

## Oscillation Criteria for Certain Nonlinear Differential Equations with Damping

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ABSTRACT. Using the integral average method, we establish some oscillation criteria for the nonlinear differential equation with damped term

$$a(t)|x'(t)|^{\sigma-1}x'(t)' + p(t)|x'(t)|^{\sigma-1}x'(t) + q(t)f(x(t)) = 0, \quad \sigma > 1,$$

where the functions  $a$ ,  $p$  and  $q$  are real-valued continuous functions defined on  $[t_0, \infty)$  with  $a(t) > 0$ ,  $f(x) \in C^1(\mathbb{R})$  and  $\frac{f'(u)}{|f^{(\sigma-1)/\sigma}(u)|} \geq k > 0$  for  $u \neq 0$ .

### 1. Preliminaries

In this paper, we consider oscillation of the nonlinear differential equation with damped term

$$(1.1) \quad (a(t)|x'(t)|^{\sigma-1}x'(t))' + p(t)|x'(t)|^{\sigma-1}x'(t) + q(t)f(x(t)) = 0,$$

where the functions  $a, p$  and  $q$  are real-valued continuous functions defined on  $[t_0, \infty)$ ,  $a(t) > 0$ ,  $f(x) \in C^1(\mathbb{R})$  and

$$(1.2) \quad \frac{f'(u)}{|f^{(\sigma-1)/\sigma}(u)|} \geq k > 0 \text{ for } u \neq 0.$$

By a solution of Eq.(1.1), we mean a function  $x(t) \in C^1[T_x, \infty)$ ,  $T_x \geq t_0$ , where  $T_x \geq t_0$  depends on the particular solution, which has the property  $a(t)|x'(t)|^{\sigma-1}x'(t) \in C^1[T_x, \infty)$  and satisfies Eq.(1.1). A solution  $x(t)$  of Eq.(1.1) is said to be nontrivial if  $\sup\{|x(t)| : t \geq T\} > 0$  for all  $T \geq T_x$ . A nontrivial solution of Eq.(1.1) is

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said to be oscillatory if it has arbitrarily large zeros, otherwise, it is said to be non-oscillatory. Eq.(1.1) is said to be oscillatory if all the solutions are oscillatory.

When  $f(x(t)) = |x(t)|^{\sigma-1}x(t)$ , where  $\sigma > 1$  is a constant, the Eq.(1.1) reduces to half-linear differential equation with damped term

$$(1.3) \quad (a(t)|x'(t)|^{\sigma-1}x'(t))' + p(t)|x'(t)|^{\sigma-1}x'(t) + q(t)|x(t)|^{\sigma-1}x(t) = 0.$$

Furthermore, If  $p(t) \equiv 0$ , Eq.(1.3) reduces to the following half-linear differential equation

$$(1.4) \quad (a(t)|x'(t)|^{\sigma-1}x'(t))' + q(t)|x(t)|^{\sigma-1}x(t) = 0.$$

The subject of oscillation for the linear and nonlinear differential equation of second order has been received much attention in the last century. For such results, the reader is referred to the papers [5], [6], [7], [8], [9]. The half-linear differential equation (1.4) has the similar properties to linear differential equation, for example, the Sturm's comparison theorem (see [3]) and separation theorem (see Elbert [2] for details) are still true for Eq.(1.4).

Numerous oscillation criteria of Eq.(1.4) and various special cases have been obtained recently (see [1], [2], [3], [4] and references cited therein). In this paper, we will obtain some oscillation criteria of Kamenev type and Yan type for Eq. (1.1) by using integral average method, which generalize oscillation criteria mentioned above.

