KYUNGPOOK Math. J. 46(2006), 189-200

The Semicontinuous Quasi-uniformity of a Frame

Maria João Ferreira and Jorge Picado
Departamento de Matemática, Universidade de Coimbra, Apartado 3008, Coimbra, Portugal
e-mail : `mjrf@mat.uc.pt` *and* `picado@mat.uc.pt`

Abstract. The semicontinuous quasi-uniformity is known to be one of the most important examples of transitive quasi-uniformities. The aim of this paper is to show that various facts in classical topology connected with the semicontinuous quasi-uniformity and semicontinuous real functions may be easily extended to pointfree topology via a construction introduced by the authors in a previous paper. Several consequences are derived.

In a previous paper [3], we established a method of constructing transitive compatible quasi-uniformities for an arbitrary frame, extending classical results of Fletcher for quasi-uniform spaces [4]. This note is a sequel to [3], which ended with the observation that, starting with the collection of all spectrum covers of a frame, one gets, via that construction, the so called semicontinuous quasi-uniformity of the frame. The present note aims to prove it. For this, we need to look at the frame analogue of semicontinuous real functions.

The familiar adjointness between functors $\mathcal{O} : \mathsf{Top} \to \mathsf{Frm}$ and $\Sigma : \mathsf{Frm} \to \mathsf{Top}$ gives us a natural isomorphism

$$\mathsf{Frm}(L, \mathcal{O}X) \xrightarrow{\sim} \mathsf{Top}(X, \Sigma L).$$

For L the frame $\mathfrak{L}(\mathbb{R})$ of reals one obtains

$$\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{O}X) \xrightarrow{\sim} \mathsf{Top}(X, \mathbb{R}),$$

since $\Sigma\mathfrak{L}(\mathbb{R})$ is homeomorphic to the space $\mathbb{R}$ of reals with euclidean topology. This means that continuous real functions $X \to \mathbb{R}$ are represented by frame homomorphisms $\mathfrak{L}(\mathbb{R}) \to \mathcal{O}X$ and hence regarding the frame homomorphisms $\mathfrak{L}(\mathbb{R}) \to L$, for a general frame L, as the *continuous real functions* on L provides a natural extension of the classical notion (see [1] for a detailed account).

Received August 19, 2004, and, in revised form, November 28, 2005.
2000 Mathematics Subject Classification: 06D22, 54C30, 54E05, 54E15, 54E55.
Key words and phrases: frame, biframe, quasi-uniform frame, biframe of reals, semicontinuous real function, transitive quasi-uniformity, functorial quasi-uniformity, spectrum cover.
The authors gratefully acknowledge financial support by CMUC/FCT.

Analogously, Li and Wang [13] introduced upper semicontinuous real functions on a frame L as frame homomorphisms $\mathfrak{L}_l(\mathbb{R}) \to L$ on the lower frame of reals $\mathfrak{L}_l(\mathbb{R})$ (it should be pointed out that we interchange, with respect to the notation used by Li and Wang, the lower and the upper frames of reals, in order to be in concordance with the usual terminology for spaces). Now, this notion is more general than the classical one and it is responsible for the need to incorporate some assumption in the statements of the pointfree generalizations of classical results (see [2] and [17] for examples). In fact, since the space $\mathbb{R}_l$ of reals with the lower topology $\{(-\infty, a) : a \in \mathbb{R}\}$ is not sober, there can be no frame L such that $\Sigma L \cong \mathbb{R}_l$. Therefore upper semicontinuous real functions $X \to \mathbb{R}$ are not represented by frame homomorphisms $\mathfrak{L}_l(\mathbb{R}) \to \mathcal{O}X$. What happens is that $\Sigma\mathfrak{L}_l(\mathbb{R})$ is homeomorphic to the space $\mathbb{R} \cup \{-\infty\}$ with the lower topology, so frame homomorphisms $\mathfrak{L}_l(\mathbb{R}) \to \mathcal{O}X$ describe upper semicontinuous functions $X \to \mathbb{R} \cup \{-\infty\}$. If one wants to describe upper semicontinuous real functions algebraically, by means of frame homomorphisms, one has to restrict to frame maps $f : \mathfrak{L}_l(\mathbb{R}) \to \mathcal{O}X$ such that $\bigvee_{p\in\mathbb{Q}} \Delta_{f(-,p)} = 1$ (see [8] for the details). Accordingly, we define an *upper semicontinuous real function* on a frame L as a frame homomorphism $f : \mathfrak{L}_l(\mathbb{R}) \to L$ such that $\bigvee_{p\in\mathbb{Q}} \Delta_{f(-,p)} = 1$. Dually, a *lower semicontinuous real function* is a frame homomorphism $g : \mathfrak{L}_u(\mathbb{R}) \to L$ such that $\bigvee_{p\in\mathbb{Q}} \Delta_{g(p,-)} = 1$.

For example, for every $x \in L$, $\chi^u_x : \mathfrak{L}_l(\mathbb{R}) \to L$ defined by $\chi^u_x(-,p) = 1$ if $1 < p$, $\chi^u_x(-,p) = x$ if $0 < p \leq 1$, and $\chi^u_x(-,p) = 0$ otherwise, is an upper semicontinuous function. Dually, $\chi^l_x : \mathfrak{L}_u(\mathbb{R}) \to L$ defined by $\chi^l_x(p,-) = 1$ if $p < 0$, $\chi^l_x(p,-) = x$ if $0 \leq p < 1$, and $\chi^l_x(p,-) = 0$ otherwise, is a lower semicontinuous function. We call χ^u_x and χ^l_x the *upper and lower characteristic functions* on x.

In this note, we start by showing briefly that, with the language of Weil entourages of [15], it is very easy and natural to define the quasi-metric quasi-uniformity $\mathcal{Q}$ of the reals. Then, with the above notion of semicontinuity, we introduce and study the semicontinuous quasi-uniformity $\mathcal{SC}(L)$ of a frame L and conclude that this is the coarsest quasi-uniformity $\mathcal{E}$ on $\mathfrak{C}L$ for which each biframe homomorphism $h : \mathfrak{L}(\mathbb{R}) \to \mathfrak{C}L$ is a uniform homomorphism $h : (\mathfrak{L}(\mathbb{R}), \mathcal{Q}) \to (\mathfrak{C}L, \mathcal{E})$. Some consequences are derived.

