

On Weakly Commutative Abundant Semigroups

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ABSTRACT. (Left or Right) Weakly commutative semigroups are described. Relationships of weakly commutative semigroups and (l- or r-) Archimedean semigroups are discussed. The structure theorems of weakly commutative semigroups and weakly commutative abundant semigroups are shown.

1. Introduction and basic concepts

The abundant semigroups are discussed in [1]. Now, a semigroup S is called left (resp. right) abundant if for $a \in S$, $(a)_{\mathcal{L}^*} \cap E(S) \neq \emptyset$ (resp. $(a)_{\mathcal{R}^*} \cap E(S) \neq \emptyset$). It is clear that S is abundant if and only if S is both left and right abundant.

Weakly commutative semigroups are described in [2]. A semigroup is called left (resp. right) weakly commutative if for any $x, y \in S$, $(xy)^n \in yS$ (resp. $(xy)^n \in Sx$) for some $n \in N^+$. It is easy to see that semigroup S is weakly commutative if and only if S is both left and right weakly commutative.

The properties of weakly commutative semigroup was firstly studied by M.Petrich in [2]. It was pointed out by Petrich that the principal filters of such kind of semigroups has a particularly simple form and in fact Petrich observed that an algebraic semigroup S is weakly commutative if and only if principal filter generated by $x \in S$ is of the form (see [2, Theorem II 5.2])

$$n(x) = \{y \in S \mid x^n \in yS^l y \text{ for some } n \in N^+\}.$$

It was then shown by Petrich in [2] that weakly commutative semigroups are semilattices of Archimedean semigroups and such semilattice decompositions may not be unique. It was pointed out in [2] that a semilattice of Archimedean semigroups may not be weakly commutative semigroup.

We call a subset T of semigroup S (weakly) l- (resp. r-) Archimedean if for any $a, b \in T$, $a^n = \mu b$ (resp. $a^n = b\mu$) for some $n \in N^+$, $(\mu \in S) \mu \in T$.

A subset T is both l- and r- Archimedean then we call T be bi- Archimedean clearly, bi- Archimedean set T is Archimedean. But the converse part of this statement is in general not true. For example, we can see the following.

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Example 1.1. Let $S = \{\frac{1}{n} \mid n \in N\}$ be a set with the following multiplication in $S^1 (\forall a, b \in S)$, i.e., S is right 0- semigroup. Then we can check that S^1 is an Archimedean semigroup. But it is not bi- Archimedean and it is not weakly commutative.

It is clear that weakly l- (resp. r-) Archimedean semigroup is in general not l- (resp. r-) Archimedean. The converse part of this statement is true.

In this paper, we study relationships of weakly commutative and (l- or r-) Archimedean. The structure theorems of weakly commutative semigroups and weakly commutative abundant semigroups are shown. Thus, we give a complete solution to the problem posed by M. Petrich in [2].

For terminologies and definitions not given in this paper, the reader is referred to Petrich [2] and A. Ei. Qallali [1]. Throughout this paper. S unless otherwise stated is always a semigroup. $SC(S)$ denote the set of all semilattice congruences on S . $\mathcal{N} \in SC(S)$ is defined by following equivalence relations:

$$\mathcal{N} = \{(x, y) \mid n(x) = x(y)\}, \quad \forall x, y \in S.$$

For $\sigma \in SC(S)$ denote the congruence class of $x \in S$ by $(x)_\sigma$. The quotient set $S/\sigma = \{(x)_\sigma \mid x \in S\}$. With the following multiplication $(x)_\sigma(y)_\sigma = (xy)_\sigma (x, y \in S)$ then S/σ is a semilattice.

2. The structure of weakly commutative semigroups

In this section the structure theorems of left (resp. right) weakly commutative semigroups and weakly commutative semigroups will be given. Thus, we provide a solution to the problem posed by M. Petrich in [2].