## 2. Main Results

In the sequel, we say that a function  $H = H(t, s)$  belongs to a function class  $\mathcal{H}$ , denoted by  $H \in \mathcal{H}$ , if  $H \in C(\Omega, R_+)$  with  $\Omega = \{(t, s) : t \geq s \geq t_0\}$ ,  $H(t, t) = 0$ ,  $H(t, s) > 0$  for  $t > s \geq t_0$ . Furthermore,  $H$  has a continuous and non-positive derivative on  $\Omega$  with respect to the second variable such that for all  $(t, s) \in \Omega$ ,

$$(2.1) \quad -\frac{\partial}{\partial s}H(t, s) = h(t, s)H(t, s),$$

where  $h \in L_{loc}(\Omega, R_+)$ . Some typical functions in  $\mathcal{H}$  are  $(t-s)^\alpha$  with  $\alpha \geq 2$  and  $\log(t/s)$ .

**Lemma 2.1** (see [14]). *Suppose  $X$  and  $Y$  are nonnegative, then*

$$(2.2) \quad qXY^{q-1} - X^q \leq (q-1)Y^q, \quad q > 1,$$

where equality holds if and only if  $X = Y$ .

Now, we will give oscillation criterion of Kamenev type.

**Theorem 2.2.** *Suppose that there exist  $H \in \mathcal{H}$  and a positive function  $\rho \in C^1[t_0, \infty)$  such that*

$$(2.3) \quad \limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \left[ H(t, s)\rho(s)q(s) - \frac{\sigma^\sigma a(s)\rho(s)R^{\sigma+1}(t, s)}{(\sigma+1)^{\sigma+1}[kH(t, s)]^\sigma} \right] ds = \infty,$$

where

$$R(t, s) = H(t, s) \left| \frac{\rho'(s)}{\rho(s)} - \frac{p(s)}{a(s)} - h(t, s) \right|.$$

Then Eq.(1.1) is oscillatory.

*Proof.* Suppose to the contrary that there is a nontrivial non-oscillatory solution  $x(t)$ . Without loss of generality, we may suppose  $x(t) > 0$  for  $t \geq t_0$ . Set

$$(2.4) \quad w(t) = \frac{\rho(t)a(t)|x'(t)|^{\sigma-1}x'(t)}{f(x(t))}, \quad t \geq t_0.$$

Using Eq.(1.1), (1.2) and (2.4), we have

$$(2.5) \quad \begin{aligned} w'(t) &= -\rho(t)q(t) + \left( \frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{a(t)} \right) w(t) - \frac{a(t)\rho(t)|x'(t)|^{\sigma+1}f'(x(t))}{f^2(x(t))} \\ &\leq -\rho(t)q(t) + \left( \frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{a(t)} \right) w(t) - \frac{k|w(t)|^{\frac{\sigma+1}{\sigma}}}{(a(t)\rho(t))^{1/\sigma}}, \quad t \geq t_0. \end{aligned}$$

For all  $t > T \geq t_0$ , replacing  $t$  in (2.5) by  $s$ , then multiplying by  $H(t, s)$  and integrating from  $T$  to  $t > T$ , we obtain

$$(2.6) \quad \begin{aligned} &\int_T^t H(t, s)\rho(s)q(s)ds \\ &\leq -\int_T^t H(t, s)w'(s)ds + \int_T^t H(t, s) \left( \frac{\rho'(s)}{\rho(s)} - \frac{p(s)}{a(s)} \right) w(s)ds \\ &\quad - \int_T^t \frac{kH(t, s)}{(a(s)\rho(s))^{1/\sigma}} |w(s)|^{\frac{\sigma+1}{\sigma}} ds \\ &= H(t, T)w(T) + \int_T^t H(t, s) \left( \frac{\rho'(s)}{\rho(s)} - \frac{p(s)}{a(s)} - h(t, s) \right) w(s)ds \\ &\quad - \int_T^t \frac{kH(t, s)}{(a(s)\rho(s))^{1/\sigma}} |w(s)|^{\frac{\sigma+1}{\sigma}} ds \\ &\leq H(t, T)w(T) + \int_T^t R(t, s)|w(s)|ds \\ &\quad - \int_T^t \frac{kH(t, s)}{(a(s)\rho(s))^{1/\sigma}} |w(s)|^{\frac{\sigma+1}{\sigma}} ds. \end{aligned}$$