1. Preliminaries

We recall that a *frame* is a complete lattice L satisfying the infinite distributive law $x \wedge \bigvee S = \bigvee\{x \wedge s \mid s \in S\}$ for all $x \in L$ and $S \subseteq L$. The category Frm of frames has as maps the homomorphisms which preserve the respective operations $\wedge$ (including the top element 1) and $\bigvee$ (including the bottom element 0). For general notions and facts concerning frames see [12] or [18]. We refer to Sections 1 and 2 of [3] for the specific background and notation on frames that we use. Here, we list briefly some details of specific relevance to this paper.

The *frame of reals* [1] is the frame $\mathfrak{L}(\mathbb{R})$ generated by all ordered pairs (p, q), $p, q \in \mathbb{Q}$, subject to the relations

(R_1) $(p,q) \wedge (r,s) = (p \vee r, q \wedge s)$,

(R_2) $(p,q) \vee (r,s) = (p,s)$ whenever $p \le r < q \le s$,

(R_3) $(p,q) = \bigvee\{(r,s) \mid p < r < s < q\}$,

(R_4) $1 = \bigvee\{(p,q) \mid p,q \in \mathbb{Q}\}$.

Let $\mathfrak{L}_l(\mathbb{R})$ be the subframe of $\mathfrak{L}(\mathbb{R})$ generated by elements $(-,q) := \bigvee\{(p,q) \mid p \in \mathbb{Q}\}$ and, dually, let $\mathfrak{L}_u(\mathbb{R})$ be the subframe of $\mathfrak{L}(\mathbb{R})$ generated by elements $(p,-) := \bigvee\{(p,q) \mid q \in \mathbb{Q}\}$. The *biframe of reals* is the biframe $(\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R}))$. A map from the set of generators of $\mathfrak{L}(\mathbb{R})$ into a biframe (L, L_1, L_2) determines a (unique) biframe homomorphism $h : (\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})) \to (L, L_1, L_2)$ if and only if it transforms the above relations (R_1)-(R_4) into identities in L and takes the generators of $\mathfrak{L}_l(\mathbb{R})$ to L_1 and the generators of $\mathfrak{L}_u(\mathbb{R})$ to L_2.

Further, for any frame L, we denote by $(\mathfrak{C}L, \nabla L, \Delta L)$ the congruence biframe defined by closed and open congruences.

Concerning frame homomorphisms, a frame homomorphism $h : L \to M$ is called *dense* if $h(x) = 0$ implies $x = 0$. For any frame homomorphism $h : L \to M$, there is its *right adjoint* $h_* : M \to L$ such that $h(x) \le y$ if and only if $x \le h_*(y)$, explicitly given by $h_*(y) := \bigvee\{a \in L \mid h(a) \le y\}$.

For a frame L consider the frame $\mathcal{D}(L \times L)$ of all non-void decreasing subsets of $L \times L$, ordered by inclusion. The coproduct $L \oplus L$ will be represented as usual [12], as the subset of $\mathcal{D}(L \times L)$ consisting of all *C-ideals*, that is, of those sets A for which $(x, \bigvee S) \in A$ whenever $\{x\} \times S \subseteq A$ and $(\bigvee S, y) \in A$ whenever $S \times \{y\} \subseteq A$. Obviously, each $x \oplus y := {\downarrow}(x,y) \cup \{(0,a),(a,0) \mid a \in L\}$ is a C-ideal.

If $A, B \in \mathcal{D}(L \times L)$, then $A \circ B := \bigvee\{x \oplus y \mid (x,z) \in A \text{ and } (z,y) \in B \text{ for some } z \neq 0\}$. Note that $(h \oplus h)(A) \circ (h \oplus h)(B) \subseteq (h \oplus h)(A \circ B)$ for every frame homomorphism h.

Given $A \in \mathcal{D}(L \times L)$, we denote by $\langle A \rangle$ the C-ideal generated by A. The following properties, taken from [15], are decisive in our approach:

Lemma 1.1. *For any A, $B \in \mathcal{D}(L \times L)$, we have*:

(a) $\langle A \rangle \circ \langle B \rangle = A \circ B$.

(b) $\langle A \rangle \cap \langle B \rangle = \langle A \cap B \rangle$.

An *entourage* [15] of L is an element E of $L \oplus L$ for which $\bigvee\{x \in L \mid (x,x) \in E\} = 1$. For a system $\mathcal{E}$ of Weil entourages of a frame L (always understood to be non-void) we write $x \overset{\mathcal{E}}{\triangleleft}_1 y$ if there exists $E \in \mathcal{E}$ such that $st_1(x,E) := \bigvee\{a \in L \mid (a,b) \in E, b \wedge x \neq 0\} \le y$. Similarly, we write $x \overset{\mathcal{E}}{\triangleleft}_2 y$ if there exists $E \in \mathcal{E}$ such that $st_2(x,E) := \bigvee\{b \in L \mid (a,b) \in E, a \wedge x \neq 0\} \le y$.

The elements $st_i(x,E)$ $(i = 1,2)$ satisfy the following properties, for every $x, y \in L$ and every $E \in \mathcal{D}(L \times L)$ [16]:

(S_1) $x \le y \Rightarrow st_i(x,E) \le st_i(y,E)$.

(S_2) $st_i(x, \langle E\rangle) = st_i(x, E)$.

(S_3) If E is a Weil entourage then $x \le st_1(x, E) \wedge st_2(x, E)$.

(S_4) For each frame map $h : L \to M$ and each $E \in L \oplus L$, $st_i(h(x), h \oplus h(E)) \le h(st_i(x, E))$.

Let $\mathcal{L}_i(\mathcal{E}) := \{x \in L \mid x = \bigvee\{y \in L \mid y \overset{\mathcal{E}}{\lhd_i} x\}\}$ $(i = 1, 2)$. A filter $\mathcal{E}$ of $WEnt(L)$ is a *quasi-uniformity* on L if $(L, \mathcal{L}_1(\mathcal{E}), \mathcal{L}_2(\mathcal{E}))$ is a biframe and, for each $E \in \mathcal{E}$, there exists an $F \in \mathcal{E}$ such that $F \circ F \subseteq E$.

