

Noor Iterations with Error for Non-Lipschitzian Mappings in Banach Spaces

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ABSTRACT. Suppose C is a nonempty closed convex subset of a real uniformly convex Banach space X . Let $T : C \rightarrow C$ be an asymptotically nonexpansive in the intermediate sense mapping. In this paper we introduced the three-step iterative sequence for such map with error members. Moreover, we prove that, if T is completely continuous then the our iterative sequence converges strongly to a fixed point of T .

1. Introduction

Let C be a subset of real normed linear space X , and let T be a self-mapping on C . T is said to be *nonexpansive* provided $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$; T is called *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ of real numbers with $\lim_{n \rightarrow \infty} k_n = 1$ such that for each $x, y \in C$ and $n \geq 1$,

$$\|T^n x - T^n y\| \leq k_n \|x - y\|.$$

T is called *asymptotically nonexpansive in the intermediate sense* [1] provided T is uniformly continuous and

$$\limsup_{n \rightarrow \infty} \sup_{x, y \in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0.$$

From the above definitions, it follows that asymptotically nonexpansive mapping must be asymptotically nonexpansive in the intermediate sense and asymptotically quasi-nonexpansive mapping. But the converges dose not holds as the following example:

Example 1.1 (see [6]). Let $X = \mathbb{R}$, $C = [-\frac{1}{\pi}, \frac{1}{\pi}]$ and $|k| < 1$. For each $x \in C$, define

$$T(x) = \begin{cases} kx \sin \frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

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Then T is an asymptotically nonexpansive in the intermediate sense but it is not asymptotically nonexpansive mapping.

The concept of asymptotically nonexpansiveness was introduced by Goebel and Kirk ([3]) in 1992. In 2001, Noor ([8], [9]) have introduced the three-step iterative sequences and he studied the approximate solutions of variational inclusions(inequalities) in Hilbert spaces. The three-step iterative approximation problems were studied extensively by Noor ([8], [9]), Glowinski and Le Tallec ([2]), Haubruge et al ([4]).

In 2002, Xu and Noor ([14]) introduced the three-step iterative for asymptotically nonexpansive mappings and they proved the following strong convergence theorem in Banach spaces;

Theorem XN ([14], Theorem 2.1). *Let X be a real uniformly convex Banach space, C be a nonempty closed, bounded convex subset of X . Let T be a completely continuous asymptotically nonexpansive self-mapping with sequence $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $[0, 1]$ satisfying;*

$$(i) \quad 0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1, \text{ and}$$

$$(ii) \quad 0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1.$$

For a give $x_0 \in C$, define

$$(1.1) \quad \begin{aligned} z_n &= \gamma_n T^n x_n + (1 - \gamma_n) x_n \\ y_n &= \beta_n T^n z_n + (1 - \beta_n) x_n \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n) x_n. \end{aligned}$$

Then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converges strongly to a fixed point of T .

Algorithm 1.1(Noor iterations with errors). Let C be a nonempty subset of normed space X and let $T : C \rightarrow C$ be a mapping. For a given $x_0 \in C$, find the sequence $\{x_{n+1}\}$ such that

$$(1.2) \quad \begin{aligned} z_n &= \alpha_n'' T^n x_n + \beta_n'' x_n + \gamma_n'' u_n \\ y_n &= \alpha_n' T^n z_n + \beta_n' x_n + \gamma_n' v_n \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n, \end{aligned}$$

where $\{\alpha_n\}, \{\alpha_n'\}, \{\alpha_n''\}, \{\beta_n\}, \{\beta_n'\}, \{\beta_n''\}, \{\gamma_n\}, \{\gamma_n'\}$ and $\{\gamma_n''\}$ are real sequences in $[0, 1]$ and $\{u_n\}, \{v_n\}$ and $\{w_n\}$ are three bounded sequences in C .

It is clear that the Mann and Ishikawa iterations processes are all special case of the Noor iterations with error.

In this paper, we will extend the process (1.1) to Noor iteration with error (1.2) for asymptotically nonexpansive in the intermediate sense and without boundedness conditions on C . The results presented in this paper generalize and extend the corresponding main results of Xu and Noor ([14]).

2. Preliminaries

For the sake of convenience, we first recall some definitions and conclusions.

Definition 2.1 (see [3]). A Banach space X is said to be *uniformly convex* if the modulus of convexity of X

$$\delta_X(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : \|x\| = \|y\| = 1, \|x - y\| = \epsilon\right\} > 0$$

for all $0 < \epsilon \leq 2$ (i.e., $\delta_X(\epsilon)$ is a function $(0, 2] \rightarrow (0, 1)$).