Taking

$$\begin{aligned} X &= [kH(t, s)]^{\sigma/(\sigma+1)} \frac{|w(s)|}{(a(s)\rho(s))^{1/(\sigma+1)}}, \quad q = \frac{\sigma+1}{\sigma}, \\ Y &= \left( \frac{\sigma}{\sigma+1} \right)^\sigma \frac{(a(s)\rho(s))^{\sigma/(\sigma+1)} R^\sigma(t, s)}{[kH(t, s)]^{\sigma^2/(\sigma+1)}}. \end{aligned}$$

According to Lemma 2.1, we obtain for  $t > T \geq t_0$ ,

$$R(t, s)|w(s)| - \frac{kH(t, s)}{(a(s)\rho(s))^{1/\sigma}}|w(s)|^{\frac{\sigma+1}{\sigma}} \leq \frac{\sigma^\sigma a(s)\rho(s)R^{\sigma+1}(t, s)}{(\sigma + 1)^{\sigma+1}[kH(t, s)]^\sigma}.$$

Hence we obtain from (2.6) that

$$\int_T^t H(t, s)\rho(s)q(s)ds \leq H(t, T)w(T) + \int_T^t \frac{\sigma^\sigma a(s)\rho(s)R^{\sigma+1}(t, s)}{(\sigma + 1)^{\sigma+1}[kH(t, s)]^\sigma} ds.$$

A simple transfiguration yields

$$(2.7) \quad \frac{1}{H(t, T)} \int_T^t \left[ H(t, s)\rho(s)q(s) - \frac{\sigma^\sigma a(s)\rho(s)R^{\sigma+1}(t, s)}{(\sigma + 1)^{\sigma+1}[kH(t, s)]^\sigma} \right] ds \leq w(T).$$

Particularly, let  $T = t_0$ , and take the upper limit as  $t \rightarrow \infty$ , we obtain a contradiction with (2.3), which complete the proof of Theorem 2.1.  $\square$

**Corollary 2.3.** *Suppose that there exist  $H \in \mathcal{H}$  and a positive function  $\rho \in C^1[t_0, \infty)$  such that*

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \left[ H(t, s)\rho(s)q(s) - \frac{a(s)\rho(s)R^{\sigma+1}(t, s)}{(\sigma + 1)^{\sigma+1}H^\sigma(t, s)} \right] ds = \infty,$$

where

$$R(t, s) = H(t, s) \left| \frac{\rho'(s)}{\rho(s)} - \frac{p(s)}{a(s)} - h(t, s) \right|.$$

Then Eq.(1.3) is oscillatory.

**Remark 2.4.** When  $p(t) \equiv 0$ , Corollary 2.3 is Theorem 1 in paper [1].

**Corollary 2.5.** *Suppose that there exist  $H \in \mathcal{H}$  and a positive function  $\rho \in C^1[t_0, \infty)$  such that*

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \frac{a(s)\rho(s)R^{\sigma+1}(t, s)}{H^\sigma(t, s)} ds < \infty$$

and

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t H(t, s)\rho(s)q(s)ds = \infty.$$

Then Eq.(1.1) is oscillatory.

**Corollary 2.6.** *Suppose there exists a constant  $\alpha \geq 2$  and a positive function  $\rho \in C^1[t_0, \infty)$  such that*

$$\limsup_{t \rightarrow \infty} \frac{1}{t^\alpha} \int_{t_0}^t [(t - s)^\alpha \rho(s)q(s) - \Phi(t, s)] ds = \infty,$$

where

$$\Phi(t, s) = \frac{\sigma^\sigma a(s)\rho(s)(t-s)^\alpha}{k^\sigma(\sigma+1)^{\sigma+1}} \left| \frac{\rho'(s)}{\rho(s)} - \frac{p(s)}{a(s)} - \frac{\alpha}{t-s} \right|^{\sigma+1}.$$

Then Eq.(1.1) is oscillatory.

