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On Applications of Weyl Fractional q-integral Operator to Generalized Basic Hypergeometric Functions

Rajendra Kumar Yadav and Sunil Dutt Purohit
Department of Mathematics and Statistics, Jai Narain Vyas University, Jodhpur-342001, India
e-mail : rkmdyadav@yahoo.co.in *and* sunil_a_purohit@yahoo.com

Abstract. Applications of Weyl fractional q-integral operator to various generalized basic hypergeometric functions including the basic analogue of Fox's H-function have been investigated in the present paper. Certain interesting special cases have also been deduced.

1. Introduction

Al-Salam [2] introduced, the q-analogue of Weyl fractional integral operator as follows:

$$K_q^\mu f(x) \ = \ \frac{q^{-\mu(\mu-1)/2}}{\Gamma_q(\mu)} \int_x^\infty (y-x)_{\mu-1} f(yq^{1-\mu}) d(y;q), \tag{1}$$

where $\mathrm{Re}(\mu) > 0$.

So that

$$K_q^0 f(x) \ = \ f(x). \tag{2}$$

Following Jackson [5], Al-Salam [2] and Agarwal [1], we have the basic integration defined as

$$\int_x^\infty f(t)d(t;q) \ = \ x(1-q)\sum_{k=1}^\infty q^{-k} f(xq^{-k}). \tag{3}$$

In view of (3), equation (1) can be expressed as

$$K_q^\mu f(x) \ = \ \frac{x^\mu (1-q) q^{-\mu(\mu+1)/2}}{\Gamma_q(\mu)} \sum_{k=0}^\infty q^{-k\mu}(1-q^{k+1})_{\mu-1} f(xq^{-\mu-k}), \tag{4}$$

where $\mathrm{Re}(\mu) > 0$.

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Further for real or complex α, and $0<|q|<1$, the q-factorial is defined as

$$(5)\qquad (\alpha;q)_n = \begin{cases} 1, & \text{if} \quad n=0 \\ (1-\alpha)(1-\alpha q)\cdots(1-\alpha q^{n-1}), & \text{if} \quad n\in N. \end{cases}$$

Also

$$(6)\qquad \Gamma_q(\alpha) = \frac{G(q^\alpha)}{(1-q)^{\alpha-1}G(q)} = \frac{(1-q)_{\alpha-1}}{(1-q)^{\alpha-1}}; \quad (\alpha\neq 0,-1,-2,\cdots)$$

and

$$(7)\qquad (x-y)_\nu = x^\nu \prod_0^\infty \left[\frac{1-(y/x)q^n}{1-(y/x)q^{\nu+n}}\right] = x^\nu\ {}_1\phi_0\left[\begin{array}{lc} q^{-\nu} & ; \\ & q, yq^\nu/x \\ - & ; \end{array}\right].$$

The generalized basic hypergeometric series cf. Gasper and Rahman [4] is given by

$$(8)\qquad {}_r\phi_s\left[\begin{array}{lc} a_1,\cdots,a_r & ; \\ & q,x \\ b_1,\cdots,b_s & ; \end{array}\right] = \sum_{n=0}^\infty \frac{(a_1,\cdots,a_r;q)_n}{(q,b_1,\cdots,b_s;q)_n}\, x^n,$$

where for convergence, we have $0<|q|<1$ and $|x|<1$ if $r=s+1$, and for any x if $r\le s$.

The abnormal type of generalized basic hypergeometric series ${}_r\phi_s(\cdot)$ is defined as

$$(9)\qquad {}_r\phi_s\left[\begin{array}{ll} a_1,\cdots,a_r & ;q,x \\ b_1,\cdots,b_s & ;q^\lambda \end{array}\right] = \sum_{n=0}^\infty \frac{(a_1,\cdots,a_r;q)_n}{(q,b_1,\cdots,b_s;q)_n}\, x^n q^{\lambda n(n+1)/2},$$

where $\lambda>0$ and $0<|q|<1$.