Lemma 2.2 (see [7]). *Let the nonnegative number sequences $\{a_n\}$, $\{b_n\}$ and $\{d_n\}$ satisfy that*

$$a_{n+1} \leq (1 + b_n)a_n + d_n, \forall n = 1, 2, \dots, \sum_{n=1}^{\infty} b_n < \infty, \sum_{n=1}^{\infty} d_n < \infty.$$

Then

- (1) $\lim_{n \rightarrow \infty} a_n$ exists.
- (2) If $\liminf_{n \rightarrow \infty} a_n = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.3 ([13], J. Schu's Lemma). *Let X be a real uniformly convex Banach space, $0 < \alpha \leq t_n \leq \beta < 1$, $x_n, y_n \in X$, $\limsup_{n \rightarrow \infty} \|x_n\| \leq a$, $\limsup_{n \rightarrow \infty} \|y_n\| \leq a$, and $\lim_{n \rightarrow \infty} \|t_n x_n + (1 - t_n)y_n\| = a$, $a \geq 0$. Then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.*

3. Main results

In this section, we prove our main theorem. First of all, we shall need the following lemmas.

Lemma 3.1. *Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X . Let T be an asymptotically nonexpansive in the intermediate sense. Put*

$$G_n = \sup_{x, y \in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0, \forall n \geq 1,$$

so that $\sum_{n=1}^{\infty} G_n < \infty$. Let $x_0 \in C$ and

$$\begin{aligned} z_n &= \alpha_n'' T^n x_n + \beta_n'' x_n + \gamma_n'' u_n \\ y_n &= \alpha_n' T^n z_n + \beta_n' x_n + \gamma_n' v_n \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n, \end{aligned}$$

where $\{\alpha_n\}$, $\{\alpha_n'\}$, $\{\alpha_n''\}$, $\{\beta_n\}$, $\{\beta_n'\}$, $\{\beta_n''\}$, $\{\gamma_n\}$, $\{\gamma_n'\}$ and $\{\gamma_n''\}$ are real sequences in $[0, 1]$ and $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are three bounded sequences in C such that

- (i) $\alpha_n + \beta_n + \gamma_n = \alpha_n' + \beta_n' + \gamma_n' = \alpha_n'' + \beta_n'' + \gamma_n'' = 1$.

$$(ii) \sum_{n=1}^{\infty} \gamma_n < \infty, \quad \sum_{n=1}^{\infty} \gamma'_n < \infty, \quad \sum_{n=1}^{\infty} \gamma''_n < \infty.$$

Then for each $p \in F(T)$, $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists.

Proof. By the Schauder fixed-point theorem [12], we obtain that $F(T) \neq \emptyset$. Let $p \in F(T)$, since $\{u_n\}, \{v_n\}$ and $\{w_n\}$ are bounded sequences in C , so we put

$$K = \sup_{n \geq 1} \|u_n - p\| \vee \sup_{n \geq 1} \|v_n - p\| \vee \sup_{n \geq 1} \|w_n - p\|.$$

For each $n \geq 1$, we note that

$$(3.1) \quad \begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n x_n T^n y_n + \beta_n x_n + \gamma_n w_n - p\| \\ &\leq \alpha_n \|T^n y_n - p\| + \beta_n \|x_n - p\| + \gamma_n \|w_n - p\| \\ &\leq \alpha_n \|y_n - p\| + G_n + \beta_n \|x_n - p\| + \gamma_n \|w_n - p\| \end{aligned}$$

and

$$(3.2) \quad \begin{aligned} \|y_n - p\| &= \|\alpha'_n T^n z_n + \beta'_n x_n + \gamma'_n v_n - p\| \\ &\leq \alpha'_n \|T^n z_n - p\| + \beta'_n \|x_n - p\| + \gamma'_n \|v_n - p\| \\ &\leq \alpha'_n \|z_n - p\| + G_n + \beta'_n \|x_n - p\| + \gamma'_n \|v_n - p\| \end{aligned}$$

and

$$(3.3) \quad \|z_n - p\| \leq \alpha''_n \|x_n - p\| + G_n + \beta''_n \|x_n - p\| + \gamma''_n \|u_n - p\|.$$