Theorem 2.1. *For a semigroup S the following statements are equivalent:*

- (1) S is left (resp. right) weakly commutative;
- (2) $(\forall a \in S) n(a) = \{y \in S \mid a^n \in yS^l (\exists n \in N^+)\}$
(resp. $n(a) = \{y \in S \mid a^m \in S^l y (\exists m \in N^+)\}$)
- (3) $(\forall a \in S) (a)_\mathcal{N} = \{y \in S \mid a^n \in yS^l \text{ and } y^m \in aS^l (\exists n, m \in N^+)\}$
(resp. $(a)_\mathcal{N} = \{y \in S \mid a^m \in S^l y \text{ and } y^n \in S^l a (\exists n, m \in N^+)\}$)
- (4) $(\forall a \in S) (a)_\mathcal{N}$ is (weakly) r- (resp. l-) Archimedean;
- (5) $(\exists \mathcal{N} \in SC(S)) S$ is semilattice of is (weakly) r- (resp. l-) Archimedean subsemigroups $\{(a)_\mathcal{N} \mid a \in S\}$.

Proof. We only prove that the part of S is left commutative.

(1) \Rightarrow (2). Let $T := \{y \in S \mid a^n \in yS^l (\exists n \in N^+)\}$.

Since $a \in aS^l$ so $T \neq \emptyset$. For $x, y \in S$, such that $xy \in T$ then $(\exists n \in N^+, u \in S^l) a^n = x(yu) yu \in S^l$, so $x \in T$. Since $(\exists m \in N^+, v \in S)$, $(x(yu))^m = (yu)v = y(uv)(uv) \in S$ by S is left weakly commutative, so we obtain that $a^{nm} = (x(yu))^m = (yu)v(nm \in N^+, uv \in S^l)$, i.e., $y \in T$ and T is a filter set.

The following, we will show that T is the smallest filter subsemigroup containing a , then $T = n(a)$ by definition of a .

Let $x \in T$, we first prove $x^k \in T$ for $k \in N^+$. Let $(\exists n - 1 \in N^+, u \in S^l)$ $a^{n-1} = xu$ and $a^n = axu$. Clearly, there exists $m \in N^+, v \in S$. Such that $((ua)x)^m = xv$ by S is left weakly commutative. Hence we imply that $a^{n(m+1)} = (axu)^{m+1} = ax(ua)^m u = axxvu = ax^2vu$. And there exists $I \in N^+, t \in S$ such that $a^{n(m+1)l} = (ax^2vu)^l = (a(x^2vu))^l = x^2vut \in x^2S^1$ by S is left weakly commutative. So we have $x^2 \in T$ and imply $x^k \in T$ for $\forall k \in N^+$.

Let $x \in T, y \in T$, we show $yx \in T$ (resp. $xy \in T$) i.e., T is semigroup and it is a filter subsemigroup containing a . So we have $n(a) \subseteq T$ by $n(a)$ is the least filter subsemigroup containing a . Let $(\exists n, m \in N^+, u, v \in S^l)$ $a^n = xu$ and $a^m = yv$ then $a^{n+m} = xuyv$. Clearly, by $xuy \in T$ and above proving we have $(xuy)^2 \in T$. Let $(\exists e \in N^+, t \in S^l)$ $a^l = (xuy)^2t = xu(yxuyt)$, by S is left weakly commutative, we imply that $(\exists k \in N^+, \lambda \in S)$ $a^l k = (xu(yxuyt))^k = yxuyt\lambda = (yx)(uyt\lambda)$ ($uyt\lambda \in S$). That is $yx \in T$. So T is a filter subsemigroup and $n(a) \subseteq T$.

We now claim that T is the smallest filter containing a . If the claim is established, then $T = n(a)$, by definition of $n(a)$. Let F be a filter of S such that $a \in F$, Let $x \in T$. Then, there exists some $k \in N^+$ such that $a^k = xu$ for some $u \in S^l$. Since F is a filter containing a , so we have $xu = a^k \in F$. Consequently, $x \in F$. Thus, our claim is established so that $T = n(a)$.