The q-binomial series is given by

$$(10)\qquad {}_1\phi_0\left[\begin{array}{lc} \alpha & ; \\ & q,x \\ - & ; \end{array}\right] = \frac{(\alpha x;q)_\infty}{(x,q)_\infty}.$$

Following Saxena et.al. [8], we have the basic analogue of Fox's H-function defined as

$$(11)\qquad H^{m_1,n_1}_{A,B}\left[\begin{array}{c|c} x;q & \begin{array}{c}(a,\alpha)\\ \\ (b,\beta)\end{array}\end{array}\right]$$

$$= \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m_1} G(q^{b_j-\beta_j s}) \prod_{j=1}^{n_1} G(q^{1-a_j+\alpha_j s})\pi x^s}{\prod_{j=m_1+1}^{B} G(q^{1-b_j+\beta_j s}) \prod_{j=n_1+1}^{A} G(q^{a_j-\alpha_j s})G(q^{1-s})\sin\pi s}\, ds,$$

where $0 \leq m_1 \leq B,\ 0 \leq n_1 \leq A$; $\alpha_i's$ and $\beta_j's$ are all positive integers, the contour C is a line parallel to $\mathrm{Re}(ws)$, with indentations if necessary, in such a manner that all poles of $G(q^{b_j-\beta_j s}),\ 1 \leq j \leq m_1$ lies to the right, and those of $G(q^{1-a_j+\alpha_j s})$, $1 \leq j \leq n_1$, to the left of C. The integral converges if $\mathrm{Re}[s\log(x) - \log\sin\pi s] < 0$ for large values of $|s|$ on the contour that is, if $|\{\arg(x) - w_2 w_1^{-1}\log|x|\}| < \pi$, where $0 < |q| < 1$, $\log q = -w = -(w_1 + iw_2)$, $w,\ w_1,\ w_2$ are definite quantities, w_1 and w_2 being real.

Also

$$G(q^a) = \left\{\prod_{n=0}^{\infty}(1-q^{a+n})\right\}^{-1} = \frac{1}{(q^a;q)_\infty}. \tag{12}$$

If we set $\alpha_j = \beta_i = 1,\ 1 \leq j \leq A,\ 1 \leq i \leq B$ in (11), then it reduces to the basic analogue of Meijer's G-function due to Saxena et.al. [8], namely

$$G_{A,B}^{m_1,n_1}\left[\ x;q\ \middle|\ \begin{matrix} a_1,\cdots,a_A \\ \\ b_1,\cdots,b_B \end{matrix}\right] \tag{13}$$

$$= \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m_1} G(q^{b_j-s}) \prod_{j=1}^{n_1} G(q^{1-a_j+s})\pi x^s}{\prod_{j=m_1+1}^{B} G(q^{1-b_j+s}) \prod_{j=n_1+1}^{A} G(q^{a_j-s})G(q^{1-s})\sin\pi s}\, ds,$$

where $0 \leq m_1 \leq B$, $0 \leq n_1 \leq A$ and $\mathrm{Re}[s\log(x) - \log\sin\pi s] < 0$.

Further, if we set $n_1 = 0,\ m_1 = B$ in (13), we obtain the basic analogue of MacRobert's E-function as

$$G_{A,B}^{B,0}\left[\ x;q\ \middle|\ \begin{matrix} a_1,\cdots,a_A \\ \\ b_1,\cdots,b_B \end{matrix}\right] = E_q[B;b_j:A;a_j:x]. \tag{14}$$

