that are not

prime.

Example 22. The semiring Z^+ of non-negative integers is an A-semiring . Let (6) denote the ideal generated by 6. Since $1 \in Z^+$, it follows that $(6) = \{6n \,|\, n \in Z^+\}$. Since $2 \notin (6)$, $3 \notin (6)$ and $2 \cdot 3 = 6 \in (6)$, it is clear that (6) is not prime. The only prime ideals in Z^+ that contain (6) are $(2), (3)$ and $\{0\} \cup \{x \in Z^+ \,|\, x > 1\}$. Since $(2) \cap (3) \cap (\{0\} \cup \{x \in Z^+ \,|\, x > 1\})$ is equal to (6), it follows that $(6) = \sqrt{(6)}$. Therefore, (6) is semi-prime.

Theorem 23. *If B is a proper ideal in an A-semiring R, then $\sqrt{B} = \{x \in R| \; \exists n \in N$ such that $x^n \in B\}$, where N denotes the natural numbers.*

Proof. Let $x \in R$ such that for some positive integer n it is valid that $x^n \in B$. Let P be a prime ideal in R that contains B. Since P is prime and $x^n \in P$, it follows that $x \in P$. Since P was an arbitrary prime ideal in R containing B, it follows that $x \in \sqrt{B}$. Conversely, assume there exists $x \in \sqrt{B}$ such that $x^n \notin B$, for any $n \in N$. Choose one such x and let $S = \{x^n \,|\, n \in N\}$. Since S is a non-empty, multiplicatively closed set in R such that $S \cap B = \emptyset$, Theorem 7 implies that there exists an ideal P in R such that $B \subset P$ and P is maximal with respect to S. Theorem 8 implies that P is prime. Since $P \cap S = \emptyset$ and $x \in S$, it follows that $x \notin P$. Thus $x \notin \sqrt{B}$, a contradiction. $\square$

Corollary 24. *If B is a proper ideal in an A-semiring R, then $\sqrt{B}$ is semi-prime.*

Proof. It is clear that the radical of B is a proper ideal in R. Theorem 19 implies $\sqrt{B} \subset \sqrt{\sqrt{B}}$. Let $x \in \sqrt{\sqrt{B}}$. Theorem 23 implies that there exists an $n \in N$ such that $x^n \in \sqrt{B}$. Moreover, Theorem 23 now implies that there exists an $m \in N$ such that $(x^n)^m \in B$. Thus, $x^{n \cdot m} \in B$ and Theorem 23 implies $x \in \sqrt{B}$. $\square$

Corollary 25. *Let A and B be proper ideals in an A-semiring R. Then*

(1) *if $A \subset B$, then $\sqrt{A} \subset \sqrt{B}$;*

(2) *if $A \cap B \neq \emptyset$, then $\sqrt{A \cap B} = \sqrt{A} \cap \sqrt{B}$.*

Proof. (1) If $x \in \sqrt{A}$, then there exists an $n \in N$ such that $x^n \in A \subset B$. Hence $x \in \sqrt{B}$.

(2) Since $A \cap B \neq \emptyset$, it is clear that $A \cap B$ is an ideal in R. Let $x \in \sqrt{A \cap B}$. Then there exists an $n \in N$ such that $x^n \in A \cap B$. Therefore, $x^n \in A$ and $x^n \in B$ and it follows that $x \in \sqrt{A}$ and $x \in \sqrt{B}$. Hence, $x \in \sqrt{A} \cap \sqrt{B}$. Consequently, $x \in \sqrt{A} \cap \sqrt{B}$ implies that there exist $n, m \in N$ such that $x^n \in A$ and $x^m \in B$. Clearly, $x^{n \cdot m} \in A \cap B$. Thus, $x \in \sqrt{A \cap B}$. $\square$

Theorem 23 gives an important characterization of the radical of an ideal in an A-semiring . Another characterization of the radical of an ideal will be developed. With the aid of this characterization, the concepts of the radical of an ideal in an A-semiring and of the Thierrin radical of an ideal can be connected.

Definition 26. Let B be an ideal in a semiring R. A prime ideal M in R will be called a *minimal prime (completely prime, resp.) divisor* of B provided;

(1) $B \subset M$;

(2) if C is a prime (completely prime, resp.) ideal in R such that $B \subsetneq C \subset M$, then $C = M$.