(2) \Rightarrow (3) ($\forall a \in S$). Let $b \in (a)_{\mathcal{N}}$ then $(a)_{\mathcal{N}} = (b)_{\mathcal{N}}$ i.e., $n(a) = n(b)$. By $b \in n(a) = \{y \in S \mid a^n \in yS^l (\exists n \in N^+)\}$ and $a \in n(b) = \{y \in S \mid b^m \in yS^l (\exists m \in N^+)\}$ we obtain $a^n \in bS^l$ and $b^m \in aS^l$. This means that

$$(a)_{\mathcal{N}} = \{y \in S \mid a^n \in yS^l \text{ and } y^m \in aS^l (\exists n, m \in N^+)\}$$

(3) \Rightarrow (4) ($\forall a \in S$). Let $x, y \in (a)_{\mathcal{N}}$ then, $x \in (a)_{\mathcal{N}} = (y)_{\mathcal{N}} = \{z \in S \mid y^n \in zS^l \text{ and } z^m \in yS^l (\exists n, m \in N^+)\}$. So we have $y^n = xu$ and $x^m = yv$ ($\exists n, m \in N^+, u, v \in S^l$). This means that $(a)_{\mathcal{N}}$ is weakly r -Archimedean. We also prove that $(a)_{\mathcal{N}}$ is r -Archimedean. For this purpose, we only prove $u, v \in (a)_{\mathcal{N}}$. Since $\mathcal{N} \in SC(S)$, $(x)_{\mathcal{N}} = (x^m)_{\mathcal{N}} = (yvx^m)_{\mathcal{N}} = (yvx)_{\mathcal{N}}$, $(y)_{\mathcal{N}} = (y^n)_{\mathcal{N}} = (xuy)_{\mathcal{N}}$, so we have $(vx)_{\mathcal{N}} = (vx^2)_{\mathcal{N}} = (vx)_{\mathcal{N}}(x)_{\mathcal{N}} = (vx)_{\mathcal{N}}(y)_{\mathcal{N}} = (yvx)_{\mathcal{N}} = (x)_{\mathcal{N}}$. Similarly, $(uy)_{\mathcal{N}} = (y)_{\mathcal{N}}$. We immediately obtain that $vx \in (x)_{\mathcal{N}} = (a)_{\mathcal{N}}$, $uy \in (y)_{\mathcal{N}} = (a)_{\mathcal{N}}$ such that $y^{n+1} = x(uy)$ and $x^{m+1} = y(vx)$. So $(a)_{\mathcal{N}}$ is r -Archimedean.

(4) \Rightarrow (5). By statement (4), we have that S is a semilattice of $\{(a)_{\mathcal{N}} \mid (a)_{\mathcal{N}} \text{ is } r\text{-Archimedean, } \forall a \in S\}$

Let $\sigma \in SC(S)$ such that $(a)_{\sigma}$ is r -Archimedean, $\forall a \in S$. Let $(a, b) \in \sigma$. Then, since $b \in (a)_{\sigma}$ so by the r -Archimedean property of $(a)_{\sigma}$, there exists $n \in N^+, u \in (a)_{\sigma}$ such that $a^n = bu$. Similarly, we have $(\exists m \in N^+, v \in (b)_{\sigma})$ $b^m = av$. Since $a^n \in n(a)$ and $b^m \in n(b)$. This leads to $b \in n(a)$ and $a \in n(b)$ by $n(a)$ and $n(b)$ are filters of S^l . Hence $n(a) = n(b)$, i.e., $(a, b) \in \mathcal{N}$ and $\sigma \in \mathcal{N}$. But, we know that \mathcal{N} is the least element of $SC(S)$ by Z. L. Gao [4]. So we also have $\mathcal{N} \subseteq \sigma$. Thus $\sigma = \mathcal{N}$.

