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On the Equivalance of Some Fixed Point Iterations

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Abstract. In this paper, we have shown that the convergence of one-step, two-step and three-step iterations is equivalent, which are known as Mann, Ishikawa and Noor iteration procedures, for a special class of Lipschitzian operators defined in a closed, convex subset of an arbitrary Banach space.

1. Introduction

The Mann iterative scheme was invented in [4], and was used to obtain convergence to a fixed point for many functions for which the Banach principle fails. In [5], Ishikawa devised a new iteration scheme to establish convergence for a Lipschitzian pseudocontractive map in a situation where the Mann iteration process failed to converge. Noor (see, [6], [7]) introduced and analyzed three-step iterative metods to study the approximate solutions of variational inclusions (inequalities) in Hilbert spaces by using the techniques of updating the solution and the auxiliary principle. Xu and Noor (see, [8]) studied the convergence of Noor iteration to fixed point of an asymptotically nonexpensive self-map defined in a closed, bounded and convex subset of an uniformly convex Banach space. In [9], Noor, Rassias and Huang proved the convergence of Noor iteration to fixed point of a pseudocontractive self-map defined in a real uniformly smooth Banach space.

B. E. Rhoades and Ş. M. Şoltuz showed that the convergence of Mann Iteration is equivalent to the convergence of Ishikawa iteration for various classes of functions at [1], [2] and [3]. A reasonable conjecture is that, whenever T is a function for which Mann iteration converges, so does the Ishikawa and Noor iterations. Given the large variety of functions and spaces, such a global statement is, of course, not provable. However, in this paper, we show that the convergence of Noor iteration is equivalent to the convergence of Mann and Ishikawa iterations for a special class of Lipschitzian operators.

Let X be an arbitrary Banach space, B a nonempty, convex subset of X and $T : B \to B$ be a operator. Let p_1, u_1 and $x_1 \in B$ be three arbitrary fixed points.

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The Mann iteration is defined by

$$p_{n+1} = (1-\alpha_n)\, p_n + \alpha_n T p_n, \quad n \geq 1, \tag{1}$$

where the sequence $(\alpha_n)_n \subset (0,1)$ is convergent, such that

$$\lim_{n\to\infty} \alpha_n = 0 \text{ and } \sum_{n=1}^{\infty} a_n = \infty. \tag{1.a}$$

The Ishikawa iteration scheme is defined by

$$\begin{aligned} u_{n+1} &= (1-\alpha_n)\, u_n + \alpha_n T v_n, \\ v_n &= (1-\beta_n)\, u_n + \beta_n T u_n, \quad n \geq 1, \end{aligned} \tag{2}$$

where the sequences $(\alpha_n)_n$ and $(\beta_n)_n \subset (0,1)$ are convergent, such that

$$\lim_{n\to\infty} \alpha_n = 0, \ \lim_{n\to\infty} \beta_n = 0 \text{ and } \sum_{n=1}^{\infty} a_n = \infty. \tag{2.a}$$

The Noor iteration scheme is defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n \\ y_n &= (1-\beta_n)x_n + \beta_n T z_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n T x_n, \quad n \geq 1, \end{aligned} \tag{3}$$

where the sequences $(\alpha_n)_n, (\beta_n)_n$ and $(\gamma_n)_n \subset (0,1)$ are convergent, such that

$$\lim_{n\to\infty} \alpha_n = 0, \ \lim_{n\to\infty} \beta_n = 0, \ \lim_{n\to\infty} \gamma_n = 0 \text{ and } \sum_{n=1}^{\infty} a_n = \infty. \tag{3.a}$$

The sequences $(\alpha_n)_n$, $(\beta_n)_n$ remains the same in three iterations. For $\gamma_n = 0$, $n \in N$, from (3) we get (2) and for $\beta_n = 0$, $\gamma_n = 0$, $n \in N$, from (2) we get (1). We denote by $F(T) = \{x^\star \in B : T(x^\star) = x^\star\}$.

We need the following lemma.

Lemma 1 ([10]). *Let $(\theta_n)_n$ be a nonnegative sequence which satisfies inequality*

$$\theta_{n+1} \leq (1-\lambda_n)\theta_n + \delta_n, \tag{4}$$

where $\lambda_n \in (0,1)$ for each $n \in N$, $\Sigma_{n=1}^{\infty}\lambda_n = \infty$ and $\delta_n = \lambda_n \epsilon_n$, $\lim \epsilon_n = 0$. Then $\lim \theta_n = 0$.

2. Main results

We are now able to give to the following results.