Regarding quasi-uniform frames, we shall need the following notions: a quasi-uniform frame $(L, \mathcal{E})$ is called *transitive* (resp. *totally bounded*) if $\mathcal{E}$ has a base consisting of transitive entourages (resp. finite entourages). For more information on transitive quasi-uniformities and totally bounded quasi-uniformities we refer to [11] and [10], respectively.

Concerning special types of maps between quasi-uniform frames, a frame homomorphism $h : (L, \mathcal{E}) \to (M, \mathcal{F})$ between quasi-uniform frames is called *uniform* (resp. a *surjection*) if $(h \oplus h)(E) \in \mathcal{F}$ for each $E \in \mathcal{E}$, (resp. if it is onto and the $(h_* \oplus h_*)(F)$, $F \in \mathcal{F}$, are entourages generating $\mathcal{E}$). Obviously, any surjection is a uniform homomorphism.

2. The quasi-metric quasi-uniformity of the reals

We start by showing that the frame $\mathfrak{L}(\mathbb{R})$ carries a natural quasi-uniformity whose underlying biframe is precisely the biframe of reals, its *quasi-metric quasi-uniformity* $\mathcal{Q}$, generated by the entourages

$$Q_n := \bigvee\Big\{(-, q) \oplus (p, -) \mid p, q \in \mathbb{Q}, 0 < q - p < \frac{1}{n}\Big\} \qquad (n \in \mathbb{N}).$$

Remark 2.1. Note that

$$Q_n = \Big\langle \bigcup\Big\{(-, q) \oplus (p, -) \mid p, q \in \mathbb{Q}, 0 < q - p < \frac{1}{n}\Big\}\Big\rangle$$

and, as can be easily proved using the fact that $\mathbb{Q}$ is dense in itself, $((r, s), (t, u))$ belongs to $\bigcup_{0<q-p<\frac{1}{n}}(-,\ q) \oplus (p,\ -)$ if and only if $s - t < \frac{1}{n}$. In the sequel we shall denote this union by Q'_n (so $Q_n = \langle Q'_n\rangle$).

Lemma 2.2. *Let $p < q$ and $p_i < q_i$ $(i \in I)$. Then*:

(a) $st_1((-, p), Q_n) \le (-, q)$, *for every natural* $n > \frac{1}{q-p}$.

(b) $st_2((q, -), Q_n) \le (p, -)$, *for every natural* $n > \frac{1}{q-p}$.

(c) $st_1(\bigvee_{i\in I}(p_i, q_i), Q_n) = st_1(\bigvee_{i\in I}(-, q_i), Q_n)$, *for every* $n \in \mathbb{N}$.

(d) $st_2(\bigvee_{i\in I}(p_i, q_i), Q_n) = st_2(\bigvee_{i\in I}(p_i, -), Q_n)$, *for every* $n \in \mathbb{N}$.

Proof. (a). By (S_2) and Remark 2.1 we need only to show that

$$\bigvee\{(r,s) \mid ((r,s),(t,u)) \in Q'_n, (t,u) \wedge (-,p) \neq 0\} \leq (-,q).$$

So let $((r,s),(t,u)) \in Q'_n$ such that $(t,u) \wedge (-,p) \neq 0$. This means that $s - t < \frac{1}{n}$ and $t < p$. Therefore, $q - t > q - p > \frac{1}{n} > s - t$, which implies $s < q$.
(c). The inequality "$\leq$" is obvious by (S_1). The reverse inequality follows from the fact that, for any $((r,s),(t,u)) \in Q'_n$ satisfying $(t,u) \wedge \bigvee_{i\in I}(-,q_i) \neq 0$, there exists $j \in I$ such that $t < q_j$, and therefore $((r,s),(t,q_j))$, which belongs to $Q'_n \subseteq Q_n$, is such that $(t,q_j) \wedge \bigvee_{i\in I}(p_i,q_i) \geq (t,q_j) \wedge (p_j,q_j) \neq 0$.
Assertions (b) and (d) may be proved in a similar way to (a) and (c), respectively.
□

Lemma 2.3. *For each $n \in \mathbb{N}$, we have*:

(a) *Q_n is an entourage of $\mathfrak{L}(\mathbb{R})$.*

(b) *$Q_{n+1} \subseteq Q_n$.*

(c) *$Q_{2n} \circ Q_{2n} \subseteq Q_n$.*

Proof. (a). Since $((p,q),(p,q)) \in Q_n$ whenever $0 < q - p < \frac{1}{n}$, it suffices to check that $\bigvee\{(p,q) \mid 0 < q-p < \frac{1}{n}\} = 1$. By ($R_2$), any (r,s) is the join of some $(p_1,q_1), \cdots, (p_m,q_m)$ where $p_1 = r < p_2 < q_1 < p_3 < \cdots < p_m < q_{m-1} < q_m = s$ and $0 < q_i - p_i < \frac{1}{n}$. Thus, by ($R_4$), $1 = \bigvee_{r,s\in\mathbb{Q}}(r,s) = \bigvee_{0<q-p<\frac{1}{n}}(p,q)$.
(b). Trivial.
(c). By Lemma 1.1 (a), it suffices to check that $Q'_{2n} \circ Q'_{2n} \subseteq Q_n$. Let $((p_1,q_1),(p_2,q_2))$ and $((p_2,q_2),(p_3,q_3))$ belong to Q'_{2n} with $p_2 < q_2$. Then, by Remark 2.1, $q_1 - p_2 < \frac{1}{2n}$ and $q_2 - p_3 < \frac{1}{2n}$. Therefore $q_1 - p_2 + q_2 - p_3 < \frac{1}{n}$, which implies that $((-,q_1 - p_2 + q_2),(p_3,-))$ belongs to Q'_n. But $((p_1,q_1),(p_3,q_3)) \leq ((-,q_1 - p_2 + q_2),(p_3,-))$, since $q_2 - p_2 > 0$. Hence $((p_1,q_1),(p_3,q_3)) \in Q'_n \subseteq Q_n$.
□

By (a) and (b) of the above lemma, the Q_n ($n \in \mathbb{N}$) form a filter base of entourages of $\mathfrak{L}(\mathbb{R})$, which, by (c), satisfies the square refinement property. Let $\mathcal{Q}$ be the corresponding filter.