(5) \Rightarrow (1). Let S be a semilattice of $\{(a)_{\mathcal{N}} \mid (a)_{\mathcal{N}} \text{ is } r\text{-Archimedean, } \forall a \in S\}$. We need to prove that S is left weakly commutative. For this purpose, we just let

$x, y \in S$, then $yx \in (xy)_{\mathcal{N}}$, and this leads to $(\exists n \in N^+) (xy)^n \in yxS^l$. Thus, S is left weakly commutative. The proof is complete. \square

By the structure theorem of left (resp. right) weakly commutative semigroups in Theorem 2.1. we can easily prove that the structure theorem of weakly commutative semigroups. In fact we have following statements.

Theorem 2.2. *Let S be a semigroup, then the following statements are equivalent:*

- (1) S is weakly commutative;
- (2) $(\forall a \in S) n(a) = \{y \in S \mid a^n \in yS^l y \ (\exists n \in N^+)\}$;
- (3) $(\forall a \in S) (a)_{\mathcal{N}} = \{y \in S \mid a^n \in yS^l y \text{ and } y^m \in aS^l a \ (\exists n, m \in N^+)\}$
- (4) $(\forall a \in S) (a)_{\mathcal{N}}$ is (weakly) bi-Archimedean;
- (5) $(\exists \mathcal{N} \in SC(S)) S$ is semilattice of is (weakly) bi-Archimedean subsemigroups $\{(a)_{\mathcal{N}} \mid a \in S\}$.

By Example 1.1 in this paper, we know that a semilattice of Archimedean semigroups may not be weakly commutative semigroup (cf. Example 1.1). This means that the semilattice of Archimedean semigroups isn't the structure characterization of weakly commutative semigroups. By Theorem 2.2 in this paper, we give a complete solution to the problem posed by M. Petrich in [2]. That is, the semilattice of (weakly) bi-Archimedean semigroups $\{(a)_{\mathcal{N}} \mid a \in S^l\}$ is the structure characterization of weakly commutative semigroup S .

3. The structure of weakly commutative abundant semigroups

In this section, we apply the statements in last section to proving of the structure of weakly commutative abundant semigroups.

Theorem 3.1. *For a semigroup S , the following statements are equivalent;*

- (1) S is left weakly commutative and right (resp. left) abundant;
- (2) $(\forall a \in S, \exists e \in (a)_{\mathcal{R}^*} \cap E(S), \exists a = ea)$ (resp. $\exists f \in (a)_{\mathcal{L}^*} \cap E(S), \exists a = af$)
 $n(a) = \{y \in S \mid (ea)^n \in yS^l \ (\exists n \in N^+)\}$
 (resp. $n(a) = \{y \in S \mid (af)^m \in yS^l \ (\exists m \in N^+)\}$);
- (3) $(\forall a \in S, \exists e \in (a)_{\mathcal{R}^*} \cap E(S), \exists a = ea)$ (resp. $\exists f \in (a)_{\mathcal{L}^*} \cap E(S), \exists a = af$)
 $(a)_{\mathcal{N}} = \{y \in S \mid (ea)^n \in yS^l \text{ and } y^m \in eaS^l \ (\exists n, m \in N^+)\}$
 (resp. $(a)_{\mathcal{N}} = \{y \in S \mid (af)^n \in yS^l \text{ and } y^m \in afS^l \ (\exists n, m \in N^+)\}$);
- (4) $(\forall a \in S, \exists e \in (a)_{\mathcal{R}^*} \cap E(S), \exists a = ea)$ (resp. $\exists f \in (a)_{\mathcal{L}^*} \cap E(S), \exists a = af$);
 $(a)_{\mathcal{N}} = (ea)_{\mathcal{N}}$ is (weakly) r -Archimedean
 (resp. $(a)_{\mathcal{N}} = (af)_{\mathcal{N}}$ is (weakly) l -Archimedean);
- (5) $(\exists \mathcal{N} \in SC(S)) S$ is a semilattice of (weakly) r - (resp. l -) Archimedean subsemigroups $\{(a)_{\mathcal{N}} = (ea)_{\mathcal{N}} \mid e \in (a)_{\mathcal{R}^*} \cap E(S), \forall a \in S\}$
 (resp. $\{(a)_{\mathcal{N}} = (af)_{\mathcal{N}} \mid e \in (a)_{\mathcal{L}^*} \cap E(S), \forall a \in S\}$).