Substituting (3.3) into (3.2),

$$(3.4) \quad \begin{aligned} \|y_n - p\| &\leq \alpha'_n \alpha''_n \|x_n - p\| + \alpha'_n G_n + \alpha'_n \beta''_n \|x_n - p\| + \alpha'_n \gamma''_n \|u_n - p\| \\ &\quad + G_n + \beta'_n \|x_n - p\| + \gamma'_n \|v_n - p\| \\ &\leq (1 - \beta'_n - \gamma'_n) \alpha''_n \|x_n - p\| + \beta'_n \|x_n - p\| + (1 - \beta'_n - \gamma'_n) \beta''_n \|x_n - p\| + m_n \\ &\leq \beta'_n \|x_n - p\| + (1 - \beta'_n) \alpha''_n \|x_n - p\| + (1 - \beta'_n) \beta''_n \|x_n - p\| + m_n \\ &= \beta'_n \|x_n - p\| + (1 - \beta'_n) (\alpha''_n + \beta''_n) \|x_n - p\| + m_n \\ &\leq \beta'_n \|x_n - p\| + (1 - \beta'_n) \|x_n - p\| + m_n \\ &= \|x_n - p\| + m_n, \end{aligned}$$

where $m_n = 2G_n + \gamma'_n \|v_n - p\| + \gamma''_n \|u_n - p\|$. Substituting (3.4) into (3.1) again, we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n (\|x_n - p\| + m_n) + G_n + \beta_n \|x_n - p\| + \gamma_n \|w_n - p\| \\ &\leq (\alpha_n + \beta_n) \|x_n - p\| + \alpha_n m_n + G_n + \gamma_n \|w_n - p\| \\ &\leq \|x_n - p\| + m_n + G_n + \gamma_n \|w_n - p\| \\ &\leq \|x_n - p\| + 3G_n + (\gamma_n + \gamma'_n + \gamma''_n) M \\ &= \|x_n - p\| + b_n, \end{aligned}$$

where $b_n = 3G_n + (\gamma_n + \gamma'_n + \gamma''_n)M$. Since $\sum_{n=1}^\infty b_n < \infty$, by Lemma 2.2, we have $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. This completes the proof. \square

Lemma 3.2. *Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X . Let T be an asymptotically nonexpansive in the intermediate sense. Put*

$$G_n = \sup_{x,y \in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0, \forall n \geq 1.$$

Let $x_0 \in C$ and for each $n \geq 0$,

$$\begin{aligned} z_n &= \alpha''_n T^n x_n + \beta''_n x_n + \gamma''_n u_n \\ y_n &= \alpha'_n T^n z_n + \beta'_n x_n + \gamma'_n v_n \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n, \end{aligned}$$

where $\{\alpha_n\}, \{\alpha'_n\}, \{\alpha''_n\}, \{\beta_n\}, \{\beta'_n\}, \{\beta''_n\}, \{\gamma_n\}, \{\gamma'_n\}, \{\gamma''_n\}$ and $\{u_n\}, \{v_n\}$ and $\{w_n\}$ are three bounded sequences in C such that

- (i) $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$.
- (ii) $\sum_{n=1}^\infty \gamma_n < \infty, \sum_{n=1}^\infty \gamma'_n < \infty, \sum_{n=1}^\infty \gamma''_n < \infty$.
- (iii) $0 \leq \alpha < \alpha_n, \alpha'_n \leq \beta < 1$. Then

- (a) $\lim_{n \rightarrow \infty} \|T^n y_n - x_n\| = 0$;
- (b) $\lim_{n \rightarrow \infty} \|T^n z_n - x_n\| = 0$.

Proof. (a). For any $p \in F(T)$, it follows from Lemma 3.1, we have $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. Let $\lim_{n \rightarrow \infty} \|x_n - p\| = a$ for some $a \geq 0$. From (3.4), we have

$$\|y_n - p\| \leq \|x_n - p\| + m_n, \forall n \geq 1.$$

Taking $\limsup_{n \rightarrow \infty}$ in both sides, we obtain

$$\limsup_{n \rightarrow \infty} \|y_n - p\| \leq \limsup_{n \rightarrow \infty} \|x_n - p\| = \lim_{n \rightarrow \infty} \|x_n - p\| = a.$$

Note that

$$\limsup_{n \rightarrow \infty} \|T^n y_n - p\| \leq \limsup_{n \rightarrow \infty} (\|y_n - p\| + G_n) = \limsup_{n \rightarrow \infty} \|y_n - p\| \leq a.$$