Saxena and Kumar [9], introduced the basic analogues of $J_\nu(x),\ Y_\nu(x),\ K_\nu(x),\ H_\nu(x)$ in terms of $H_q(\cdot)$ function as under:

$$J_\nu(x;q) = \{G(q)\}^2 H_{0,3}^{1,0}\left[\ \frac{x^2(1-q)^2}{4};q\ \middle|\ \begin{matrix} - \\ \\ (\frac{\nu}{2},1),\ (\frac{-\nu}{2},1),\ (1,1) \end{matrix}\right], \tag{15}$$

where $J_\nu(x;q)$ denotes the q-analogue of Bessel function of first kind $J_\nu(x)$.

$$Y_\nu(x;q) \tag{16}$$

$$= \{G(q)\}^2 H_{1,4}^{2,0}\left[\ \frac{x^2(1-q)^2}{4};q\ \middle|\ \begin{matrix} (\frac{-\nu-1}{2},1) \\ \\ (\frac{\nu}{2},1),\ (\frac{-\nu}{2},1),\ (\frac{-\nu-1}{2},1),\ (1,1) \end{matrix}\right],$$

where $Y_\nu(x;q)$ denotes the q-analogue of the Bessel function $Y_\nu(x)$.

(17) $$K_\nu(x;q) = (1-q)H_{0,3}^{2,0}\left[\frac{x^2(1-q)^2}{4};q \;\middle|\; \begin{matrix} - \\ (\frac{\nu}{2},1),\ (\frac{-\nu}{2},1),\ (1,1) \end{matrix}\right],$$

where $K_\nu(x;q)$ denotes the basic analogue of the Bessel function of the third kind $K_\nu(x)$.

(18) $$H_\nu(x;q) = \left(\frac{1-q}{2}\right)^{1-\alpha} H_{1,4}^{3,1}\left[\frac{x^2(1-q)^2}{4};q \;\middle|\; \begin{matrix} (\frac{1+\alpha}{2},1) \\ (\frac{\nu}{2},1),\ (\frac{-\nu}{2},1),\ (\frac{1+\alpha}{2},1),\ (1,1) \end{matrix}\right],$$

where $H_\nu(x;q)$ is the basic analogue of Struve's function $H_\nu(x)$.

In view of the definition (11), we can express the following elementary basic ($q-$) functions in terms of the basic analogue of H-function as:

(19) $$e_q(-x) = G(q)H_{0,2}^{1,0}\left[x(1-q);q \;\middle|\; \begin{matrix} - \\ (0,1),(1,1) \end{matrix}\right],$$

(20) $$\sin_q(x) = \sqrt{\pi}(1-q)^{-1/2}\{G(q)\}^2 H_{0,3}^{1,0}\left[\frac{x^2(1-q)^2}{4};q \;\middle|\; \begin{matrix} - \\ (\frac{1}{2},1),\ (0,1),\ (1,1) \end{matrix}\right],$$

(21) $$\cos_q(x) = \sqrt{\pi}(1-q)^{-1/2}\{G(q)\}^2 H_{0,3}^{1,0}\left[\frac{x^2(1-q)^2}{4};q \;\middle|\; \begin{matrix} - \\ (0,1),\ (\frac{1}{2},1),\ (1,1) \end{matrix}\right],$$

(22) $$\sinh_q(x) = \frac{\sqrt{\pi}}{i}(1-q)^{-1/2}\{G(q)\}^2 H_{0,3}^{1,0}\left[-\frac{x^2(1-q)^2}{4};q \;\middle|\; \begin{matrix} - \\ (\frac{1}{2},1),\ (0,1),\ (1,1) \end{matrix}\right],$$

(23) $$\cosh_q(x) = \sqrt{\pi}(1-q)^{-1/2}\{G(q)\}^2 H_{0,3}^{1,0}\left[-\frac{x^2(1-q)^2}{4};q \;\middle|\; \begin{matrix} - \\ (0,1)\ (\frac{1}{2},1),\ (1,1) \end{matrix}\right].$$

A detailed account of various functions expressible in terms of Meijer's G-function or Fox's H-function can be found in research monographs by Saxena and Mathai [6] and [7].