Theorem 2. *Let B be a nonempty closed convex subset of an arbitrary Banach space X, x_1 be a point in B and T be a Lipschitzian self-map of B with Lipschitz constant $L \leq 1$. Let $(u_n)_n$ and $(x_n)_n$ be defined by* (2) *and* (3) *with $(\alpha_n)_n$, $(\beta_n)_n$ and $(\gamma_n)_n$ satisfying* (3.a). *Then $(\|Tx_n - v_n\|)_n$ is bounded.*

Proof. If we prove that $(\|x_n\|)_n$ is bounded, then proof is obvious. From (3) we have

$$\begin{aligned}
\|x_{n+1}\| &= \|\alpha_n Ty_n + (1-\alpha_n)x_n\| \\
&\leq \alpha_n L\|\beta_n Tz_n + (1-\beta_n)x_n\| + (1-\alpha_n)\|x_n\| \\
&\leq \alpha_n L\left[\beta_n L\|z_n\| + (1-\beta_n)\|x_n\|)\right] + (1-\alpha_n)\|x_n\| \\
&= \alpha_n L^2\ \beta_n\|\gamma_n Tx_n + (1-\gamma_n)x_n\| + \alpha_n L(1-\beta_n)\|x_n\| + (1-\alpha_n)\|x_n\| \\
&\leq \alpha_n L^2\beta_n\left[\gamma_n\|Tx_n\| + (1-\gamma_n)\|x_n\|\right] + \alpha_n L(1-\beta_n)\|x_n\| + (1-\alpha_n)\|x_n\| \\
&\leq \|x_n\|\alpha_n L^3\beta_n\gamma_n + \|x_n\|\alpha_n L^2\beta_n(1-\gamma_n) + \alpha_n L(1-\beta_n)\|x_n\| + (1-\alpha_n)\|x_n\| \\
&= \|x_n\|\left[\alpha_n L^3\beta_n\gamma_n + \alpha_n L^2\beta_n(1-\gamma_n) + \alpha_n L(1-\beta_n) + (1-\alpha_n)\right] \\
&\leq \|x_n\|\left[\alpha_n\beta_n\gamma_n + \alpha_n\beta_n - \alpha_n\beta_n\gamma_n + \alpha_n - \alpha_n\beta_n + 1 - \alpha_n\right] \\
&= \|x_n\|.
\end{aligned}$$

A simple induction lead us to $\|x_{n+1}\| \leq \|x_n\| \leq \cdots \leq \|x_1\|$. □

Theorem 3. *Let B be a nonempty closed convex subset of an arbitrary Banach space X and be $p_1 = u_1 = x_1 \in B$. Let T be an Lipschitzian self-map of B with Lipschitz constant $L \leq 1$ and $x^\star$ be the fixed point of T. Let $(p_n)_n$, $(u_n)_n$ and $(x_n)_n$ be defined by* (1), (2) *and* (3) *with $(\alpha_n)_n$, $(\beta_n)_n$ and $(\gamma_n)_n$ satisfying* (1.a), (2.a) *and* (3.a), *respectively. Then the following are equivalent*:

(i) *the Mann iteration* (1) *converges to $x^\star \in F(T)$,*

(ii) *the Ishikawa iteration* (2) *converges to $x^\star \in F(T)$,*

(iii) *the Noor iteration* (3) *converges to $x^\star \in F(T)$.*

Proof. (i) ⇔ (ii) (see, [1]). That (iii) implies (ii) is obvious setting $\gamma_n = 0$ in (3). We prove that (ii) implies (iii). From (2) and (3),

$$\begin{aligned}
&\|x_{n+1} - u_{n+1}\| \\
=\ & \|\alpha_n Ty_n + (1-\alpha_n)x_n - \alpha_n Tv_n - (1-\alpha_n)u_n\| \\
\leq\ & (1-\alpha_n)\|x_n - u_n\| + \alpha_n\|Ty_n - Tv_n\| \\
\leq\ & (1-\alpha_n)\|x_n - u_n\| + \alpha_n L\|y_n - v_n\| \\
=\ & (1-\alpha_n)\|x_n - u_n\| + \alpha_n L\|\beta_n Tz_n + (1-\beta_n)x_n - \beta_n Tu_n - (1-\beta_n)u_n\| \\
\leq\ & (1-\alpha_n)\|x_n - u_n\| + \alpha_n L\left((1-\beta_n)\|x_n - u_n\| + \beta_n\|Tz_n - Tu_n\|\right) \\
\leq\ & (1-\alpha_n)\|x_n - u_n\| + \alpha_n L(1-\beta_n)\|x_n - v_n\| + \alpha_n\beta_n L^2\|z_n - u_n\| \\
=\ & \|x_n - u_n\|\left[(1-\alpha_n) + \alpha_n L(1-\beta_n)\right] + \alpha_n\beta_n L^2\|(1-\gamma_n)(u_n - x_n) + \gamma_n(u_n - Tx_n)\| \\
\leq\ & \|x_n - u_n\|\left[(1-\alpha_n) + \alpha_n L(1-\beta_n)\right] + \alpha_n\beta_n L^2\left[(1-\gamma_n)\|u_n - x_n\| + \gamma_n\|u_n - Tx_n\|\right] \\
=\ & \|x_n - u_n\|\{1 - \alpha_n\left[1 - L(1-\beta_n(1 - L(1-\gamma_n)))\right]\} + \alpha_n\beta_n\gamma_n L^2\|Tx_n - u_n\|.
\end{aligned}$$