**Example 2.7.** Consider the following half-linear differential equation

$$\left(|x'(t)|^{\sigma-1}x'(t)\right)' + \frac{1}{2t}|x'(t)|^{\sigma-1}x'(t) + t^{-\lambda}(\epsilon + \sin t)|x(t)|^{\sigma-1}x(t) = 0$$

with  $\epsilon > 0$  and  $\sigma > 1$ , then  $k = \sigma$ . Taking  $\alpha = \sigma + 1$  in Corollary 2.6 and  $\rho(t) = t^{1/2}$ , we obtain

$$\Phi(t, s) = \frac{a(s)\rho(s)(t-s)^\alpha}{(\sigma+1)^{\sigma+1}} \left| \frac{\rho'(s)}{\rho(s)} - \frac{p(s)}{a(s)} - \frac{\alpha}{t-s} \right|^{\sigma+1} = s^{1/2}.$$

If  $\lambda \leq \frac{3}{2}$ , then it is easy to see that

$$\lim_{t \rightarrow \infty} \int_{t_0}^t s^{-\lambda+\frac{1}{2}}(\epsilon + \sin s)ds = \infty, \quad \epsilon > 0.$$

Now, using Lemma 1 in [5] we obtain

$$\lim_{t \rightarrow \infty} \frac{1}{t^{\sigma+1}} \int_{t_0}^t (t-s)^{\sigma+1} s^{-\lambda+\frac{1}{2}}(\epsilon + \sin s)ds = \infty, \quad \epsilon > 0.$$

Apply Corollary 2.6, note that

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{t^\alpha} \int_{t_0}^t ((t-s)^\alpha \rho(s)q(s) - \Phi(t, s)) ds \\ &= \limsup_{t \rightarrow \infty} \frac{1}{t^{\sigma+1}} \int_{t_0}^t \left[ (t-s)^{\sigma+1} s^{-\lambda+\frac{1}{2}}(\epsilon + \sin s) - s^{1/2} \right] ds \\ &= \infty. \end{aligned}$$

Hence the equation is oscillatory for  $\epsilon > 0$  and  $\lambda \leq \frac{3}{2}$ .

Apply Example 2.7 and Corollary 2.6, we note that if we let  $\rho(t) = \exp(\int_{t_0}^t \frac{p(s)}{a(s)} ds)$  and  $\alpha = \sigma + 1$ , then  $\Phi(t, s) = (\frac{\sigma}{k})^\sigma a(s) \exp(\int_{t_0}^s \frac{p(\tau)}{a(\tau)} d\tau)$ . So we obtain the following Corollary.

**Corollary 2.8.** Suppose that the coefficients of Eq.(1.1) satisfy

$$\limsup_{t \rightarrow \infty} \frac{1}{t^{\sigma+1}} \int_{t_0}^t \exp\left(\int_{t_0}^s \frac{p(\tau)}{a(\tau)} d\tau\right) \left( (t-s)^{\sigma+1} q(s) - \left(\frac{\sigma}{k}\right)^\sigma a(s) \right) ds = \infty.$$

Then Eq.(1.1) is oscillatory.

Next, we will give oscillation criterion of Yan type.

**Theorem 2.9.** *Suppose that there exist  $H \in \mathcal{H}$ ,  $\rho \in C^1[t_0, \infty)$  with  $\rho(t) > 0$ , and  $\psi \in C[t_0, \infty)$  such that*

$$(2.9) \quad 0 < \inf_{s \geq t_0} \left[ \liminf_{t \rightarrow \infty} \frac{H(t, s)}{H(t, t_0)} \right] \leq \infty,$$

$$(2.10) \quad \limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \frac{a(s)\rho(s)R^{\sigma+1}(t, s)}{H^\sigma(t, s)} ds < \infty,$$