Theorem 27. *If B is a proper ideal in an A-semiring R, then there exists a minimal prime divisor of B. Moreover, if P is a prime ideal in R such that $B \subset P$, then there exists a minimal prime divisor M of B such that $B \subset M \subset P$.*

Proof. Since B is a proper ideal in an A-semiring R, it follows that there exists a prime ideal in R containing B. Pick any such ideal and call it P. Let $F = \{C \subset P \,|\, B \subset C$ and C is a prime ideal in $R\}$. Define the relation $\leq$ on F by $C_1 \leq C_2 \Leftrightarrow C_2 \subset C_1$. F is a partially ordered set under the relation $\leq$. Let $\{C_i\}_{i \in I}$ be a non-empty chain in F. Clearly, $\cap_{i \in I} C_i$ is an ideal in R. It is clear that $B \subset \cap_{i \in I} C_i \subset P$. It will follow that $\cap_{i \in I} C_i$ is an element in F if it can be shown that $\cap_{i \in I} C_i$ is prime. Let $a, b \in R$ such that $ab \in \cap_{i \in I} C_i$. Clearly, $ab \in C_i$ for any $i \in I$. Suppose that there exists an $i_0 \in I$ such that $a \notin C_{i_0}$. Since C_{i_0} is prime, it is clear that $b \in C_{i_0}$. Let $C_i \in \{C_i\}_{i \in I}$. If $C_i \subset C_{i_0}$, then $b \in C_i$; otherwise, C_i prime implies $a \in C_i \subset C_{i_0}$, a contradiction. If $C_{i_0} \subset C_i$, it is clear that $b \in C_i$. Since $\{C_i\}_{i \in I}$ is a chain in F, it follows that $b \in C_i$ for any $i \in I$, i.e., $b \in \cap_{i \in I} C_i$. Consequently, $\cap_{i \in I} C_i$ is prime. Therefore $\cap_{i \in I} C_i \in F$ and $\cap_{i \in I} C_i$ is an upper bound of the chain $\{C_i\}_{i \in I}$. Zorn's lemma implies that F has a maximal element. Pick one such element and call it M. Let D be a prime ideal in R such that $B \subsetneq D \subset M$. Clearly, $D \in F$ and $M \leq D$. Since M is maximal in F, it follows that $D = M$. Therefore, M is a minimal prime divisor of B. $\qquad\square$

Theorem 28. *If B is a proper ideal in an A-semiring R, then the radical of B is the intersection of all minimal prime divisors of B.*

Proof. Let $\{B_i\}_{i \in I}$ denote the collection of all prime ideals in R containing B. Let $I^* = \{i \in I \,|\, B_i$ is a minimal prime divisor of $B\}$. It must be shown that $\sqrt{B} = \cap_{i \in I^*} B_i$. Let $x \in \sqrt{B}$. Definition 18 implies $x \in \cap_{i \in I} B_i$. Thus, $x \in B_i$ for any $i \in I$ and it follows that $x \in B_i$ for any $i \in I^*$. Therefore, $x \in \cap_{i \in I^*} B_i$. Conversely, assume $\cap_{i \in I^*} B_i \nsubseteq \sqrt{B}$. Then there exists an $x \in \cap_{i \in I^*} B_i$ such that $x \notin \sqrt{B}$. Hence there exists a prime ideal B_{i_0} in R such that $x \notin B_{i_0}$. Since $B \subset B_{i_0}$ and B_{i_0} is prime, by Theorem 27, there exists a minimal prime divisor $B_j \in \{B_i\}_{i \in I^*}$ such that $B \subset B_j \subset B_{i_0}$. However, $B_j \in \{B_i\}_{i \in I^*}$ and $x \in \cap_{i \in I^*} B_i$ implies that $x \in B_j \subset B_{i_0}$, a contradiction. $\qquad\square$

Iséki ([10]) developed the Thierrin radical of an ideal in an arbitrary semiring as follows:

Definition 29. Let B be an ideal in a semiring R. An element x in R will be called a *T-element* for B if $x = x_1 x_2 \cdots x_n$ such that $x_1^2 x_2^2 \cdots x_n^2 \in B$ for some $x_i \in R$ and for some positive integer n.

The set of all T-elements for the ideal B will be denoted by $T^1(B)$. The ideal generated by $T^1(B)$ will be denoted by $T_1(B)$. If $m > 1$, then $T_m(B)$ is defined recursively as follows: $T_m(B) = T_1(T_{m-1}(B))$. It is clear that $T_m(B) \subset T_{m+1}(B)$ for each positive integer m.