From Theorem 2, $\|Tx_n - u_n\|$ is bounded, and so there is M such that $\|Tx_n - u_n\| \leq M$. Then, we obtain

$$\begin{aligned} & \|x_{n+1} - u_{n+1}\| \\ \leq \ & \|x_n - u_n\|\{1 - \alpha_n\left[1 - L(1 - \beta_n(1 - L(1 - \gamma_n)))\right]\} + \alpha_n\beta_n\gamma_n L^2 M \\ \leq \ & \|x_n - u_n\|\{1 - \alpha_n\left[1 - L(1 - \beta_n(1 - L(1 - \gamma_n)))\right]\} + \alpha_n\beta_n\gamma_n M. \end{aligned}$$

Obviously, $\{1 - L(1 - \beta_n(1 - L(1 - \gamma_n)))\} \in (0, 1)$. We say

$$\theta_n := \|x_n - u_n\|,$$

$$\lambda_n := \alpha_n\left[1 - L(1 - \beta_n(1 - L(1 - \gamma_n)))\right]$$

and

$$\delta_n := \alpha_n\beta_n\gamma_n M,$$

for each $n \in N$, then the inequality of Lemma 1 is satisfied. Therefore,

$$\lim \|x_n - u_n\| = 0.$$

Since (ii) is true

$$\begin{aligned} \|x_n - x^\star\| &\leq \|x_n - u_n\| + \|u_n - x^\star\| \\ \lim \|x_n - x^\star\| &\leq \lim \|x_n - u_n\| + \lim \|u_n - x^\star\|, \end{aligned}$$

which implies that $\lim \|x_n - x^\star\| = 0$. □

3. The equivalence between T-stability

Let $F(T) := \{x^* \in X : x^* = T(x^*)\}$, $x^* \in F(T)$. Consider

$$\eta_n = \left\|p_{n+1} - (1 - \alpha_n)\,p_n - \alpha_n T p_n\right\|, \tag{5}$$

$$\mu_n = \left\|u_{n+1} - (1 - \alpha_n)\,u_n - \alpha_n T v_n\right\|, \tag{6}$$

$$\xi_n = \left\|x_{n+1} - (1 - \alpha_n)\,x_n - \alpha_n T y_n\right\|. \tag{7}$$

Definition 4. If $\lim_{n\to\infty}\eta_n = 0$(respectively $\lim_{n\to\infty}\mu_n = 0$, $\lim_{n\to\infty}\xi_n = 0$) implies that $\lim_{n\to\infty} p_n = x^*$ (respectively $\lim_{n\to\infty} u_n = x^*$, $\lim_{n\to\infty} x_n = x^*$), then (1) (respectively (2)and (3)) is said to be T-stable.

It is obvious if we take the limit in (1), respectively (2) and (3).

Remark 5. Let X be an arbitrary Banach space with B a nonempty, convex and closed subset. Let $T : B \to B$ be a map. If the Mann (respectively Ishikawa and Noor) iteration converges, then $\lim_{n\to\infty}\eta_n = 0$(respectively $\lim_{n\to\infty}\mu_n = 0$ and $\lim_{n\to\infty}\xi_n = 0$).