$$(2.11) \quad \limsup_{t \rightarrow \infty} \frac{1}{H(t, T)} \int_T^t \left[ H(t, s)\rho(s)q(s) - \frac{\sigma^\sigma a(s)\rho(s)R^{\sigma+1}(t, s)}{(\sigma+1)^{\sigma+1}[kH(t, s)]^\sigma} \right] ds \geq \psi(T)$$

for all  $T \geq t_0$ , where  $R(t, s)$  is defined as in Theorem 2.2. Moreover,

$$(2.12) \quad \int_{t_0}^{\infty} \frac{\psi_+^{\frac{\sigma+1}{\sigma}}(s)}{(a(s)\rho(s))^{1/\sigma}} ds = \infty,$$

where  $\psi_+(t) = \max\{\psi(t), 0\}$ . Then Eq.(1.1) is oscillatory.

*Proof.* Followed the proof of Theorem 2.2, we get (2.7). This and (2.11) imply

$$(2.13) \quad \psi(T) \leq w(T)$$

and

$$(2.14) \quad \limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t H(t, s)\rho(s)q(s) ds \geq \psi(t_0).$$

Let

$$\begin{aligned} F(t) &= \frac{1}{H(t, t_0)} \int_{t_0}^t R(t, s)|w(s)| ds, \\ G(t) &= \frac{1}{H(t, t_0)} \int_{t_0}^t \frac{kH(t, s)}{(a(s)\rho(s))^{1/\sigma}} |w(s)|^{\frac{\sigma+1}{\sigma}} ds, \end{aligned}$$

then from (2.6) and (2.14),

$$(2.15) \quad \begin{aligned} \liminf_{t \rightarrow \infty} [G(t) - F(t)] &\leq w(t_0) - \limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t H(t, s)\rho(s)q(s) ds \\ &\leq w(t_0) - \psi(t_0) < \infty. \end{aligned}$$

Now, we claim that

$$(2.16) \quad \int_{t_0}^{\infty} \frac{|w(s)|^{\frac{\sigma+1}{\sigma}}}{(a(s)\rho(s))^{1/\sigma}} ds < \infty.$$

From (2.9), there exists  $\xi > 0$  such that

$$(2.17) \quad \inf_{s \geq t_0} \left[ \liminf_{t \rightarrow \infty} \frac{H(t, s)}{H(t, t_0)} \right] > \xi > 0.$$

If (2.16) is not true, then, for every  $\mu > 0$ , there exists  $t_1 > t_0$  such that

$$\int_{t_0}^t \frac{|w(s)|^{\frac{\sigma+1}{\sigma}}}{(a(s)\rho(s))^{1/\sigma}} ds > \frac{\mu}{k\xi}, \quad t \geq t_1.$$

From (2.17), there exists  $t_2 > t_1$  such that for  $t > t_2$ ,  $H(t, t_1)/H(t, t_0) > \xi$  and

$$\begin{aligned} G(t) &= \frac{1}{H(t, t_0)} \int_{t_0}^t \frac{kH(t, s)}{(a(s)\rho(s))^{1/\sigma}} |w(s)|^{\frac{\sigma+1}{\sigma}} ds \\ &= \frac{k}{H(t, t_0)} \int_{t_0}^t -\frac{\partial H(t, s)}{\partial s} \left\{ \int_{t_0}^s \frac{|w(\tau)|^{\frac{\sigma+1}{\sigma}}}{(a(\tau)\rho(\tau))^{1/\sigma}} ds \right\} ds \\ &\geq \frac{k}{H(t, t_0)} \int_{t_1}^t -\frac{\partial H(t, s)}{\partial s} \left\{ \int_{t_0}^s \frac{|w(\tau)|^{\frac{\sigma+1}{\sigma}}}{(a(\tau)\rho(\tau))^{1/\sigma}} ds \right\} ds \\ &\geq \frac{\mu}{\xi H(t, t_0)} \int_{t_1}^t -\frac{\partial H(t, s)}{\partial s} ds = \frac{\mu H(t, t_1)}{\xi H(t, t_0)} > \mu, \quad t > t_2. \end{aligned}$$