Proposition 2.4. $(\mathfrak{L}(\mathbb{R}),\mathcal{Q})$ *is a quasi-uniform frame whose underlying biframe is the biframe of reals.*

Proof. It remains to prove that $(\mathfrak{L}(\mathbb{R}), \mathcal{L}_1(\mathcal{Q}), \mathcal{L}_2(\mathcal{Q}))$ is the biframe of reals.

By Lemma 2.2 (a), $(-,p) \overset{\mathcal{Q}}{\triangleleft}_1 (-,q)$ whenever $p < q$, so, for each $(-,q) \in \mathfrak{L}_l(\mathbb{R})$, we have

$$(-,q) = \bigvee_{p<q}(-,p) \leq \bigvee\{x \in \mathfrak{L}(\mathbb{R}) \mid x \overset{\mathcal{Q}}{\triangleleft}_1 (-,q)\} \leq (-,q),$$

which shows the inclusion $\mathfrak{L}_l(\mathbb{R}) \subseteq \mathcal{L}_1(\mathcal{Q})$. In order to show the reverse inclusion consider $x \in \mathcal{L}_1(\mathcal{Q})$. Then $x = \bigvee\{y \in \mathfrak{L}(\mathbb{R}) \mid y \overset{\mathcal{Q}}{\triangleleft}_1 x\}$. But $y = \bigvee_{i\in I}(p_i,q_i)$ for some pairs (p_i, q_i), $p_i < q_i$. Therefore, by Lemma 2.2(c), $y \leq \bigvee_{i\in I}(-, q_i) \overset{\mathcal{Q}}{\lhd}_1 x$, and consequently, $x \leq \bigvee\{z \in \mathfrak{L}_l(\mathbb{R}) \mid z \overset{\mathcal{Q}}{\lhd}_1 x\} \leq x$, which shows that $x \in \mathfrak{L}_l(\mathbb{R})$.

The equality $\mathcal{L}_2(\mathcal{Q}) = \mathfrak{L}_u(\mathbb{R})$ may be shown analogously, using assertions (b) and (d) of Lemma 2.2. □

Recall from [7] that a quasi-uniform frame $(L, \mathcal{E})$ is *complete* if every dense surjection $(M, \mathcal{F}) \to (L, \mathcal{E})$ is an isomorphism. We end this section with the proof that $\mathfrak{L}(\mathbb{R})$ is complete in its quasi-metric quasi-uniformity.

Proposition 2.5. $(\mathfrak{L}(\mathbb{R}), \mathcal{Q})$ *is complete.*

Proof. Let $h : (M, \mathcal{E}) \to (\mathfrak{L}(\mathbb{R}), \mathcal{Q})$ be a dense surjection. Since h is dense, we show h is an isomorphism by simply exhibiting a right inverse g for it. Let $g(p, q) := h_*(p, q)$, with h_* the right adjoint of h. By the properties of h_* and of dense surjections, this turns the conditions (R_1)-(R_4) into identities in M (we omit the details, that are straightforward) and therefore, it defines a frame homomorphism $g : \mathfrak{L}(\mathbb{R}) \to M$. This gives us the right inverse for h, as $hh_* = id$ because h is onto. □

3. The semicontinuous quasi-uniformity of a frame

Let L be a frame. In the following, $\mathfrak{S}$ will denote the collection of all upper semicontinuous real functions on L.

By the isomorphism $\nabla_L : L \to \nabla L$, we may regard each $f \in \mathfrak{S}$ as a frame homomorphism $f : \mathfrak{L}_l(\mathbb{R}) \to \nabla L$ and then, since each element of ∇L is complemented in $\mathfrak{C}L$ with complement in ΔL, we have a map $\mathfrak{L}_u(\mathbb{R}) \to \Delta L$ given by $(p, -) \mapsto \bigvee_{q>p} \neg f(-, q)$. This defines a lower semicontinuous real function g on $\mathfrak{C}L$ and f and g extend immediately to a biframe map $h_f : \mathcal{L}(\mathbb{R}) \to \mathfrak{C}L$ defined by $h_f(p, q) = f(-, q) \wedge g(p, -)$ ([8], Proposition 3.2). For each $f \in \mathfrak{S}$ and each $n \in \mathbb{N}$ let

$$E_{f,n} := \bigvee_{0<q-p<\frac{1}{n}} h_f(-, q) \oplus h_f(p, -).$$

Lemma 3.1. *For any* $f_1, \cdots, f_k \in \mathfrak{S}$, $n_1, \cdots, n_k \in \mathbb{N}$ *and* $\theta \in \mathfrak{C}L$, *we have*:

(a) $st_1(\theta, \bigcap_{i=1}^k E_{f_i,n_i}) \in \nabla L$.

(b) $st_2(\theta, \bigcap_{i=1}^k E_{f_i,n_i}) \in \Delta L$.

Proof. (a). By Lemma 1.1 (b) and property (S_2), we have

$$\begin{aligned} st_1(\theta, \bigcap_{i=1}^{k} E_{f_i,n_i}) &= st_1(\theta, \bigcap_{i=1}^{k} < E'_{f_i,n_i} >) = st_1(\theta, < \bigcap_{i=1}^{k} E'_{f_i,n_i} >) = st_1(\theta, \bigcap_{i=1}^{k} E'_{f_i,n_i}) \\ &= \bigvee\{\alpha \mid (\alpha,\beta) \in \bigcap_{i=1}^{k} E'_{f_i,n_i}, \beta \wedge \theta \neq 0\}, \end{aligned}$$

where E'_{f_i,n_i} denotes the union $\bigcup_{0<q-p<\frac{1}{n_i}}(h_{f_i}(-,q) \oplus h_{f_i}(p,-))$. But, for each such pairs (α,β), we have $(\alpha,\beta) \leq h_{f_i}(-,q_i) \oplus h_{f_i}(p_i,-)$ for some $0 < q_i - p_i < \frac{1}{n_i}$, thus $\alpha \leq \bigwedge_{i=1}^{k} h_{f_i}(-,q_i)$ and $\bigwedge_{i=1}^{k} h_{f_i}(p_i,-) \wedge \theta \neq 0$; on the other hand, if p_i and q_i are rationals such that $0 < q_i - p_i < \frac{1}{n_i}$ and $\bigwedge_{i=1}^{k} h_{f_i}(p_i,-) \wedge \theta \neq 0$ then

$$\Big(\bigwedge_{i=1}^{k} h_{f_i}(-,q_i), \bigwedge_{i=1}^{k} h_{f_i}(p_i,-)\Big) \in \bigcap_{i=1}^{k} E'_{f_i,n_i}.$$