Proof. We only prove that the part of S is left weakly commutative and right abundant.

(1) \Rightarrow (2). Let $a \in S$, since S is right abundant. There exists $e \in (a)_{\mathcal{R}^*} \cap E(S)$ such that $(a, e) \in \mathcal{R}^*$. Clearly, $a = ea$ [1. Definition]. Hence $(a)_{\mathcal{N}} = (ea)_{\mathcal{N}}$. Now, applying Theorem 2.1 in this paper, we immediately obtain

$$(\forall a \in S) n(a) = \{y \in S \mid (ea)^n \in yS^l (\exists n \in N^+)\};$$

(2) \Rightarrow (3). Using $(\forall a \in S) (a)_{\mathcal{N}} = (ea)_{\mathcal{N}}$ and Theorem 2.1 we get statement (3).

(3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1). By using similar technique of proof given in (2) \Rightarrow (3) we also obtain the proof of above any section. \square

Corollary 3.2. *For a semigroup S , if $\mathcal{R}^* \subseteq N$, then the following statements are equivalent:*

- (1) S is left weakly commutative and right abundant;
- (2) $\forall a \in S, \exists e \in (a)_{\mathcal{R}^*} \cap E(S), \ni (a)_{\mathcal{N}} = (e)_{\mathcal{N}}, n(a) = \{y \in S \mid a \in yS^l\}$
- (3) $\forall a \in S, \exists e \in (a)_{\mathcal{R}^*} \cap E(S), \ni (a)_{\mathcal{N}} = (e)_{\mathcal{N}},$
 $(a)_{\mathcal{N}} = \{y \in S \mid a \in yS^l \text{ and } y \in aS^l\}$
- (4) $\forall a \in S, \exists e \in (a)_{\mathcal{R}^*} \cap E(S), \ni (a)_{\mathcal{N}} = (e)_{\mathcal{N}}, (a)_{\mathcal{N}}$ is right simply.
- (5) $(\exists N \in SC(S)) S$ is the semilattice of right simply subsemigroups
 $\{(a)_{\mathcal{N}} = (ea)_{\mathcal{N}} \mid e \in (a)_{\mathcal{R}^*} \cap E(S), \forall a \in S\}$.

Proof. (1) \Rightarrow (2). Let $(a, e) \in \mathcal{R}^* \subseteq N$, then $n(a) = n(e) = \{y \in S \mid e \in yS^l\}$.

(2) \Rightarrow (3). Using $(a)_{\mathcal{N}} = (ea)_{\mathcal{N}}$ and Theorem 2.1. we get statement (3).

(3) \Rightarrow (4). We prove that $(\forall a \in S) (a)_{\mathcal{N}}$ is right simply.

(i) Clearly $(a)_{\mathcal{N}}$ is subsemigroup of S and $(a)_{\mathcal{N}} = \{y \in S \mid a^n \in yS^l \text{ and } y^m \in aS^l\}$ by statement (3).

(ii) $\mathcal{R} = \mathcal{N}$. Clearly $\mathcal{R} \subseteq \mathcal{N}$ by [2]. Let $(a, b) \in N$ then $b \in (a)_{\mathcal{N}}$. So $a \in bS^l$ and $b \in aS^l$ by statement (3). Hence, we imply $R(a) = R(b)$, that is, $(a, b) \in \mathcal{R}$ and we have $\mathcal{R} = \mathcal{N}$.

(iii) Let I is a right ideal of S . then $I = \bigcup \{(x)_{\mathcal{N}} \mid x \in I\}$ by [2]. Now, we apply statements (i)-(iii) to proving S is right simply semigroup.