Proof. Let $\lim_{n\to\infty} p_n = x^*$. Then from (5) we have

$$\begin{aligned} 0 &\leq \eta_n \leq \|p_{n+1} - p_n\| + \alpha_n \|p_n - Tp_n\| \\ &\leq \|p_{n+1} - x^*\| + \|p_n - x^*\| + \alpha_n \|p_n - x^*\| + \alpha_n \|x^* - Tp_n\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

□

We are able now to prove the following result:

Theorem 6. *Let B be a closed convex subset of an arbitrary Banach space X and (p_n), (u_n) and (x_n) be defined by* (1), (2) *and* (3) *with $(\alpha_n)_n$, $(\beta_n)_n$ and $(\gamma_n)_n$ satisfying* (1.*a*), (2.*a*) *and* (3.*a*), *respectively. Let an Lipschitzian self-map of B with Lipschitz constant $L \leq 1$ and $x^\star$ be the fixed point of T. If $p_1 = u_1 = x_1 \in B$, then the following three assertions are equivalent*:

(i) *Mann iteration* (1) *is T-stable,*

(ii) *Ishikawa iteration* (2) *is T-stable,*

(iii) *Noor iteration* (3) *is T-stable.*

Proof. (ii)⇔(i) (see, [3]). From Definition 4, (iii)⇔(ii) means that $\lim_{n\to\infty} \xi_n = 0 \Leftrightarrow \lim_{n\to\infty} \mu_n = 0$ and so, $\lim_{n\to\infty} \xi_n = 0 \Longrightarrow \lim_{n\to\infty} \mu_n = 0$ is obvious by setting $\gamma_n = 0$ in (3).

Conversely, suppose that (1) is T-stable. Using Definition 4, we get

$$\lim_{n\to\infty} \eta_n = 0 \Rightarrow \lim_{n\to\infty} p_n = x^*.$$

Theorem 3 assure that $\lim_{n\to\infty} p_n = x^* \Rightarrow \lim_{n\to\infty} u_n = x^\star$. Using Remark 5 we have $\lim_{n\to\infty} \mu_n = 0$. Thus we get $\lim_{n\to\infty} \eta_n = 0 \Rightarrow \lim_{n\to\infty} \mu_n = 0$. Similarly, suppose that (2) is T-stable. Again, we get

$$\lim_{n\to\infty} \mu_n = 0 \Rightarrow \lim_{n\to\infty} u_n = x^*.$$

From Theorem 3, $\lim_{n\to\infty} u_n = x^* \Rightarrow \lim_{n\to\infty} x_n = x^*$ and so, we have $\lim_{n\to\infty} \xi_n = 0$. Finally, we get $\lim_{n\to\infty} \mu_n = 0 \Rightarrow \lim_{n\to\infty} \xi_n = 0$. □

Example 7. Let $X = R$, $B = [0,2]$ and $T : B \to B$, $Tx = \frac{2}{x+1}$. Obviously, T is a Lipschitzian self-map of B with Lipschitz constant $L \leq 1$ and T has a fixed point $x^\star = 1 \in B$. Let's choice the sequences $(\alpha_n)_n$, $(\beta_n)_n$ and $(\gamma_n)_n$ such that $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{1}{n+2}$ and $\gamma_n = \frac{1}{n+3}$, and $p_1 = u_1 = x_1 = 0.1 \in B$. Then, conditions of Theorem 3 are satisfied. A few step of iterations (1), (2) and (3) are following.

	Mann Iteration	Ishikawa Iteration	Noor Iteration
$n=1$	0.1000000	0.1000000	0.1000000
$n=2$	0.9590909	0.6478261	0.7155459
$n=3$	0.9796879	0.8044693	0.8446285
$n=4$	0.9873310	0.8714129	0.8983083
$n=5$	0.9911398	0.9071817	0.9267462
$n=6$	0.9933581	0.9289299	0.9439700
$n=7$	0.9947829	0.9433176	0.9553407
$n=8$	0.9957620	0.9534205	0.9633149
$n=9$	0.9964688	0.9608368	0.9691638
$n=10$	0.9969988	0.9664715	0.9736051
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n=100$	0.9999098	0.9989281	0.9991564
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
∞	1	1	1

Table 1

Under the same condition, if we choice $p_1 = u_1 = x_1 = 2.0 \in B$, then we obtain following table.

	Mann Iteration	Ishikawa Iteration	Noor Iteration
$n=1$	2.000000	2.000000	2.000000
$n=2$	1.333333	1.391304	1.387097
$n=3$	1.174603	1.223495	1.219731
$n=4$	1.110879	1.149243	1.146236
$n=5$	1.078198	1.108717	1.106315
$n=6$	1.058894	1.083740	1.081787
$n=7$	1.046394	1.067063	1.065444
$n=8$	1.037761	1.055275	1.053908
$n=9$	1.031506	1.046579	1.045408
$n=10$	1.026805	1.039947	1.038930
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n=100$	1.000810	1.001291	1.001256
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
∞	1	1	1

Table 2

Table 1 and Table 2 have been obtained by FORTRAN 90 Programming Language.

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