Hence $st_1(\theta, \bigcap_{i=1}^{k} E_{f_i,n_i})$ coincides with

$$\bigvee\{\bigwedge_{i=1}^{k} h_{f_i}(-,q_i) \mid q_i \in \mathbb{Q} \text{ such that } \exists\, p_i \in \mathbb{Q} : 0 < q_i - p_i < \frac{1}{n_i}, \bigwedge_{i=1}^{k} h_{f_i}(p_i,-) \wedge \theta \neq 0\},$$

which clearly belongs to ∇L.

(b). Similarly, $st_2(\theta, \bigcap_{i=1}^{k} E_{f_i,n_i})$ coincides with

$$\bigvee\{\bigwedge_{i=1}^{k} h_{f_i}(p_i,-) \mid p_i \in \mathbb{Q} \text{ such that } \exists\, q_i \in \mathbb{Q} : 0 < q_i - p_i < \frac{1}{n_i}, \bigwedge_{i=1}^{k} h_{f_i}(-,q_i) \wedge \theta \neq 0\}.$$

Since $h_{f_i}(p,-) = \bigvee_{q>p} \neg h_{f_i}(-,q) \in \Delta L$, for each $p \in \mathbb{Q}$, $st_2(\theta, \bigcap_{i=1}^{k} E_{f_i,n_i})$ also belongs to ΔL. □

Further, for each upper characteristic function $\chi^u_{\nabla_x}$ $(x \in L)$ and each $n \in \mathbb{N}$, we have:

Lemma 3.2.

(a) $st_1(\nabla_x, E_{\chi^u_{\nabla_x},n}) = \nabla_x$.

(b) $st_2(\Delta_x, E_{\chi^u_{\nabla_x},n}) = \Delta_x$.

Proof. The proof follows immediately from the definition of $\chi^u_{\nabla_x}$ and properties (S_2) and (S_3). □

Proposition 3.3. $\{E_{f,n} \mid f \in \mathfrak{S}, n \in \mathbb{N}\}$ *is a subbase for a quasi-uniformity* $\mathcal{E}_{\mathfrak{S}}$ *on* $\mathfrak{C}L$*, with underlying biframe* $(\mathfrak{C}L, \nabla L, \Delta L)$.

Proof. Since each $E_{f,n}$ coincides with $(h_f \oplus h_f)(Q_n)$, it follows immediately from Lemma 2.3 that the $E_{f,n}$ form a subbase of a filter $\mathcal{E}_{\mathfrak{S}}$ of entourages satisfying the square refinement property.

Let $\theta \in \mathfrak{C}L$. Then $\theta = \bigvee_{i\in I}(\nabla_{x_i} \wedge \Delta_{y_i})$ for some $x_i, y_i \in L$. Lemma 3.2 implies that $\nabla_{x_i} \overset{\mathcal{E}_{\mathfrak{S}}}{\lhd_1} \nabla_{x_i}$ and $\Delta_{y_i} \overset{\mathcal{E}_{\mathfrak{S}}}{\lhd_2} \Delta_{y_i}$. Consequently, each ∇_{x_i} belongs to $\mathcal{L}_1(\mathcal{E}_{\mathfrak{S}})$ and each Δ_{y_i} belongs to $\mathcal{L}_2(\mathcal{E}_{\mathfrak{S}})$ and we may conclude that $\mathfrak{C}L$ is generated by $\mathcal{L}_1(\mathcal{E}_{\mathfrak{S}})$ and $\mathcal{L}_2(\mathcal{E}_{\mathfrak{S}})$, that is, $(\mathfrak{C}L, \mathcal{L}_1(\mathcal{E}_{\mathfrak{S}}), \mathcal{L}_2(\mathcal{E}_{\mathfrak{S}}))$ is a biframe.

Finally, the compatibility: $\nabla L \subseteq \mathcal{L}_1(\mathcal{E}_{\mathfrak{S}})$ again by Lemma 3.2(a). The reverse inclusion $\mathcal{L}_1(\mathcal{E}_{\mathfrak{S}}) \subseteq \nabla L$ follows immediately from Lemma 3.1(a), because for each $\theta \in \mathcal{L}_1(\mathcal{E}_{\mathfrak{S}})$, $\theta = \bigvee\{\alpha \in \mathfrak{C}L \mid \alpha \overset{\mathcal{E}_{\mathfrak{S}}}{\lhd_1} \theta\}$ and $\alpha \overset{\mathcal{E}_{\mathfrak{S}}}{\lhd_1} \theta$ means that there exist $f_1, \cdots, f_k \in \mathfrak{S}$ and $n_1, \cdots, n_k \in \mathbb{N}$ such that $\alpha \le st_1(\alpha, \bigcap_{i=1}^k E_{f_i,n_i}) \le \theta$.

Similarly, $\mathcal{L}_2(\mathcal{E}_{\mathfrak{S}}) = \Delta L$, by Lemma 3.1(b) and 3.2(b). $\square$

It is easy to see that $\mathcal{E}_{\mathfrak{S}}$ is the coarsest quasi-uniformity $\mathcal{E}$ on $\mathfrak{C}L$ for which each biframe map $h : \mathfrak{L}(\mathbb{R}) \to \mathfrak{C}L$ is a uniform homomorphism $h : (\mathfrak{L}(\mathbb{R}), \mathcal{Q}) \to (\mathfrak{C}L, \mathcal{E})$ ([2], Prop. 5.13). We call it the *semicontinuous quasi-uniformity* for L and denote it by $\mathcal{SC}(L)$.

It is also clear from its proof that Proposition 3.3 may be generalized to any collection $\mathcal{C}$ containing all upper characteristic functions.