The object of the present paper is to evaluate Weyl fractional basic integral operator involving basic analogue of the H-function and various other basic hypergeometric functions. The results deduced are believed to be a new contribution to the theory of basic hypergeometric functions.

2. Main results

In this section we shall evaluate the following q-fractional integrals of Weyl type involving basic hypergeometric function ${}_r\phi_s(\cdot)$ and basic analogue of Fox's H-function

$$(24)\qquad K_q^{\mu}\left\{x^{-\lambda}{}_r\phi_s\left[\begin{matrix} a_1,\cdots,a_r & ; & \\ & & q,-a/x \\ b_1,\cdots,b_s & ; & \end{matrix}\right]\right\}$$

$$= \frac{\Gamma_q(\lambda-\mu)}{\Gamma_q(\lambda)}\, x^{\mu-\lambda}q^{\mu\lambda-\mu(\mu+1)/2}{}_{r+1}\phi_{s+1}\left[\begin{matrix} a_1,\cdots,a_r,q^{\lambda-\mu} & ; & \\ & & q,-aq^{\mu}/x \\ b_1,\cdots,b_s,q^{\lambda} & ; & \end{matrix}\right],$$

where $\mathrm{Re}(\lambda-\mu)>0$.

$$(25)\qquad K_q^{\mu}\left\{H_{A,B}^{m_1,n_1}\left[\begin{matrix} x;q \end{matrix}\left|\begin{matrix}(a,\alpha)\\ \\ (b,\beta)\end{matrix}\right.\right]\right\}$$

$$= x^{\mu}(1-q)^{\mu}q^{-\mu(\mu+1)/2}\times H_{A+1,B+1}^{m_1+1,n_1}\left[\begin{matrix} xq^{-\mu};q \end{matrix}\left|\begin{matrix}(a,\alpha),(0,1)\\ \\ (-\mu,1),(b,\beta)\end{matrix}\right.\right],$$

where $\mathrm{Re}(\mu)>0$, $0\le m_1\le B$, $0\le n_1\le A$ and $\mathrm{Re}[s\log(x)-\log\sin\pi s]<0$.

Proof of (24). In view of (4) and (8) L.H.S. of (24) becomes

$$x^{\mu}(1-q)^{\mu}q^{-\mu(\mu+1)/2}\sum_{k=0}^{\infty}\frac{(q^{\mu};q)_k}{(q;q)_k}\,q^{-\mu k}\,(xq^{-\mu-k})^{-\lambda}\sum_{n=0}^{\infty}\frac{(a_1,\cdots,a_r;q)_n}{(q,b_1,\cdots,b_s;q)_n}\,(-aq^{\mu+k}/x)^n.$$

On interchanging the order of summations, we obtain

$$x^{\mu-\lambda}(1-q)^{\mu}q^{\mu\lambda-\mu(\mu+1)/2}\sum_{n=0}^{\infty}\frac{(a_1,\cdots,a_r;q)_n}{(q,b_1,\cdots,b_s;q)_n}\,(-aq^{\mu}/x)^n\sum_{k=0}^{\infty}\frac{(q^{\mu};q)_k}{(q;q)_k}\,q^{(n+\lambda-\mu)k}.$$

On summing the inner ${}_1\phi_0(\cdot)$ series, it yields to

$$x^{\mu-\lambda}(1-q)^{\mu}q^{\mu\lambda-\mu(\mu+1)/2}\sum_{n=0}^{\infty}\frac{(a_1,\cdots,a_r;q)_n}{(q,b_1,\cdots,b_s;q)_n\,(q^{n+\lambda-\mu};q)_{\mu}}\,(-aq^{\mu}/x)^n$$

which on further simplification, reduces to R.H.S. of (24).