Definition 30. If B is an ideal in a semiring R, then the *Thierrin radical* of B is denoted by $T^*(B)$ and defined by $T^*(B) = \cup_{m=1}^\infty T_m(B)$.

Iséki proved the following characterization of the Thierrin radical.

Theorem 31. *If B is an ideal in a semiring R, the Thierrin radical of B is the intersection of all minimal completely prime divisors of B.*

The following theorem will show that the radical of a proper ideal in an A-semiring is a specialization of the Thierrin radical.

Theorem 32. *If B is a proper ideal in an A-semiring R, then $\sqrt{B} = T^*(B)$.*

Proof. Theorem 5 implies that the notion of prime ideal and completely prime ideal are equivalent in a commutative semiring. Thus, the collection of all minimal prime divisors of B is identical to the collection of all minimal completely prime divisors of B. The theorem follows from Theorem 28 and Theorem 31. □

The remainder of this paper will be devoted to the development of further properties of ideals in commutative semirings.

Theorem 33. *Let R be a semiring with commutative addition and let $P_1, P_2, \cdots, P_n$ be prime, k-ideals in R. If A is an ideal in R such that $A \not\subset P_i$, $i = 1, 2, \cdots, n$, then there exists $a \in A$ such that $a \notin P_i$, $i = 1, 2, \cdots, n$.*

Proof. Without loss of generality, it may be assumed that there is no inclusion among the P_i. Assume that there exists $i_0 \in \{1, 2, \cdots, n\}$ such that

$$P_1 P_2 \cdots P_{i_0-1} P_{i_0+1} \cdots P_n A \subset P_{i_0}.$$

Pick $p_j \in P_j$ such that $p_j \notin P_{i_0}$ for any $j \neq i_0$, and pick $x \in A$ such that $x \notin P_{i_0}$. Thus,

$$a_{i_0} = p_1 p_2 \cdots p_{i_0-1} p_{i_0+1} \cdots p_n x \in P_{i_0}.$$

Since $p_1 \notin P_{i_0}$ (assuming $i_0 \neq 1$) and P_{i_0} is prime, it follows that $p_2 \cdots p_{i_0-1} p_{i_0+1} \cdots p_n x \in P_{i_0}$. Repeating this argument a finite number of times implies that $x \in P_{i_0}$, a contradiction. Therefore, for each $i = 1, 2, \cdots, n$, there exists $a_i \in P_1 P_2 \cdots P_{i-1} P_{i+1} \cdots P_n A$ such that $a_i \notin P_i$. Choose such an a_i, for each $i_0 \in \{1, 2, \cdots, n\}$. Since A is an ideal, it is clear that $\sum_{i=1}^n a_i \in A$. Let $a = \sum_{i=1}^n a_i$. Assume there exists $j_0 \in \{1, 2, \cdots, \}$ such that $a \in P_{j_0}$. Since P_{j_0} is an ideal and $a_i = p_1 \cdots p_{i-1} p_{i+1} \cdots p_{j_0} \cdots p_n x$ for any $i \neq j_0$, it follows that $a_i \in P_{j_0}$ for any $i \neq j_0$. Therefore, $\sum_{i \neq j_0} a_i \in P_{j_0}$. Consequently, P_{j_0} is a k-ideal and $[\sum_{i \neq j_0} a_i] + a_{j_0} = a \in P_{j_0}$ imply $a_{j_0} \in P_{j_0}$, a contradiction. Hence, there exists an $a \in A$ such that $a \notin P_i$ for each $i = 1, 2, \cdots, n$. □

Definition 34. Let P be a proper ideal in a semiring R. If $a, b \in R, ab \in P$ and $a \notin P$ implies $b^n \in P$ for some positive integer n, then P is said to be *primary*.

The following theorem is an immediate consequence of Theorem 23.

Theorem 35. *If P is a proper ideal in an A-semiring R, then the following statements are equivalent*:

(1) *P is primary;*

(2) *if $a, b \in R$ such that $a \notin P$ and $ab \in P$, then $b \in \sqrt{P}$;*

(3) *if $a, b \in R$ such that $ab \in P$ and $b \notin \sqrt{P}$, then $a \in P$.*

Allen [1] has presented the notion of a Q-ideal I in the semiring R and constructed the quotient semiring R/I. The results proven in [1] will be used in the next results.