Therefore,

$$(2.18) \quad \lim_{t \rightarrow \infty} G(t) = \infty.$$

Let  $\{t_k\} \subset [t_0, \infty)$  be a sequence such that  $t_k \rightarrow \infty$  as  $k \rightarrow \infty$  and

$$(2.19) \quad \lim_{k \rightarrow \infty} [G(t_k) - F(t_k)] = \liminf_{t \rightarrow \infty} [G(t) - F(t)].$$

From (2.15), there exists a constant  $M > 0$  such that

$$(2.20) \quad G(t_k) - F(t_k) \leq M$$

for all sufficiently large  $k$ . Since (2.18) ensures that

$$(2.21) \quad \lim_{k \rightarrow \infty} G(t_k) = \infty,$$

(2.20) implies

$$(2.22) \quad \lim_{k \rightarrow \infty} F(t_k) = \infty.$$

Taking into account (2.20) and (2.21), we derive for  $k$  sufficiently large

$$\frac{F(t_k)}{G(t_k)} - 1 \geq -\frac{M}{G(t_k)} > -\frac{1}{2}.$$

Therefore,  $\frac{F(t_k)}{G(t_k)} > \frac{1}{2}$  for all large  $k$ , which together with (2.22) implies

$$(2.23) \quad \lim_{k \rightarrow \infty} \frac{F^{\sigma+1}(t_k)}{G^\sigma(t_k)} = \infty.$$

Moreover, by the Hölder inequality, we have

$$\begin{aligned} F(t_k) &= \frac{1}{H(t_k, t_0)} \int_{t_0}^{t_k} R(t_k, s) |w(s)| ds \\ &= \frac{1}{H(t_k, t_0)} \int_{t_0}^{t_k} \frac{[kH(t_k, s)]^{\frac{\sigma}{\sigma+1}}}{(a(s)\rho(s))^{\frac{1}{\sigma+1}}} |w(s)| \frac{(a(s)\rho(s))^{\frac{1}{\sigma+1}} R(t_k, s)}{[kH(t_k, s)]^{\frac{\sigma}{\sigma+1}}} ds \\ &\leq \left\{ \frac{1}{H(t_k, t_0)} \int_{t_0}^{t_k} \frac{kH(t_k, s)}{(a(s)\rho(s))^{\frac{1}{\sigma}}} |w(s)|^{\frac{\sigma+1}{\sigma}} ds \right\}^{\frac{\sigma}{\sigma+1}} \\ &\quad \times \left\{ \frac{1}{k^\sigma H(t_k, t_0)} \int_{t_0}^{t_k} \frac{a(s)\rho(s)R^{\sigma+1}(t_k, s)}{H^\sigma(t_k, s)} ds \right\}^{\frac{1}{\sigma+1}} \\ &= [G(t_k)]^{\frac{\sigma}{\sigma+1}} \left\{ \frac{1}{k^\sigma H(t_k, t_0)} \int_{t_0}^{t_k} \frac{a(s)\rho(s)R^{\sigma+1}(t_k, s)}{H^\sigma(t_k, s)} ds \right\}^{\frac{1}{\sigma+1}}. \end{aligned}$$

So we have

$$\frac{F^{\sigma+1}(t_k)}{G^\sigma(t_k)} \leq \frac{1}{k^\sigma H(t_k, t_0)} \int_{t_0}^{t_k} \frac{a(s)\rho(s)R^{\sigma+1}(t_k, s)}{H^\sigma(t_k, s)} ds.$$