Let $\phi \neq I$ is a right ideal of $(a)_{\mathcal{N}}$ and $y \in (a)_{\mathcal{N}}$. For $z \in I$ by $(z)_{\mathcal{N}} = (a)_{\mathcal{N}}$ and statement (iii) then $z^2S^l = \bigcup \{(t)_{\mathcal{N}} \mid t \in z^2S^l\}$ Since $z^2 \in z^2S^l$ so $(a)_{\mathcal{N}} = (z)_{\mathcal{N}} = (z^2)_{\mathcal{N}} \subseteq z^2S^l$. Hence, for $y \in (a)_{\mathcal{N}}$. Let $y = z^2u = z(zu) (\exists u \in S^l)$ by $\mathcal{N} \in SC(S)$ we have $(y, yz^2u) \in \mathcal{N}$ and

$$\begin{aligned} (zu)_{\mathcal{N}} &= (z)_{\mathcal{N}}(u)_{\mathcal{N}} = (y)_{\mathcal{N}}(u)_{\mathcal{N}}, & ((z)_{\mathcal{N}} = (y)_{\mathcal{N}}) \\ &= (yz^2u)_{\mathcal{N}}(u)_{\mathcal{N}}, & ((y)_{\mathcal{N}} = (yz^2u)_{\mathcal{N}}) \\ &= (yz^2u^2)_{\mathcal{N}} = (yz^2u)_{\mathcal{N}} = (y)_{\mathcal{N}} = (a)_{\mathcal{N}} \end{aligned}$$

so $zu \in (a)_{\mathcal{N}}$. By $z \in I$ and $zu \in (a)_{\mathcal{N}}$ we have $y = z(zu) \in I$ by I is right ideal of S . So $(a)_{\mathcal{N}}$ is right simply.

(4) \Rightarrow (5). It is clear by using similar technique of proof given in Theorem 2.1(5). The proof is completed. \square

Applying Theorem 3.1, we may obtain the following theorem.

Theorem 3.3. *For a semigroup S , the following statements are equivalent:*

- (1) S is weakly commutative abundant;
- (2) $(\forall a \in S, \exists e \in (a)_{\mathcal{L}^*} \cap E(S), f \in (a)_{\mathcal{R}^*} \cap E(S), \ni a = ae = fa)$
 $n(a) = \{y \in S \mid (ae)^n \in yS^l y \text{ and } (fa)^m \in yS^l y (\exists n, m \in N^+)\}$;
- (3) $(\forall a \in S, \exists e \in (a)_{\mathcal{L}^*} \cap E(S), f \in (a)_{\mathcal{R}^*} \cap E(S), \ni a = ae = fa)$
 $(a)_{\mathcal{N}} = \{y \in S \mid (ae)^{n_1} \in yS^l y, (fa)^{n_2} \in yS^l y \text{ and } y^{m_1} \in aeS^l ae, y^{m_2} \in faS^l fa (\exists n_i, m_i \in N^+ i = 1, 2)\}$
- (4) $(\forall a \in S, \exists e \in (a)_{\mathcal{L}^*} \cap E(S), f \in (a)_{\mathcal{R}^*} \cap E(S), \ni a = ae = fa)$
 $(a)_{\mathcal{N}}$ is (weakly) bi-Archimedean;
- (5) $(\exists \mathcal{N} \in SC(S)) S$ is a semilattice of (weakly) bi-Archimedean subsemigroups
 $\{(a)_{\mathcal{N}} = (ae)_{\mathcal{N}} = (fa)_{\mathcal{N}} \mid e \in (a)_{\mathcal{L}^*} \cap E(S), f \in (a)_{\mathcal{R}^*} \cap E(S), \forall a \in S\}$.