Lemma 36. *Let R be a semiring with zero and commutative addition, and let P be a Q-ideal in R. If $q \in Q$ and $q + P$ is the zero in R/P, then $q + P = P$.*

Proof. Let $\varphi : R \to R/P$ be the natural homomorphism of R onto R/P. It is clear that $\ker(\varphi) = \{x \in R \,|\, x + P$ is a subset of $q + P\}$. If $x \in \ker(\varphi)$, then

$$x = x + 0 \in x + P \subset q + P.$$

Thus, $\ker(\varphi) \subset q + P$. Since $q \in \ker(\varphi)$ and $\ker(\varphi)$ is an ideal, it follows that $q + q \in \ker(\varphi)$. Thus, there exists an $i \in P$ such that $q + q = q + i$. Since $0 \in \ker(\varphi)$, there exists a $j \in P$ such that $0 = q + j$. Thus, $q = q + 0 = q + (q + j) = (q + q) + j = (q + i) + j = (q + j) + i = 0 + i = i$. Hence, $q \in P$ and it is clear that $q + P \subset P$. Since $0 \in \ker(\varphi)$, it is clear that $P = 0 + P \subset q + P$. Thus, $P = q + P$. □

Theorem 37. *Let R be a semiring with zero and commutative addition, and let P be a proper ideal in R. If P is a Q-ideal, then P is primary if and only if every zero divisor in R/P is nilpotent.*

Proof. Let q be the unique element in Q such that $q + P$ is the zero in R/P. Let P be primary and let $q_1 + P$ and $q_2 + P$ be element in R/P such that $q_1 + P \neq q + P$ and $(q_1 + P) \odot (q_2 + P) = q + P$. If $q_1 \in P$, by Lemma 36, $q_1 \in q + P$ and it would follow that $q_1 \in (q_1 + P) \cap (q + P)$, a contradiction. Thus, $q_1 \notin P$. $(q_1 + P) \odot (q_2 + P) = q + P$ implies $q_1 q_2 + P \subset q + P$. Hence, $q_1 q_2 \in q + P$ and Lemma 36 implies $q_1 q_2 \in P$. Since P is primary, there exists an $n \in Z^+$ such that $q_2^n \in P$. Lemma 36 implies $q_2^n \in q + P$. Since $q_2 \in q_2 + P$, it is clear that $q_2^n \in (q_2 + P)^n$. Thus, $q_2^n \in (q_2 + P)^n \cap (q + P)$, and it follows that $(q_2 + P)^n = q + P$. Since P is primary, every zero divisor in R/P is nilpotent.

Conversely, suppose that every zero divisor in R/P is nilpotent. Let $a, b \in R$ such that $a \notin P$ and $ab \in P$. Lemma 36 implies $a \notin q + P$ and $ab \in q + P$. Since P is a Q-ideal, $a + P \subset q_1 + P$ and $b + P \subset q_2 + P$ for some $q_1, q_2 \in Q$. Thus,

$a = q_1 + i_1$ and $b = q_2 + i_2$ for some $i_1, i_2 \in P$. Since $a \in q_1 + P$ and $a \notin q + P$, it is clear that $q_1 + P \neq q + P$. Clearly,

$$ab = q_1 q_2 + q_1 i_2 + i_1 q_2 + i_1 i_2 \in q_1 q_2 + P.$$

Let q^* be the unique element in Q such that $q_1 q_2 + P \subset q^* + P$. Since $ab \in (q + P) \cap (q^* + P)$, it is clear that $q = q^*$ and $q + P = (q_1 + P) \odot (q_2 + P)$. Since $q_2 + P$ is a zero divisor in R/P, there exists an $n \in Z^+$ such that $(q_2 + P)^n = q + P$. Lemma 36 implies $(q_2 + P)^n = P$. Since $b \in q_2 + P$, it is clear that $b^n \in (q_2 + P)^n$. Thus, $b^n \in P$, and it follows that P is primary. □

Theorem 38. *If P is a primary ideal in an A-semiring R, then $\sqrt{P}$ is prime.*

Proof. Let $a, b \in R$ such that $a \notin \sqrt{P}$ and $ab \in \sqrt{P}$. Since $ab \in \sqrt{P}$, by Theorem 23, there exists an $n \in Z^+$ such that $(ab)^n \in P$. Since R is commutative, it follows that $a^n b^n \in P$. Since $a \notin \sqrt{P}$, by Theorem 23, $a^n \notin P$. Since P is primary, $a^n \notin P$ and $a^n b^n \in P$. By Theorem 35, we have $b^n \in \sqrt{P}$. Thus, there exists an $m \in Z^+$ such that $(b^n)^m \in P$. Clearly, $p = nm \in Z^+$ and $b^p = (b^n)^m \in P$. Therefore, $b \in \sqrt{P}$ and it follows that $\sqrt{P}$ is prime. □

The following is an immediate consequence of Theorem 38.