But this is impossible because of (2.10) and (2.23). Thus, (2.16) is true. Therefore, from (2.13), we get

$$\int_{t_0}^\infty \frac{[\psi_+(s)]^{\frac{\sigma+1}{\sigma}}}{(a(s)\rho(s))^{1/\sigma}} ds \leq \int_{t_0}^\infty \frac{|w(s)|^{\frac{\sigma+1}{\sigma}}}{(a(s)\rho(s))^{1/\sigma}} ds < \infty,$$

which contradicts with condition (2.12). The proof of Theorem 2.9 is complete.  $\square$

**Corollary 2.10.** *Suppose there exist constant  $\alpha \geq 2$  and  $\psi \in C[t_0, \infty)$  such that*

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{t^\alpha} \int_{t_0}^t \Phi(t, s) ds &< \infty, \\ \limsup_{t \rightarrow \infty} \frac{1}{t^\alpha} \int_T^t [(t-s)^\alpha \rho(s)q(s) - \Phi(t, s)] ds &\geq \psi(T) \end{aligned}$$



for all  $T \geq t_0$ , Where  $\Phi(t, s)$  is the same as that in Corollary 2.6. Then Eq.(1.1) is oscillatory provided (2.12) is fulfilled.

**Theorem 2.11.** Suppose that there exist  $H \in \mathcal{H}$ ,  $\rho \in C^1[t_0, \infty)$  with  $\rho(t) > 0$ , and  $\psi \in C[t_0, \infty)$  such that

$$(2.24) \quad \liminf_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t H(t, s)\rho(s)q(s)ds < \infty,$$

and (2.9) are satisfied. For all  $T \geq t_0$ ,

$$\liminf_{t \rightarrow \infty} \frac{1}{H(t, T)} \int_T^t \left[ H(t, s)\rho(s)q(s) - \frac{\sigma^\sigma a(s)\rho(s)R^{\sigma+1}(t, s)}{(\sigma + 1)^{\sigma+1}[kH(t, s)]^\sigma} \right] ds \geq \psi(T)$$

and (2.12) hold, where  $R(t, s)$  is defined as in Theorem 2.2. Then Eq.(1.1) is oscillatory.

*Proof.* Followed the proof of Theorem 2.9, we get (2.13). Using (2.24), we conclude that

$$\begin{aligned} \limsup_{t \rightarrow \infty} [G(t) - F(t)] &\leq w(t_0) - \liminf_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t H(t, s)\rho(s)q(s)ds \\ &< \infty. \end{aligned}$$

Let  $\{t_k\} \subset [t_0, \infty)$  be a sequence such that  $t_k \rightarrow \infty$  as  $k \rightarrow \infty$  and

$$\lim_{k \rightarrow \infty} [G(t_k) - F(t_k)] = \limsup_{t \rightarrow \infty} [G(t) - F(t)].$$

As the proof of Theorem 2.9, we get (2.16). The remainder proof is the same as that of Theorem 2.9, which completes the proof.  $\square$

**Example 2.12.** Consider the following differential equation

$$(t^\lambda |x'(t)|^{\sigma-1} x'(t))' - t^{\lambda-1} |x'(t)|^{\sigma-1} x'(t) + t^\mu \cos t |x(t)|^{\sigma-1} x(t) = 0, \quad t \geq 1,$$

where  $\lambda, \mu$  and  $\sigma$  are real numbers,  $\sigma > 1$  is an integer,  $\mu < 1, \lambda \leq \mu(\sigma + 1)$ . Here we take  $H(t, s) = (t - s)^{\sigma+1}, \rho(t) = \frac{1}{t}$ , then  $\Phi(t, s) = s^{\lambda-1}$ . Now, condition (2.24) holds for  $\mu \leq 1$ . Since  $\mu < 1$  and  $\lambda \leq \mu(\sigma + 1)$ , we note that  $\lambda < \sigma + 1$ . For arbitrary small  $\epsilon > 0$ , there exists a  $t_1 > 1$  such that for  $T \geq t_1$

$$\begin{aligned} &\liminf_{t \rightarrow \infty} \frac{1}{t^{\sigma+1}} \int_T^t [(t - s)^{\sigma+1} s^{\mu-1} \cos s - s^{\lambda-1}] ds \\ &> \liminf_{t \rightarrow \infty} \frac{T^{\mu-1}}{t^{\sigma+1}} \int_T^t (t - s)^{\sigma+1} \cos s ds - \frac{\lambda t^\lambda}{t^{\sigma+1}} \\ &> -T^{\mu-1} \sin T - \epsilon \end{aligned}$$