Corollary 3.4. *Let $\mathcal{C}$ be a collection of upper semicontinuous real functions $f : \mathfrak{L}_l(\mathbb{R}) \to \nabla L$, containing all upper characteristic functions $\chi^u_{\nabla_x}$ $(x \in L)$. Then*

$$\{E_{f,n} \mid f \in \mathcal{C}, n \in \mathbb{N}\}$$

is a subbase for a quasi-uniformity $\mathcal{E}_{\mathcal{C}}$ on $\mathfrak{C}L$, with underlying biframe $(\mathfrak{C}L, \nabla L, \Delta L)$.

Proposition 3.3 and its Corollary 3.4 are the pointfree version of results from [14].

Example 3.5. Recall from [3] that the *Frith quasi-uniformity* $\mathcal{F}$ of $\mathfrak{C}L$ is the quasi-uniformity with subbase $\{(\nabla_x \oplus 1) \vee (1 \oplus \Delta_x) \mid x \in L\}$. This is the pointfree analogue of the Pervin quasi-uniformity. For each upper characteristic function $\chi^u_{\nabla_x}$,

$$E_{\chi^u_{\nabla_x},n} = \bigvee_{0<q-p<\frac{1}{n}} (\chi^u_{\nabla_x}(-,q) \oplus \chi^l_{\Delta_x}(p,-)) = (\nabla_x \oplus 1) \vee (1 \oplus \Delta_x).$$

Indeed: for every those p, q, $\chi^u_{\nabla_x}(-,q) = 1$ implies $\chi^l_{\Delta_x}(p,-) \le \Delta_x$ and $\chi^l_{\Delta_x}(p,-) = 1$ implies $\chi^u_{\nabla_x}(-,q) \le \nabla_x$; on the other hand, there exist clearly p, q for which $\chi^u_{\nabla_x}(-,q) \oplus \chi^l_{\Delta_x}(p,-) = \nabla_x \oplus 1$ ($p = \frac{-1}{4n}$ and $q = \frac{1}{4n}$, for instance) and there exist p, q for which $\chi^u_{\nabla_x}(-,q) \oplus \chi^l_{\Delta_x}(p,-) = 1 \oplus \Delta_x$ ($p = 1 - \frac{1}{4n}$ and $q = 1 + \frac{5}{4n}$, for instance).

Thus, for $\mathcal{C} = \{\chi^u_{\nabla_x} \mid x \in L\}$, $\mathcal{E}_{\mathcal{C}}$ and the Frith quasi-uniformity $\mathcal{F}$ have a common subbase. Hence $\mathcal{E}_{\mathcal{C}} = \mathcal{F}$.

Proposition 3.6. *Let $g : \mathfrak{C}L \to \mathfrak{C}M$ be a biframe homomorphism. Then g is a* *uniform homomorphism* $(\mathfrak{C}L, \mathcal{SC}(L)) \to (\mathfrak{C}M, \mathcal{SC}(M))$.

Proof. Let $E_{f,n} \in \mathcal{SC}(L)$, for some upper semicontinuous real function f on L and $n \in \mathbb{N}$. Then, we have

$$\begin{aligned}(g \oplus g)(E_{f,n}) &= (g \oplus g)\Big(\bigvee_{0<q-p<\frac{1}{n}} h_f(-,q) \oplus h_f(p,-)\Big)\\ &= \Big(\bigvee_{0<q-p<\frac{1}{n}} gh_f(-,q) \oplus gh_f(p,-)\Big).\end{aligned}$$

But, evidently, gh_f is the biframe extension $h_{g_1 f}$ of the upper semicontinuous real function $g_1 f : \mathfrak{L}(\mathbb{R}) \to M$ (where g_1 denotes the restriction of g to ∇L, regarded as a frame homomorphism from L to M). Hence $(g \oplus g)(E_{f,n}) = E_{g_1 f,n} \in \mathcal{SC}(M)$. $\square$

We end this section by showing that $\mathcal{SC}(L)$ is transitive and that it can be obtained by our general construction in [3]. This is the pointfree version of a theorem of Fletcher and Lindgren [5].

Recall from [3] that a *spectrum cover* of L is a cover $A := \{a_n \mid n \in \mathbb{Z}\}$ of L such that $a_n \le a_{n+1}$ for each $n \in \mathbb{Z}$, and $\bigvee_{n\in\mathbb{Z}} \Delta_{a_n} = 1$ (which implies, in particular, that $\bigwedge_{n\in\mathbb{Z}} a_n = 0$). As we proved in [3], the collection $\mathcal{A}$ of spectrum covers of L is an example of a family of interior-preserving covers, for which the following general procedure works. For each $A \in \mathcal{A}$, let

$$R_A := \bigcap_{a\in A} (\nabla_a \oplus 1) \vee (1 \oplus \Delta_a)$$

and let $\mathcal{E}_{\mathcal{A}}$ be the filter of entourages of $\mathfrak{C}L$ generated by $\{R_A \mid A \in \mathcal{A}\}$. Then $\mathcal{E}_{\mathcal{A}}$ is a quasi-uniformity on $\mathfrak{C}L$ satisfying $\mathcal{L}_1(\mathcal{E}_{\mathcal{A}}) = \nabla L$ and $\mathcal{L}_2(\mathcal{E}_{\mathcal{A}}) = \Delta L$.

Here is a proof of the result announced in [3] that this quasi-uniformity is precisely the semicontinuous quasi-uniformity.

Theorem 3.7. *Let $\mathcal{A}$ be the collection of all spectrum covers of L. Then $\mathcal{E}_{\mathcal{A}}(L) = \mathcal{SC}(L)$.*

Proof. It suffices to show that $\{R_A \mid A \in \mathcal{A}\}$ and $\{E_{f,n} \mid f \in \mathfrak{S}, n \in \mathbb{N}\}$ are equivalent subbases.