Corollary 3.4. *For a semigroup S , the following statements are equivalent:*

- (1) S is weakly commutative superabundant;
- (2) $(\forall a \in S, \exists e \in (a)_{\mathcal{H}^*} \cap E(S), \ni a = ae = ea)$
 $n(a) = \{y \in S \mid (ae)^n \in yS^l y \text{ and } (ea)^m \in yS^l y (\exists n, m \in N^+)\}$;
- (3) $(\forall a \in S, \exists e \in (a)_{\mathcal{H}^*} \cap E(S), \ni a = ae = ea)$
 $(a)_{\mathcal{N}} = \{y \in S \mid (ae)^{n_1} \in yS^l y, (ea)^{n_2} \in yS^l y \text{ and } y^{m_1} \in aeS^l ae, y^{m_2} \in eaS^l ea (\exists n_i, m_i \in N^+ i = 1, 2)\}$
- (4) $(\forall a \in S, \exists e \in (a)_{\mathcal{H}^*} \cap E(S), \ni a = ae = fa)$ $(a)_{\mathcal{N}}$ is (weakly) bi-Archimedean;
- (5) S is the semilattice of (weakly) bi-Archimedean subsemigroups
 $\{(a)_{\mathcal{N}} = (ae)_{\mathcal{N}} = (ea)_{\mathcal{N}} \mid e \in (a)_{\mathcal{H}^*} \cap E(S), \forall a \in S\}$.

Inclosing this paper, we cite Example 3.5 to illustrate that the applying of Theorem 3.2 in this paper.

Example 3.5. Let $S = \{a, b, c, d, e\}$ be a set with Cayley table shown below:

	a	b	c	d	e
a	e	d	b	c	e
b	b	d	b	d	b
c	c	d	c	d	c
d	d	d	d	d	d
e	e	d	c	d	e

Then, by using the method of Theorem 3.2. we can verify that S is a semigroup with the following propositions:

- (1) $(a)_{\mathcal{N}} = \{a, e\}; \quad (b)_{\mathcal{N}} = \{b, d\}; \quad (c)_{\mathcal{N}} = \{c\};$
 $(a)_{\mathcal{L}^*} = \{a, e\}; \quad (b)_{\mathcal{L}^*} = \{b, c\}; \quad (d)_{\mathcal{L}^*} = \{d\};$
 $(a)_{\mathcal{R}^*} = \{a, e\}; \quad (c)_{\mathcal{R}^*} = \{c\}; \quad (d)_{\mathcal{R}^*} = \{b, d\};$

$$E(g) = \{c, d, e\} = \text{Re } g(s);$$

$$\mathcal{N} = \{(a, a), (b, b), (c, c), (d, d), (e, e), (b, d), (d, b), (a, e), (e, a)\};$$

$$\tau = \mathcal{N} \cup \{(a, c), (c, a), (c, e), (e, c)\};$$

$$\mathcal{N}^* = \mathcal{N} \cup \{(b, c), (c, b), (d, c), (c, d)\};$$

$$\sigma = S \times S.$$

- (2) S is weakly commutative abundant but is not commutative;

(3) \mathcal{N} is the unique element of $SC(S)\{\mathcal{N}, \tau, \mathcal{N}^*, \sigma\}$ such that $S = (e)_{\mathcal{N}} \cup (c)_{\mathcal{N}} \cup (d)_{\mathcal{N}}$ in which $(a)_{\mathcal{N}}$ is bi-Archimedean, for $x = e$ or c or d .

- (4) Since $a \notin \text{Re } g(s)$ so S is not regular.

(5) $(a)_{\mathcal{N}^*} = \{a, e\}$ and $(b)_{\mathcal{N}^*} = \{b, c, d\}$ are weakly commutative abundant, but $(b)_{\mathcal{N}^*}$ is not bi-Archimedean. That is, S is a semilattice of weakly commutative abundant subsemigroups $(a)_{\mathcal{N}^*}$ and $(b)_{\mathcal{N}^*}$ but is not a semilattice of (weakly) bi-Archimedean subsemigroups for $\mathcal{N}^* \in SC(S)$.

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