Set  $\psi(t) = -t^{\mu-1} \sin t - \epsilon$ , select an integer  $N$  large enough such that  $(2N + 1)\pi + \frac{\pi}{4} > t_1$ , for  $n > N$  and  $t \in [(2n + 1)\pi + \frac{\pi}{4}, (2n + 1)\pi + \frac{3}{4}\pi]$ , we obtain  $\psi(t) \geq \frac{\sqrt{2}}{4}t^{\mu-1}$ . Now

$$\int_{t_0}^{\infty} \frac{\psi_{+\frac{\sigma}{\sigma}}^{\frac{\sigma+1}{\sigma}}(s)}{(a(s)\rho(s))^{1/\sigma}} ds \geq \sum_{n=N}^{\infty} \left(\frac{\sqrt{2}}{4}\right)^{\frac{\sigma+1}{\sigma}} \int_{(2n+1)\pi+\frac{\pi}{4}}^{(2n+1)\pi+\frac{3}{4}\pi} s^{\beta} ds,$$

where  $\beta = \frac{\sigma+1}{\sigma}(\mu-1) - \frac{\lambda-1}{\sigma}$ . Since  $\sigma > 1 > 0$  and  $\lambda \leq \mu(\sigma+1)$ , we see that  $\beta \geq -1$  and hence by Theorem 2.11, the equation is oscillatory.

**Theorem 2.13.** *Suppose that there exist  $H \in \mathcal{H}$ ,  $\rho \in C^1[t_0, \infty)$  with  $\rho(t) > 0$ , and  $\psi_1, \psi_2 \in C[t_0, \infty)$  such that (2.9) and*

$$(2.25) \quad \limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \frac{a(s)\rho(s)R^{\sigma+1}(t, s)}{H^{\sigma}(t, s)} ds > \psi_2(T),$$

$$(2.26) \quad \lim_{t \rightarrow \infty} \frac{1}{H(t, T)} \int_T^t \frac{a(s)\rho(s)R^{\sigma+1}(t, s)}{H^{\sigma}(t, s)} ds \leq \psi_1(T)$$

for all  $T \geq t_0$ , where  $R(t, s)$  is defined as in Theorem 2.2. Moreover,

$$(2.27) \quad \lim_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \frac{H(t, s)}{(a(s)\rho(s))^{1/\sigma}} \left[ \psi_2(s) - \frac{\psi_1(s)}{(\sigma+1)^{\sigma+1}} \right]_+^{\frac{\sigma+1}{\sigma}} ds = \infty.$$

Then Eq.(1.1) is oscillatory.

*Proof.* Followed the proof of Theorem 2.2, we get (2.7). This and (2.25), (2.26) imply

$$(2.28) \quad \psi_2(T) - \frac{\psi_1(T)}{(\sigma+1)^{\sigma+1}} \leq w(T).$$

Since  $G(t)$  is nondecreasing in  $t$  (see [5, Lemma 1] for details), we obtain

$$\lim_{t \rightarrow \infty} G(t) \leq \infty.$$

Now, we claim that

$$(2.29) \quad \lim_{t \rightarrow \infty} G(t) < \infty.$$

If (2.29) is not true, we obtain (2.18). Followed the process of the proof of Theorem 2.9, we obtain a contradiction with (2.26). So (2.29) is true. Then (2.28) and (2.29) lead to the desired contradiction to (2.27), which complete the proof.  $\square$

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