This completes the proof of (24). □

Proof of (25). To prove (25), we consider L.H.S. of (25) and make use of (4) and (11) to obtains

$$x^{\mu}(1-q)^{\mu}q^{-\mu(\mu+1)/2}\sum_{k=0}^{\infty}\frac{q^{-\mu k}(q^{\mu};q)_k}{(q;q)_k}$$

$$\times\frac{1}{2\pi i}\int_C\frac{\prod_{j=1}^{m_1}G(q^{b_j-\beta_j s})\prod_{j=1}^{n_1}G(q^{1-a_j+\alpha_j s})\pi(xq^{-\mu-k})^s}{\prod_{j=m_1+1}^{B}G(q^{1-b_j+\beta_j s})\prod_{j=n_1+1}^{A}G(q^{a_j-\alpha_j s})G(q^{1-s})\sin\pi s}\,ds.$$

On interchanging the order of summation and integration, which is valid under the conditions stated with (11), we obtain

$$\frac{x^{\mu}(1-q)^{\mu}q^{-\mu(\mu+1)/2}}{2\pi i}\int_C\frac{\prod_{j=1}^{m_1}G(q^{b_j-\beta_j s})\prod_{j=1}^{n_1}G(q^{1-a_j+\alpha_j s})\pi(xq^{-\mu})^s}{\prod_{j=m_1+1}^{B}G(q^{1-b_j+\beta_j s})\prod_{j=n_1+1}^{A}G(q^{a_j-\alpha_j s})G(q^{1-s})\sin\pi s}$$

$$\times\sum_{k=0}^{\infty}\frac{q^{-k(\mu+s)}(q^{\mu};q)_k}{(q;q)_k}\,ds.$$

On summing inner series ${}_1\phi_0(\cdot)$, it yields after some simplifications

$$\frac{x^{\mu}(1-q)^{\mu}q^{-\mu(\mu+1)/2}}{2\pi i}\int_C\frac{\prod_{j=1}^{m_1}G(q^{b_j-\beta_j s})G(q^{-\mu-s})\prod_{j=1}^{n_1}G(q^{1-a_j+\alpha_j s})\pi(xq^{-\mu})^s}{\prod_{j=m_1+1}^{B}G(q^{1-b_j+\beta_j s})\prod_{j=n_1+1}^{A}G(q^{a_j-\alpha_j s})G(q^{-s})G(q^{1-s})\sin\pi s}\,ds.$$

In view of (11), we arrive at the R.H.S. of (25).

This completes the proof of (25). □

In the next section we shall evaluate some basic integrals of Weyl type, involving basic hypergeometric functions and various other basic functions expressible in terms of the basic analogue of Fox's H-function as the applications of (24) and (25).

3. Applications of the Main Results

In this section we shall make use of (24) and (25) to deduce the following fifteen results given in the table as under:

Table-1

Eq. No.	$f(x)$	$K_q^{\mu} f(x) = \frac{q^{-\mu(\mu-1)/2}}{\Gamma_q(\mu)} \int_x^{\infty} (y-x)_{\mu-1} f(yq^{1-\mu}) d(y;q)$ Re$(\mu) > 0$
26.	$x^{-\lambda}$	$\frac{\Gamma_q(\lambda-\mu)}{\Gamma_q(\lambda)} x^{\mu-\lambda} q^{\mu\lambda-\mu(\mu+1)/2}$ Re$(\lambda-\mu) > 0$.
27.	$x^{-\lambda} e_q(a/x)$	$\frac{\Gamma_q(\lambda-\mu)}{\Gamma_q(\lambda)} x^{\mu-\lambda} q^{\mu\lambda-\mu(\mu+1)/2} {}_1\phi_1 \left[\begin{matrix} q^{\lambda-\mu} & ; & \\ & & q, aq^{\mu}/x \\ q^{\lambda} & ; & \end{matrix}\right]$ Re$(\lambda-\mu) > 0$.
28.	$x^{-\lambda}(x+aq^{-\nu})_{\nu}$	$\frac{\Gamma_q(\lambda-\mu-\nu)}{\Gamma_q(\lambda-\nu)} x^{\mu-\lambda+\nu} q^{\mu\lambda-\mu\nu-\mu(\mu+1)/2} \times {}_2\phi_1 \left[\begin{matrix} q^{-\nu}, q^{\lambda-\nu-\mu} & ; & \\ & & q, -aq^{\mu}/x \\ q^{\lambda-\nu} & ; & \end{matrix}\right]$ Re$(\lambda-\nu-\mu) > 0$
29.	$x^{-\lambda} {}_r\phi_s \begin{matrix} a_1, \cdots, a_r & ; q, a/x \\ b_1, \cdots, b_s & ; q^{\delta} \end{matrix}$	$\frac{\Gamma_q(\lambda-\mu)}{\Gamma_q(\lambda)} x^{\mu-\lambda} q^{\mu\lambda-\mu(\mu+1)/2} \times {}_{r+1}\phi_{s+1} \begin{matrix} a_1, \cdots, a_r, q^{\lambda-\mu} & ; q, aq^{\mu}/x \\ b_1, \cdots, b_s, q^{\lambda} & ; q^{\delta} \end{matrix}$ Re$(\lambda-\mu) > 0,\ \delta > 0$
30.	$G_{A,B}^{m_1,n_1} \left[x; q \left\| \begin{matrix} a_1, \cdots, a_A \\ b_1, \cdots, b_B \end{matrix} \right. \right]$	$x^{\mu}(1-q)^{\mu} q^{-\mu(\mu+1)/2} \times G_{A+1,B+1}^{m_1+1,n_1} \left[xq^{-\mu}; q \left\| \begin{matrix} a_1, \cdots, a_A, 0 \\ -\mu, b_1, \cdots, b_B \end{matrix} \right. \right]$ $0 \le m_1 \le B$, $0 \le n_1 \le A$ and Re$[s\log(x) - \log\sin\pi s] < 0$
31.	$E_q[B; b_j : A; a_j : x]$	$x^{\mu}(1-q)^{\mu} q^{-\mu(\mu+1)/2} \times G_{A+1,B+1}^{B+1,0} \left[xq^{-\mu}; q \left\| \begin{matrix} a_1, \cdots, a_A, 0 \\ -\mu, b_1, \cdots, b_B \end{matrix} \right. \right]$ $0 \le m_1 \le B$, $0 \le n_1 \le A$ and Re$[s\log(x) - \log\sin\pi s] < 0$