Next, consider

$$\|T^n y_n - p + \gamma_n(w_n - x_n)\| \leq \|T^n y_n - p\| + \gamma_n \|w_n - x_n\|.$$

Thus,

$$(3.5) \quad \limsup_{n \rightarrow \infty} \|T^n y_n - p + \gamma_n(w_n - x_n)\| \leq a.$$

Also, $\|x_n - p + \gamma_n(w_n - x_n)\| \leq \|x_n - p\| + \gamma_n\|w_n - x_n\|$,
gives that

$$(3.6) \quad \limsup_{n \rightarrow \infty} \|x_n - p + \gamma_n(w_n - x_n)\| \leq a,$$

and

$$\begin{aligned} a &= \lim_{n \rightarrow \infty} \|x_{n+1} - p\| \\ &= \lim_{n \rightarrow \infty} \|\alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n - p\| \\ &= \lim_{n \rightarrow \infty} \|\alpha_n T^n y_n + (1 - \alpha_n)x_n - \gamma_n x_n + \gamma_n w_n - (1 - \alpha_n)p - \alpha_n p\| \\ &= \lim_{n \rightarrow \infty} \|\alpha_n T^n y_n - \alpha_n p + \alpha_n \gamma_n w_n - \alpha_n \gamma_n x_n + (1 - \alpha_n)x_n - (1 - \alpha_n)p \\ &\quad - \gamma_n x_n + \gamma_n w_n - \alpha_n \gamma_n w_n + \alpha_n \gamma_n x_n\| \\ &= \lim_{n \rightarrow \infty} \|\alpha_n(T^n y_n - p + \gamma_n(w_n - x_n)) + (1 - \alpha_n)(x_n - p + \gamma_n(w_n - x_n))\|. \end{aligned}$$

By J. Schu's Lemma, we have

$$(3.7) \quad \lim_{n \rightarrow \infty} \|T^n y_n - x_n\| = 0.$$

This completes the proof of (a).

(b). For each $n \geq 1$,

$$\begin{aligned} \|x_n - p\| &\leq \|x_n - T^n y_n\| + \|T^n y_n - p\| \\ &\leq \|x_n - T^n y_n\| + \|y_n - p\| + G_n. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \|x_n - T^n y_n\| = 0 = \lim_{n \rightarrow \infty} G_n$, we obtain that

$$a = \lim_{n \rightarrow \infty} \|x_n - p\| \leq \liminf_{n \rightarrow \infty} \|y_n - p\|.$$

It follows that

$$a \leq \liminf_{n \rightarrow \infty} \|y_n - p\| \leq \limsup_{n \rightarrow \infty} \|y_n - p\| \leq a.$$

This implies that

$$\lim_{n \rightarrow \infty} \|y_n - p\| = a.$$

On the other hand, we note that

$$\begin{aligned} \|z_n - p\| &\leq \|\alpha_n'' T^n x_n + \beta_n'' x_n + \gamma_n'' u_n - p\| \\ &\leq \alpha_n'' \|x_n - p\| + G_n + \beta_n'' \|x_n - p\| + \gamma_n'' \|u_n - p\| \\ &\leq \alpha_n'' \|x_n - p\| + G_n + (1 - \alpha_n'') \|x_n - p\| + \gamma_n'' \|u_n - p\| \\ &\leq \|x_n - p\| + G_n + \gamma_n'' \|u_n - p\|. \end{aligned}$$

By boundedness of $\{u_n\}$ and $\lim_{n \rightarrow \infty} G_n = 0 = \lim_{n \rightarrow \infty} \gamma_n''$, we have

$$\limsup_{n \rightarrow \infty} \|z_n - p\| \leq \limsup_{n \rightarrow \infty} \|x_n - p\| = a,$$

and

$$\limsup_{n \rightarrow \infty} \|T^n z_n - p\| \leq \limsup_{n \rightarrow \infty} (\|z_n - p\| + G_n) \leq a$$

Next, consider

$$\|T^n z_n - p + \gamma_n'(v_n - x_n)\| \leq \|T^n z_n - p\| + \gamma_n' \|v_n - x_n\|.$$