Corollary 39. *If P is a primary ideal in an A-semiring R, then $\sqrt{P}$ is the unique minimal prime divisor of P.*

If P is a primary ideal in an A-semiring R, Theorem 38 implies that $\sqrt{P}$ is prime. From this result a natural question arises. What conditions on the semiring R and $\sqrt{P}$ will be sufficient to imply that P is primary? The following theorem will answer this question.

Theorem 40. *Let R be a commutative semiring with an identity 1, and let P be a proper ideal in R. If $\sqrt{P}$ is maximal in R, then P is primary.*

Proof. Let $a, b \in R$ such that $ab \in P$ and $b \notin \sqrt{P}$. If it can be shown that $a \in P$, by Theorem 35, P is primary. An inspection will show that $\sqrt{P} \cup Rb \cup (\sqrt{P} + Rb)$ is an ideal in R that properly contains $\sqrt{P}$. Since $\sqrt{P}$ is maximal in R, it follows that $R = \sqrt{P} \cup Rb \cup (\sqrt{P} + Rb)$. If $1 \in Rb$, then $1 = rb$ for some $r \in R$. Hence, $a = a \cdot 1 = a(rb) = r(ab)$. Since P is an ideal and $ab \in P$, it is clear that $a \in P$. Suppose that $1 \in Rb + \sqrt{P}$. Then $1 = p + rb$ for some $p \in \sqrt{P}$ and for some $r \in R$. Clearly, $p \in \sqrt{P}$ implies $p^n \in P$ for some $n \in Z^+$. The binomial theorem implies the following:

$$\begin{aligned}
1 &= 1^n \\
&= (p + rb)^n = p^n + \binom{n}{1}p^{n-1}(rb)^1 + \binom{n}{2}p^{n-2}(rb)^2 + \cdots + (rb)^n \\
&= p^n + \binom{n}{1}p^{n-1}r^1 b^1 + \binom{n}{2}p^{n-2}r^2 b^2 + \cdots + r^n b^n \\
&= p^n + [\binom{n}{1}p^{n-1}r + \binom{n}{2}p^{n-2}r^2 b + \cdots + r^n b^{n-1}]b.
\end{aligned}$$

Thus, $1 = p' + r'b$ where $p' = p^n \in P$ and

$$r' = \binom{n}{1}p^{n-1}r + \binom{n}{2}p^{n-2}r^2b + \cdots + r^n b^{n-1} \in R.$$

It is clear that $a = a \cdot 1 = a(p' + r'b) = ap' + r'(ab)$. Since $p' \in P$, $ab \in P$ and P is an ideal, it follows that $a \in P$. Therefore, P is primary. $\qquad\square$

The following theorem will conclude this paper.

Theorem 41. *Let $P_1, P_2, \cdots, P_n$ be primary ideals in an A-semiring R. If $\sqrt{P_i} = P$ for each $i = 1, 2, \cdots, n$, then $\cap_{i=1}^{n} P_i$ is primary and $\sqrt{\cap_{i=1}^{n} P_i} = P$.*

Proof. Let $x \in P$. Clearly, $\sqrt{P_i} = P$ implies $x^{m_i} \in P_i$ for some $m_i \in Z^+$ where $i = 1, 2, \cdots, n$. Let $m = \max\{m_i\}_{i=1}^{m}$. Thus, $x^m = x^{m-m_i}x^{m_i}$ and $x^{m_i} \in P_i$ for each i. Since each P_i is an ideal, it follows that $x^m \in P_i$ for each i. Thus, $x^m \in \cap_{i=1}^{n} P_i$. Therefore, $x \in \sqrt{\cap_{i=1}^{n} P_i}$.

Conversely, Theorem 25 implies that $\sqrt{\cap_{i=1}^{n} P_i} \subset \sqrt{P_i} = P$. Thus, $\sqrt{\cap_{i=1}^{n} P_i} = P$. The following argument will show that $\cap_{i=1}^{n} P_i$ is primary. Let $a, b \in R$ such that $ab \in \cap_{i=1}^{n} P_i$ and $b \notin P$. Since each P_i is primary, $ab \in P_i$, and $b \notin P = \sqrt{P_i}$, it follows that $a \in P_i$ for each $i = 1, 2, \cdots, n$. Thus, $a \in \cap_{i=1}^{n} P_i$. $\qquad\square$

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