continue

32.	$J_\nu(x;q)$	$x^\mu(1-q)^\mu\{G(q)\}^2 q^{-\mu(\mu+1)/2}\times H_{1,4}^{2,0}\left[\frac{x^2(1-q)^2}{4q^{2\mu}};q \middle\vert \begin{matrix}(0,2)\\ (-\mu,2),(\frac{\nu}{2},1),(\frac{-\nu}{2},1),(1,1)\end{matrix}\right]$
33.	$Y_\nu(x;q)$	$x^\mu(1-q)^\mu\{G(q)\}^2 q^{-\mu(\mu+1)/2}\times H_{2,5}^{3,0}\left[\frac{x^2(1-q)^2}{4q^{2\mu}};q \middle\vert \begin{matrix}(\frac{-\nu-1}{2},1),(0,2)\\ (-\mu,2),(\frac{\nu}{2},1),(\frac{-\nu}{2},1),(\frac{-\nu-1}{2},1),(1,1)\end{matrix}\right]$
34.	$K_\nu(x;q)$	$x^\mu(1-q)^{\mu+1} q^{-\mu(\mu+1)/2}\times H_{1,4}^{3,0}\left[\frac{x^2(1-q)^2}{4q^{2\mu}};q \middle\vert \begin{matrix}(0,2)\\ (-\mu,2),(\frac{\nu}{2},1),(\frac{-\nu}{2},1),(1,1)\end{matrix}\right]$
35.	$H_\nu(x;q)$	$x^\mu(1-q)^{\mu+1-\alpha}2^{\alpha-1} q^{-\mu(\mu+1)/2}\times H_{2,5}^{4,1}\left[\frac{x^2(1-q)^2}{4q^{2\mu}};q \middle\vert \begin{matrix}(\frac{\alpha+1}{2},1),(0,2)\\ (-\mu,2),(\frac{\nu}{2},1),(\frac{-\nu}{2},1),(\frac{\alpha+1}{2},1),(1,1)\end{matrix}\right]$
36.	$e_q(-x)$	$x^\mu(1-q)^\mu G(q) q^{-\mu(\mu+1)/2}\times H_{1,3}^{2,0}\left[xq^{-\mu}(1-q);q \middle\vert \begin{matrix}(0,1)\\ (-\mu,1),(0,1),(1,1)\end{matrix}\right]$
37.	$\sin_q(x)$	$x^\mu\sqrt{\pi}(1-q)^{\mu-1/2}\{G(q)\}^2 q^{-\mu(\mu+1)/2}\times H_{1,4}^{2,0}\left[\frac{x^2(1-q)^2}{4q^{2\mu}};q \middle\vert \begin{matrix}(0,2)\\ (-\mu,2),(\frac{1}{2},1),(0,1),(1,1)\end{matrix}\right]$
38.	$\cos_q(x)$	$x^\mu\sqrt{\pi}(1-q)^{\mu-1/2}\{G(q)\}^2 q^{-\mu(\mu+1)/2}\times H_{1,4}^{2,0}\left[\frac{x^2(1-q)^2}{4q^{2\mu}};q \middle\vert \begin{matrix}(0,2)\\ (-\mu,2),(0,1),(\frac{1}{2},1),(1,1)\end{matrix}\right]$

continue

39.	$\sinh_q(x)$	$x^{\mu}\frac{\sqrt{\pi}}{i}(1-q)^{\mu-1/2}\{G(q)\}^2q^{-\mu(\mu+1)/2}\times$ $H^{2,0}_{1,4}\left[\frac{-x^2(1-q)^2}{4q^{2\mu}};q\left\|\begin{matrix}(0,2)\\ \\(-\mu,2),(\frac{1}{2},1),(0,1),(1,1)\end{matrix}\right.\right]$
40.	$\cosh_q(x)$	$x^{\mu}\sqrt{\pi}(1-q)^{\mu-1/2}\{G(q)\}^2q^{-\mu(\mu+1)/2}\times$ $H^{2,0}_{1,4}\left[\frac{-x^2(1-q)^2}{4q^{2\mu}};q\left\|\begin{matrix}(0,2)\\ \\(-\mu,2),(0,1),(\frac{1}{2},1),(1,1)\end{matrix}\right.\right]$

We shall now give proof of some of the results mentioned in the table.

Proof of (27). We set $f(x)=x^{-\lambda}e_q(a/x)$ in (4) and make use of the q-exponential series

$$e_q(x) \;=\; {}_0\phi_0[\_;\_;q,x] \tag{26}$$

to obtain

$$K_q^{\mu}\{x^{-\lambda}e_q(a/x)\} \;=\; K_q^{\mu}\{x^{-\lambda}{}_0\phi_0[\_;\_;q,a/x]\}$$

which in view of (24) with $r=s=0$ and a replaced by $-a$ reduces to (27). □

Proof of (28). If we take $f(x)=x^{-\lambda}(x+aq^{-\nu})_{\nu}$ where $\mathrm{Re}(\lambda-\nu-\mu)>0$ in (4) and make use of (7), we have

$$K_q^{\mu}\{x^{-\lambda}(x+aq^{-\nu})_{\nu}\} \;=\; K_q^{\mu}\left\{x^{-\lambda+\nu}\;{}_1\phi_0\left[\begin{matrix}q^{-\nu} & ; & \\ & & q,-a/x\\ \_ & ; & \end{matrix}\right]\right\}$$

which on using (24) with $r=1,\ s=0$ and λ replaced by $\lambda-\nu$ yields in R.H.S. of (28). □

Proof of (29) follows similarly as of (24).