Let $A := \{a_n \mid n \in \mathbb{Z}\} \in \mathcal{A}$. For each $p \in \mathbb{Q}$ let $n(p)$ be the largest integer contained in p. Then, immediately, $f_A : \mathfrak{L}_l(\mathbb{R}) \to \mathfrak{C}L$ given by $f_A(-,p) = \nabla_{a_{n(p)}}$ belongs to $\mathfrak{S}$. It is also easy to see that

$$E_{f_A,1} = \bigvee_{n\in\mathbb{Z}} (\nabla_{a_n} \oplus \Delta_{a_{n-1}}) \subseteq \bigcap_{n\in\mathbb{Z}} ((\nabla_{a_n} \oplus 1) \vee (1 \oplus \Delta_{a_{n-1}})) = R_A.$$

Let $E_{f,m} \in \mathcal{SC}(L)$. Then $\bigvee_{n\in\mathbb{Z}} \neg f(-,\frac{n}{2m}) = 1$. Therefore, considering, for each $n \in \mathbb{Z}$, the $a_n \in L$ such that $f(-,\frac{n}{2m}) = \nabla_{a_n}$ we get a spectrum cover $A := \{a_n \mid n \in \mathbb{Z}\}$ of L. Now it suffices to check that $R_A \subseteq E_{f,m}$. So, let

$$(\alpha,\beta) \in R_A = \bigcap_{n\in\mathbb{Z}} \Big((f(-,\frac{n}{2m}) \oplus 1) \cup (1 \oplus \neg f(-,\frac{n}{2m}))\Big).$$

This means that, for some partition $\mathbb{Z}_1\dot{\cup}\mathbb{Z}_2$ of $\mathbb{Z}$, we have $\alpha \leq \bigwedge_{n\in\mathbb{Z}_1} f(-,\frac{n}{2m})$ and

$$\beta \leq \bigwedge_{n\in\mathbb{Z}_2} \neg f(-,\frac{n}{2m}) = \neg(\bigvee_{n\in\mathbb{Z}_2} f(-,\frac{n}{2m})) = \neg f(\bigvee_{n\in\mathbb{Z}_2}(-,\frac{n}{2m})).$$

Then, in order to prove that $(\alpha,\beta)\in E_{f,m}$, it remains to show that $\bigwedge_{n\in\mathbb{Z}_1} f(-,\frac{n}{2m}) \leq f(-,q)$ and $\neg f(\bigvee_{n\in\mathbb{Z}_2}(-,\frac{n}{2m})) \leq h_f(p,-)$, for some p,q such that $0<q-p<\frac{1}{m}$.

If $\mathbb{Z}_2$ has a greatest element $\overline{n}$, $\overline{n}+1\in\mathbb{Z}_1$ and $\bigwedge_{n\in\mathbb{Z}_1} f(-,\frac{n}{2m}) \leq f(-,\frac{\overline{n}+1}{2m})$. Take $q=\frac{\overline{n}+1}{2m}$ and $p=\frac{\overline{n}}{2m}-\epsilon$ for some rational $\epsilon\in(0,\frac{1}{2m})$. Clearly, $0<q-p<\frac{1}{m}$ and

$$\neg f(\bigvee_{n\in\mathbb{Z}_2}(-,\frac{n}{2m})) = \neg f(-,\frac{\overline{n}}{2m}) \leq h_f(p,-)$$

because

$$h_f(p,-)\vee f(-,\frac{\overline{n}}{2m}) = h_f((p,-)\vee(-,\frac{\overline{n}}{2m}) = h_f(1)=1.$$

If $\mathbb{Z}_2$ has no greatest element, we have $\neg f(\bigvee_{n\in\mathbb{Z}_2}(-,\frac{n}{2m})) = \neg f(1)=0$, which implies $\beta=0$. Then $(\alpha,\beta)\in E_{f,m}$ trivially. □

Corollary 3.8. *$\mathcal{SC}(L)$ is a transitive quasi-uniformity.*

Remark 3.9. It is clear that we can substitute the congruence biframe $(\mathfrak{C}L,\nabla L,\Delta L)$, in every result of this section, by a more general strictly zero-dimensional biframe (L,L_1,L_2) (the proofs could be effected in a perfect similar way). For instance, if L_1 is the part of L whose elements are all complemented with complements in L_2, Corollary 3.4 could be formulated in the following way:

Let $\mathcal{C}$ be a collection of upper semicontinuous real functions $f:\mathfrak{L}_l(\mathbb{R})\to L_1$, containing all upper characteristic functions χ^u_x ($x\in L_1$). Then $\{E_{f,n}\mid f\in\mathcal{C}, n\in\mathbb{N}\}$ is a subbase for a quasi-uniformity on L, with underlying biframe (L,L_1,L_2).

4. Some consequences

We say that an upper semicontinuous real function $f:\mathfrak{L}_l(\mathbb{R})\to L$ is *bounded above* if $f(-,p)=1$ for some $p\in\mathbb{Q}$. Since every upper characteristic function is bounded above, the discussion in Example 3.5 immediately leads to the following result:

Proposition 4.1. *Let $\mathcal{C}$ be the collection of all bounded above upper semicontinuous real functions on L. Then $\{E_{f,n}\mid f\in\mathcal{C}, n\in\mathbb{N}\}$ is a subbase for $\mathcal{F}$.*

Note that the proof of the corresponding classical result (in [9], Theorem 2, or [6], Proposition 2.10) is not so direct and simple as the proof above.

Recall that a continuous real function h is *bounded* [1] if $h(p,q)=1$ for some $p,q\in\mathbb{Q}$.

Lemma 4.2. *If $(\mathfrak{C}L,\mathcal{E})$ is a totally bounded quasi-uniform frame then every uniform* *homomorphism* $h : (\mathfrak{L}(\mathbb{R}), \mathcal{Q}) \to (\mathfrak{C}L, \mathcal{E})$ *is bounded.*

