Durrmyer Type Summation Integral Operators

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Abstract. In the present paper, we give the applications of the optimum bound for Bernstein basis functions. It is noted that using the optimum bound the main results of Aniol and Taberska [Ann. Soc. Math. Pol. Seri, Commentat. Math., 30(1990), 9-17], [Approx. Theory and its Appl. 11:2(1995), 94-105] and V. Gupta [Soochow J. Math., 23(1)(1997) 115-118] can be improved which were not pointed out earlier.

1. Introduction

Durrmeyer [4] introduced the integral modification of the Bernstein polynomials to approximate Lebesgue integrable functions on the interval $[0, 1]$. The operators introduced by Durrmeyer are defined as

$$(1.1) \qquad B_n(f, x) = (n + 1) \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n,k}(t) f(t) dt, \quad x \in [0, 1],$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1 - x)^{n-k}$. Guo [5] estimated the rate of convergence for the operator (1.1) for functions of bounded variation, using some results of probability theory. After this Aniol and Taberska (see e.g. [1] and [2]) generalized and extended the results of Guo [5]. Recently Gupta [6] introduced a slight but interesting integral modification of the Bernstein polynomials and studied the rate of convergence for functions of bounded variation. The operators introduced by Gupta [6] are defined by

$$(1.2) \qquad P_n(f, x) = \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 b_{n,k}(t) f(t) dt, \quad x \in [0, 1],$$

where $p_{n,k} = (-1)^k \dfrac{x^k}{k!} \Phi_n^{(k)}(x)$, $b_{n,k}(t) = (-1)^{k+1} \dfrac{t^k}{k!} \Phi_n^{(k+1)}(t)$, $b_{n,n}(t) = 0$ and $\Phi_n(x) = (1 - x)^n$.

We can easily check, by simple computation that the terms $p_{n,k}(x)$ in the above definition (1.1) and (1.2) are same. Although the rate of convergence for the operator (1.2) is same as that of the usual Durrmeyer operators (1.1), but some approximation properties and analysis become simpler for the integral modification of Bernstein polynomials defined by (1.2). For Bernstein basis function, Guo [5] proved the following inequality:

$$(1.3) \qquad p_{n,k}(x) \leq C\{nx(1-x)\}^{-1/2}, \quad C = 2.5, \quad x \in (0,1) \text{ and } 0 \leq k \leq n.$$

Aniol and Toberska ([1] and [2]) have considered the above constant $C = 2$, they have taken this constant from an old paper of Herzog and Hill [7]. Later Bastien and Rogalski [3] gave the optimum bound for Bernstein basis function which we mention in the form of following lemma:

Lemma 1.1 ([3]). *For $x \in (0,1)$, $0 \leq k \leq n$ and for all $n \in \mathbb{N}$, we have*

$$p_{n,k}(x) \leq \frac{1}{\sqrt{2enx(1-x)}},$$

where the estimate coefficient $1/\sqrt{2e}$ and the estimation order $n^{-1/2}$ are the best possible.

In the present paper, we give some applications of this bound to sharp the earlier known results due to Aniol and Taberska [1], [2] and Gupta [6].

2. Applications

In this section, we give some applications of our theorem.

(i). For the Durrmeyer operator (1.1), by using Lemma 1.1 and proceeding along the lines of Aniol and Taberska [1], we have the sharp estimate as follows:

Theorem 2.1. *Suppose $f \in L_0^\infty$ (the class of all complex valued functions bounded and measurable on $[0,1]$) and at a fixed point $x \in (0,1)$, the one sided limits $f(x\pm0)$ exist. Let a, b be arbitrary numbers such that $x \leq a \leq 1$, $1 - x \leq b \leq 1$. Then if $nx^2 \geq 4a^2$ and $n(1-x)^2 \geq 4b^2$, we have*

$$\left| B_n(f,x) - \frac{1}{2}\{f(x+) + f(x-)\} \right| \leq \left(\frac{4\sqrt{2e}+1}{2\sqrt{2e}} \right)\{nx(1-x)\}^{-1/2}|f(x+) - f(x-)|$$

$$+ \frac{1}{a^2}\left\{ 8\sum_{j=1}^{\mu} \frac{v_j\left(g_x; x - \frac{j_a}{\sqrt{n}}, x\right)}{j^3} + \frac{2v_\mu(g_x, 0, x)}{\mu^2} \right\}$$

$$+ \frac{1}{b^2}\left\{ 8\sum_{j=1}^{\sigma} \frac{v_j\left(g_x; x, x + \frac{j_b}{\sqrt{n}}\right)}{j^3} + \frac{2v_\sigma(g_x; x, l)}{\sigma^2} \right\},$$

where $\mu = \left[\dfrac{x\sqrt{n}}{a}\right]$, $\sigma = \left[\dfrac{(1-x)\sqrt{n}}{b}\right]$ and g_x is as defined in Theorem 1 of [1].

It is remarked here that the estimate of Aniol and Taberska [1, Theorem 1], i.e., $\frac{7}{2}\{nx(1-x)\}^{-1/2}|f(x+) - f(x-)|$ can be improved to $\left(\dfrac{4\sqrt{2e}+1}{2\sqrt{2e}}\right)\{nx(1-x)\}^{-1/2}|f(x+) - f(x-)|$.

In the other paper of Aniol and Taberska [2], for the case of Bernstein-Durrmeyer operator, by using Lemma 1.1, the term $|\Delta_n(x)| \le 7\{nx(1-x)\}^{-1/2}$ considered in [2, p. 102] can be improved to $|\Delta_n(x)| \le \dfrac{4\sqrt{2e}+1}{\sqrt{2e}}\{nx(1-x)\}^{-1/2}$.

(ii). For the operator (1.2), by theorem 1.1 and proceeding along the lines of [5],our sharp estimate is as follows:

Theorem 2.2. *Let f be a function of bounded variation on the interval $[0,1]$. Then for $x \in [0,1]$ and n sufficiently large, we have*

$$\left|P_n(f,x) - \frac{1}{2}\{f(x+) + f(x-)\}\right|$$

$$\le \quad \frac{5\left(x(1-x)\right)^{-1}}{n}\sum_{k=1}^{n}V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x) + \frac{1}{2\sqrt{2enx(1-x)}}|f(x+) - f(x-)|,$$

where $V_a^b(g_x)$ is the total variation of g_x on $[a,b]$ as defined in [6].

It is remarked that the estimate of main theorem of [6], i.e., $\dfrac{5\{x(1-x)\}^{-1/2}}{4\sqrt{n}}|f(x+) - f(x-)|$ can be improved to $\dfrac{2\{2ex(1-x)\}^{-1/2}}{\sqrt{n}}|f(x+) - f(x-)|$.

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