Thus

$$\limsup \|T^n z_n - p + \gamma_n'(v_n - x_n)\| \leq a.$$

Also,

$$\|x_n - p + \gamma_n'(v_n - x_n)\| \leq \|x_n - p\| + \gamma_n' \|v_n - x_n\|$$

gives that

$$\limsup \|x_n - p + \gamma_n'(v_n - x_n)\| \leq a$$

and

$$\begin{aligned} a = \lim_{n \rightarrow \infty} \|y_n - p\| &= \lim_{n \rightarrow \infty} \|\alpha_n' T^n z_n + \beta_n' x_n + \gamma_n' v_n - p\| \\ &= \lim_{n \rightarrow \infty} \|\alpha_n' [T^n z_n - x_n + \gamma_n'(v_n - x_n)] \\ &\quad + (1 - \alpha_n') [x_n - p + \gamma_n'(v_n - x_n)]\|. \end{aligned}$$

By J. Schu's Lemma, we have

$$(3.8) \quad \lim_{n \rightarrow \infty} \|T^n z_n - x_n\| = 0.$$

This completes the proof of (b). □

Theorem 3.3. Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X . Let T be a completely continuous asymptotically non-expansive in the intermediate sense. Put

$$G_n = \sup_{x, y \in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0, \forall n \geq 1.$$

Let $x_0 \in C$ and for each $n \geq 0$,

$$\begin{aligned} z_n &= \alpha_n'' T^n x_n + \beta_n'' x_n + \gamma_n'' u_n \\ y_n &= \alpha_n' T^n z_n + \beta_n' x_n + \gamma_n' v_n \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n, \end{aligned}$$

where $\{\alpha_n\}, \{\alpha_n'\}, \{\alpha_n''\}, \{\beta_n\}, \{\beta_n'\}, \{\beta_n''\}, \{\gamma_n\}, \{\gamma_n'\}$ and $\{\gamma_n''\}$ are real sequences in $[0, 1]$ and $\{u_n\}, \{v_n\}$ and $\{w_n\}$ are three bounded sequences in C such that

- (i) $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$.
- (ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$, $\sum_{n=1}^{\infty} \gamma''_n < \infty$.
- (iii) $0 < \alpha \leq \alpha_n$, $\alpha'_n \leq \beta < 1$. Then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converges strongly to a fixed point of T .

Proof. It follows from Lemma 3.2, that

$$\lim_{n \rightarrow \infty} \|T^n y_n - x_n\| = 0 = \lim_{n \rightarrow \infty} \|T^n z_n - x_n\|$$

and this implies that,

$$\|x_{n+1} - x_n\| \leq \alpha_n \|T^n y_n - x_n\| + \gamma_n \|w_n - x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Thus

$$\begin{aligned} \|T^n x_n - x_n\| &\leq \|T^n x_n - T^n y_n\| + \|T^n y_n - x_n\| \\ &\leq \|x_n - y_n\| + G_n + \|T^n y_n - x_n\| \\ &\leq \alpha'_n \|x_n - T^n z_n\| + G_n + \gamma'_n \|v_n - x_n\| \\ &\quad + \|T^n y_n - x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \tag{9}$$

Since

$$\begin{aligned} \|x_n - Tx_n\| &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - T^{n+1}x_{n+1}\| \\ &\quad + \|T^{n+1}x_{n+1} - T^{n+1}x_n\| + \|T^{n+1}x_n - Tx_n\|, \end{aligned}$$

and by uniform continuity of T and (3.9), we have

$$\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0. \tag{10}$$

By Lemma 3.1, $\{x_n\}$ is a bounded. It follows by our assumption that T is completely continuous, there exists a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ such that $Tx_{n_k} \rightarrow p \in C$ as $k \rightarrow \infty$. Moreover, by (3.10), we have $\|Tx_{n_k} - x_{n_k}\| \rightarrow 0$ which implies that $x_{n_k} \rightarrow p$ as $k \rightarrow \infty$. By (3.10) again, we have

$$\|p - Tp\| = \lim_{k \rightarrow \infty} \|x_{n_k} - Tx_{n_k}\| = 0.$$

It show that $p \in F(T)$. Furthermore, since $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. Therefore $\lim_{n \rightarrow \infty} \|x_n - p\| = 0$, that is $\{x_n\}$ converges to some fixed point of T .

$$\|y_n - x_n\| \leq \alpha'_n \|T^n z_n - x_n\| + \gamma'_n \|v_n - x_n\| \rightarrow 0,$$

and

$$\|z_n - x_n\| \leq \alpha''_n \|T^n x_n - x_n\| + \gamma''_n \|u_n - x_n\| \rightarrow 0.$$

Therefore $\lim_{n \rightarrow \infty} y_n = p = \lim_{n \rightarrow \infty} z_n$. This completes the proof. \square

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