Proof of (30). Setting $\alpha_j=\beta_i=1$, $1\le j\le A$, $1\le i\le B$ in (25) and making use of (13), we obtain

$$K_q^{\mu}\left\{G^{m_1,n_1}_{A,B}\left[x;q\left|\begin{matrix}a_1,\cdots,a_A\\ \\ b_1,\cdots,b_B\end{matrix}\right.\right]\right\} \tag{27}$$
$$=\; x^{\mu}(1-q)^{\mu}q^{-\mu(\mu+1)/2}\times G^{m_1+1,n_1}_{A+1,B+1}\left[xq^{-\mu};q\left|\begin{matrix}a_1,\cdots,a_A,0\\ \\ -\mu,b_1,\cdots,b_B\end{matrix}\right.\right],$$

where $0 \le m_1 \le B$, $0 \le n_1 \le A$ and $\mathrm{Re}[s \log(x) - \log \sin \pi s] < 0$. □

Results (31) follows from (42) with $n_1 = 0$, $m_1 = B$.

Proof of (32). We take $f(x) = J_\nu(x;q)$ in (4) and on making use of (15), we get

$$K_q^\mu\{J_\nu(x;q)\} \;=\; x^\mu(1-q)^\mu q^{-\mu(\mu+1)/2}\{G(q)\}^2 \sum_{k=0}^{\infty} \frac{q^{-\mu k}(q^\mu;q)_k}{(q;q)_k} \times \frac{1}{2\pi i}\int_C \frac{G(q^{\nu/2-s})\pi(xq^{-\mu-k})^{2s}(1-q)^{2s}}{G(q^{1+\nu/2+s})G(q^s)G(q^{1-s})4^s \sin \pi s}\, ds.$$

On interchanging the order of summation and integration, which is valid under the conditions stated with (11), we obtain

$$\frac{x^\mu(1-q)^\mu q^{-\mu(\mu+1)/2}\{G(q)\}^2}{2\pi i}\int_C \frac{G(q^{\nu/2-s})\pi x^{2s}(1-q)^{2s}}{G(q^{1+\nu/2+s})G(q^s)G(q^{1-s})\sin \pi s(2q^\mu)^{2s}} \sum_{k=0}^{\infty} \frac{q^{-k(\mu+2s)}(q^\mu;q)_k}{(q;q)_k}\, ds.$$

On summing the inner ${}_1\phi_0(\cdot)$ series, it yields after certain simplification

$$\frac{x^\mu(1-q)^\mu q^{-\mu(\mu+1)/2}\{G(q)\}^2}{2\pi i}\int_C \frac{G(q^{\nu/2-s})G(q^{-\mu-2s})\pi\, x^{2s}(1-q)^{2s}}{G(q^{1+\nu/2+s})G(q^s)G(q^{-2s})G(q^{1-s})\sin \pi s(2q^\mu)^{2s}}\, ds.$$

In view of (11), we arrive at the R.H.S. of (32). □

Proofs of (33)-(40) follow similarly.

Finally, if we let $q \to 1^-$ in the results (24), (26)-(28) and (30), and make use of the limit formulas

$$\lim_{q\to 1^-} \frac{(q^\alpha;q)_n}{(1-q)^n} \;=\; (\alpha)_n, \tag{28}$$

where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$ and

$$\lim_{q\to 1^-} \Gamma_q(\alpha) \;=\; \Gamma(\alpha). \tag{29}$$

We respectively obtain various results mentioned in Erdelyi [3] [table (13.2), pp. 201-212]. All these results are believed to be a new contribution to the theory of q-hypergeoemetric functions.

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