Proof. Let $h : (\mathfrak{L}(\mathbb{R}), \mathcal{Q}) \to (\mathfrak{C}L, \mathcal{E})$ be a uniform homomorphism. For each $n \in \mathbb{N}$, $(h \oplus h)(Q_n) \in \mathcal{E}$, so there exists a finite cover $\{\alpha_1, \cdots, \alpha_k\}$ of $\mathfrak{C}L$ such that $\bigvee_{i=1}^{k}(\alpha_i \oplus \alpha_i)$ is contained in $(h \oplus h)(Q_n)$. For each $i \in \{1, \cdots, k\}$, $\bigvee_{p \in \mathbb{Q}}(h(-,p) \wedge \alpha_i) = \alpha_i \neq 0$, thus there exists $p_i \in \mathbb{Q}$ such that $h(-,p_i) \wedge \alpha_i \neq 0$. Consequently, $\alpha_i \leq st_1(h(-,p_i), (h \oplus h)(Q_n))$, which, using property (S_4), implies that $\alpha_i \leq h(st_1((-,p_i), Q_n)) \leq h(-,q_i)$ for every $q_i > p_i$. Hence $1 \leq \bigvee_{i=1}^{k} h(-,q_i)$ for every $q_i > p_i$. Choose $q_i \in \mathbb{Q}$ $(i = 1, \cdots, k)$ such that $q_i > p_i$ and let $q \in \mathbb{Q}$ be the largest of these q_i. Immediately, $h(-,q) = 1$. Similarly, we may guarantee the existence of $p \in \mathbb{Q}$ such that $h(p,-) = 1$. Then $h(p,q) = h(p,-) \wedge h(-,q) = 1$ and h is bounded. $\square$

This allows us to get the pointfree counterpart of a theorem of Hunsaker and Lindgren [9]:

Theorem 4.3. *Let $(\mathfrak{C}L, \mathcal{E})$ be a totally bounded quasi-uniform frame. Then there exists a collection $\mathcal{C}$ of bounded above upper semicontinuous real functions $f : \mathfrak{L}_l(\mathbb{R}) \to L$ such that $\{E_{f,n} \mid f \in \mathcal{C}, n \in \mathbb{N}\}$ is a subbase for $\mathcal{E}$.*

Proof. Let $(\mathfrak{C}L, \mathcal{E})$ be a totally bounded quasi-uniform frame. Every uniform homomorphism $h : (\mathfrak{L}(\mathbb{R}), \mathcal{Q}) \to (\mathfrak{C}L, \mathcal{E})$, which is bounded by Lemma 4.2, restricts to a bounded above upper semicontinuous $f_h : \mathfrak{L}_l(\mathbb{R}) \to \nabla L \cong L$. Let $\mathcal{C}$ be the collection of every such maps. Since $\mathcal{C}$ contains all upper characteristic functions $\chi^u_{\nabla_x}$ $(x \in L)$, by Corollary 3.4 $\{E_{f,n} \mid f \in \mathcal{C}, n \in \mathbb{N}\}$ is a subbase for a quasi-uniformity $\mathcal{E}_{\mathcal{C}}$ on $\mathfrak{C}L$. Evidently, $E_{f_h,n} = (h \oplus h)(Q_n) \in \mathcal{E}$, because h is uniform, thus $\{E_{f,n} \mid f \in \mathcal{C}, n \in \mathbb{N}\}$ is a subbase for $\mathcal{E}$. $\square$

Again, by putting an arbitrary strictly zero-dimensional biframe in the place of $(\mathfrak{C}L, \nabla L, \Delta L)$, we could get, similarly, the following:

Let $(L, \mathcal{E})$ be a totally bounded quasi-uniform frame, whose underlying biframe $(L, \mathcal{L}_1(\mathcal{E}), \mathcal{L}_2(\mathcal{E}))$ is strictly zero-dimensional. Then there exists a collection $\mathcal{C}$ of bounded above upper semicontinuous real functions $f : \mathfrak{L}_l(\mathbb{R}) \to \mathcal{L}_1(\mathcal{E})$ such that $\{E_{f,n} \mid f \in \mathcal{C}, n \in \mathbb{N}\}$ is a subbase for $\mathcal{E}$.

References

[1] B. Banaschewski, The real numbers in pointfree topology, Textos de Matemática, Série B, Vol. 12, University of Coimbra, 1997.

[2] M. J. Ferreira, Sobre a construção de estruturas quase-uniformes em topologia sem pontos, Doctoral dissertation, University of Coimbra, 2004.

[3] M. J. Ferreira and J. Picado, *A method of constructing compatible frame quasi-uniformities*, Kyungpook Math. J., **44**(2004), 415-442.

[4] P. Fletcher, *On totally bounded quasi-uniform spaces*, Arch. Math., **21**(1970), 396–401.

[5] P. Fletcher and W. F. Lindgren, *Quasi-uniformities with a transitive base*, Pacific J. Math., **43**(1972), 619-631.

[6] P. Fletcher and W. F. Lindgren, Quasi-uniform spaces, Marcel Dekker, New York, 1982.

[7] J. Frith, W. Hunsaker and J. Walters-Wayland, *The Samuel compactification of a quasi-uniform frame*, Topology Proc., **23**(1998), 115–126.

[8] J. Gutiérrez García and J. Picado, *On the algebraic representation of semicontinuity*, J. Pure and Appl. Algebra (to appear).

[9] W. Hunsaker and W. F. Lindgren, *Construction of quasi-uniformities*, Math. Ann., **188**(1970), 39–42.

[10] W. Hunsaker and J. Picado, *A note on totally boundedness*, Acta Math. Hungar., **88**(2000), 25–34.

[11] W. Hunsaker and J. Picado, *Frames with transitive structures*, Appl. Categ. Structures, **10**(2002), 63–79.

[12] P. T. Johnstone, Stone Spaces, Cambridge Studies in Advanced Mathematics, Vol. 3, Cambridge University Press, Cambridge, 1982.

[13] Li Yong-ming and Wang Guo-jun, *Localic Katětov-Tong insertion theorem and localic Tietze extension theorem*, Comment. Math. Univ. Carolinae, **38**(1997), 801-814.

[14] R. Nielsen and C. Sloyer, *Quasi-uniformizability*, Math. Ann., **182**(1969), 273–274.

[15] J. Picado, *Weil uniformities for frames*, Comment. Math. Univ. Carolinae, **36**(1995), 357-370.

[16] J. Picado, Frame quasi-uniformities by entourages, in: *Symposium on Categorical Topology (University of Cape Town 1994)*, Department of Mathematics, University of Cape Town, 1999, pp. 161–175.

[17] J. Picado, *A new look at localic interpolation theorems*, Topology Appl., (to appear).

[18] J. Picado, A. Pultr and A. Tozzi, Locales, in: Categorical Foundations - Special Topics in Order, Topology, Algebra and Sheaf Theory, M. C. Pedicchio and W. Tholen (eds.), Encyclopedia of Mathematics and its Applications, Vol. 94, Cambridge University Press, 2004, 49-101.