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Commutative Semigroups whose Proper Homomorphic Images are All of Smaller Cardinality

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Abstract. We characterize those commutative semigroups S such that every non-isomorphic homomorphic image of S has smaller cardinality than S. We also characterize commutative groups with the same property.

In [3] Kaplansky posed the following problem for an infinite commutative group G: Show that every proper (not isomorphic) homomorphic image of G is finite if and only if G is an infinite cyclic group. In [2] Jensen and Miller characterized all infinite commutative semigroups S such that every proper homomorphic image of S is finite; they called such semigroups *homomorphically finite* or *HF semigroups*. In this note we characterize those infinite commutative semigroups S such that every proper homomorphic image of S is of smaller cardinality than S. We call such semigroups *H-smaller*. Surprisingly, the H-smaller semigroups are precisely those in Jensen and Miller's Theorem. As part of the proof of this fact we also generalize the exercise in Kaplansky by showing that, if G is an infinite commutative group, then every proper homomorphic image of G is of smaller cardinality than G if and only if G is an infinite cyclic group.

For any semigroup S let S^0, S^1, and $S^{0,1}$ denote S with zero adjoined, S with identity adjoined, and S with both zero and identity adjoined, respectively. The group of integers is denoted $\mathbb{Z}$. The symbol $\mathbb{N}'$ stands for any subsemigroup of $(\mathbb{N}, +)$, the semigroup of positive integers under addition. We now state Jensen and Miller's theorem.

Theorem 1 [2, Theorem 3]**.** *Let S be an infinite commutative semigroup. Then every proper homomorphic image of S is finite if and only if S is either $\mathbb{Z}$, $\mathbb{Z}^0$, $\mathbb{N}'$, $(\mathbb{N}')^0$, $(\mathbb{N}')^1$, or $(\mathbb{N}')^{0,1}$.*

We let $|X|$ denote the cardinality of X for any set X. Throughout the rest of this note S will denote an infinite commutative H-smaller semigroup. Our result follows easily from the following lemmas, which are taken almost without change from [2].

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Lemma 2. *If I is a nonzero ideal of S, then $|I| = |S|$.*

Proof. If $I \neq 0$, then $|S/I| < |S|$. But $S = (S \setminus I) \cup I$ so that $|S| = |S \setminus I| + |I| = |S/I| + |I|$, which implies that $|S| = |I|$. □

Lemma 3.

(a) *If S has no zero, then S embeds in a group.*

(b) *If S has a zero, then $S \setminus \{0\}$ embeds in a group.*

Proof. It suffices to show that S or $S \setminus \{0\}$ is cancellative. First we show that if $0 \in S$ then S has no nonzero nilpotent elements. Let $s \in S$ be nilpotent of index n. Then the ideal $s^{n-1}S^1 = \{s^{n-1}t \mid t \in S^1 \setminus sS^1\}$ satisfies $|s^{n-1}S^1| \leq |S^1/sS^1| < |S|$. By Lemma 2 it follows that $s = 0$.

Now we show that if $0 \in S$ then $S \setminus 0$ is closed under multiplication. Let $s, t \in S$ with $st = 0$, and assume that $s \neq 0$. Then $(sS \cap tS)^2 = 0$ so that $sS \cap tS = 0$. Hence tS embeds in S/sS so $|tS| < |S|$, and Lemma 2 implies that $tS = 0$. In particular, $t^2 = 0$, so by the previous paragraph $t = 0$.

Finally we show that S or $S \setminus 0$ is cancellative. Let $0 \neq s \in S$ and define the congruence ρ_s by the following: if $a, b \in S$ then $a \rho_s b$ if and only if $as = bs$. By the previous paragraphs $sS \neq 0$. Then $|S| = |sS| = |S/\rho_s|$ so that ρ_s is the identity congruence, and hence $a = b$. □

Lemma 4. *The group of quotients of S or $S \setminus \{0\}$ is countable.*

Proof. Let G be the group of quotients of S or $S \setminus \{0\}$. We first show that G is H-smaller. Clearly, $|G| = |S|$. Let ρ be a congruence on G which is not 1-1. Suppose that $\frac{a}{b} \rho \frac{c}{d}$ for distinct $\frac{a}{b}$, $\frac{c}{d} \in G$. Then $((\frac{a}{b})bd)\rho((\frac{c}{d})bd)$; i.e., $ad \; \rho \; bc$ and $ad \neq bc$. Thus, ρ is not 1-1 on S so that $|S/\rho| < |S|$, and hence $|G/\rho| < |G|$.

It is now easy to see that G is countable. Let $g \in G$ be any non-identity element and let $K = \langle g \rangle$ be the group generated by g. Then $|G| = |K||G/K|$ and $|G/K| < |G|$, so $|G| = |K|$ where K is countable. □

Corollary 5. *Let G be an infinite commutative group. Then G is H-smaller if and only if $G \cong \mathbb{Z}$.*

Proof. This follows from the proof of the previous lemma. □

Theorem 6. *Let S be an infinite commutative semigroup. Then the following are equivalent*:

(1) *S is H-smaller*;

(2) *S is HF*;

(3) *S is one of the following*: $\mathbb{Z}$, $\mathbb{Z}^0$, $\mathbb{N}'$, $(\mathbb{N}')^0$, $(\mathbb{N}')^1$, *or* $(\mathbb{N}')^{0,1}$.

Proof. (1) ⇒ (2). By Lemma 4, S is countable and H-smaller, and hence HF.
(2) ⇒ (1). This follows by definition.
(2) ⇔ (3). This is Theorem 1, Jensen and Miller's Theorem. □

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