KYUNGPOOK Math. J. 46(2006), 181-184

On Partitioning Ideals of Semirings

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Abstract. We prove the following results:

(1) Let R be a strongly euclidean semiring. Then an ideal A of $R_{n\times n}$ is a partitioning ideal if and only if it is a subtractive ideal.

(2) A monic ideal M of $R[x]$, where R is a strongly euclidean semiring, is a partitioning ideal if and only if it is a subtractive ideal.

1. Introduction

Throughout this paper all semirings are with a multiplicative identity. Z^+ will denote the set of all non negative integers. For the terminology, we refer [1], [2] and [4]. A right ideal I of a semiring R is called subtractive if a, $a+b \in I$, $b \in R$ then $b \in I$. An ideal I of a semiring R is called partitioning ideal if there exists a subset Q of R such that:

1. $R = \cup\{q + I : q \in Q\}$.
2. If $q_1,\ q_2 \in Q$ then $q_1 = q_2$ if and only if $(q_1 + I) \cap (q_2 + I) \neq \emptyset$.

A commutative semiring R is called strongly euclidean [5] if there exists a function $d : R - \{0\} \to Z^+$ such that (1) $d(ab) \geq d(a)$ for all $a, b \in R - \{0\}$, and (2) if $a, b \in R$ with $b \neq 0$ then there exist unique $q, r \in R$ such that $a = bq + r$ where either $r = 0$ or $d(r) < d(b)$. Let $R = (Z^+, +, \cdot)$. Then R is a strongly euclidean semiring. Every strongly euclidean semiring is a euclidean semiring [4].

Lemma 1.1 [4, Corollary 8.23, p. 102]**.** *If I is a partitioning ideal of a semiring R then I is a subtractive ideal of R.*

The converse of the above lemma is not true. The following example is suggested by the referee.

Received April 23, 2004, and, in revised form, May 17, 2005.
2000 Mathematics Subject Classification: 16Y60.
Key words and phrases: semiring, subtractive ideal, partitioning ideal, quotient semiring, strongly euclidean semiring, full matrix semiring, polynomial semiring, monic ideal, coefficient ideal.

Let $R = (Z^+, \text{ gcd}, \text{ lcm})$. Then the ideal $2Z^+$ of R is subtractive but not partitioning.

Lemma 1.2 [4, Proposition 12.14, p. 138]**.** *If I is a subtractive right ideal of a right euclidean semiring R then I is a principal right ideal of R.*

Lemma 1.3 [5]**.** *If I is a principal ideal of a strongly euclidean semiring R then I is a partitioning ideal of R.*

From the above Lemmas, we obtain:

Theorem 1.4. *Let R be a strongly euclidean semiring. Then the following statements are equivalent.*

1. *I is a partitioning ideal of R.*
2. *I is a subtractive ideal of R.*
3. *I is a principal ideal of R.*

2. Full matrix semirings

The full matrix semirings of all $n \times n$ matrices over a semiring R will be denoted by $R_{n\times n}$. We will show that if R is a strongly euclidean semiring then an ideal A of $R_{n\times n}$ is a partitioning ideal if and only if it is a subtractive ideal.

The following lemma can be proved easily.

Lemma 2.1. *I is a subtractive ideal of a semiring R if and only if $I_{n\times n}$ is a subtractive ideal of $R_{n\times n}$.*

Lemma 2.2. *If I is a partitioning ideal of a semiring R then $I_{n\times n}$ is a partitioning ideal of $R_{n\times n}$.*

Proof. Let I be a partitioning ideal of R. Then there exists a subset Q of R such that $R = \cup\{q+I : q \in Q\}$ and if $q_1, q_2 \in Q$ then $q_1 = q_2$ if and only if $(q_1+I)\cap(q_2+I) \neq \emptyset$. Let $[a_{ij}] \in R_{n\times n}$, where $a_{ij} \in R$ for each i and j. Hence there exist $q_{ij} \in Q$, $x_{ij} \in I$ such that $a_{ij} = q_{ij} + x_{ij}$. Now $[a_{ij}] = [q_{ij}] + [x_{ij}] \in [q_{ij}] + I_{n\times n}$. So $[a_{ij}] \in \cup\{[q_{ij}] + I_{n\times n} : [q_{ij}] \in Q_{n\times n}\}$. Now $R_{n\times n} = \cup\{[q_{ij}] + I_{n\times n} : [q_{ij}] \in Q_{n\times n}\}$. Let $[q_{ij}], [q'_{ij}] \in Q_{n\times n}$, where $q_{ij}, q'_{ij} \in Q$. Suppose $([q_{ij}]+I_{n\times n}) \cap ([q'_{ij}]+I_{n\times n}) \neq \emptyset$. Let $[a_{ij}] \in ([q_{ij}] + I_{n\times n}) \cap ([q'_{ij}] + I_{n\times n})$. Then $a_{ij} \in (q_{ij} + I) \cap (q'_{ij} + I)$. Hence $q_{ij} = q'_{ij}$. Now, $[q_{ij}] = [q'_{ij}]$. So $I_{n\times n}$ is a partitioning ideal of $R_{n\times n}$. □

Theorem 2.3. *Let R be a strongly euclidean semiring. Then an ideal A of $R_{n\times n}$ is a partitioning ideal if and only if it is a subtractive ideal.*

Proof. Let A be an ideal of $R_{n\times n}$. Then $A = I_{n\times n}$ for some ideal I of R. Suppose A is a partitioning ideal of $R_{n\times n}$. Then A is a subtractive ideal of $R_{n\times n}$, by Lemma 1.1. Conversely, suppose $A = I_{n\times n}$ is a subtractive ideal of $R_{n\times n}$. Then I is a subtractive ideal of R. By Theorem 1.4, I is a partitioning ideal of R. Hence $I_{n\times n}$ is a partitioning ideal of $R_{n\times n}$, by Lemma 2.2. □

3. Polynomial semirings

In this section, we will show that a monic ideal M of a polynomial semiring $R[x]$ where R is a strongly euclidean semiring, is a partitioning ideal if and only if it is a subtractive ideal. An ideal M of $R[x]$ where R is a commutative semiring, is called a monic ideal if $\sum_{i=0}^{n} a_i x^i \in M$ implies $a_i x^i \in M$ for each i. Let A be an ideal of a commutative semiring $R[x]$. Define $A_i = \{a \in R :$ there exists $f \in A$ such that ax^i is a term of $f\}$. Then A_i is an ideal of R. It is called a coefficient ideal of R.

Lemma 3.1 [2]**.** *Let M be a monic ideal of $R[x]$ where R is a commutative semiring. Then M is a subtractive ideal of $R[x]$ if and only if M_i is a subtractive ideal of R for each i.*

Lemma 3.2. *Let M be a monic ideal of $R[x]$ where R is a commutative semiring. If M_i is a partitioning ideal of R for each i then M is a partitioning ideal of $R[x]$.*

Proof. Let M_i be a partitioning ideal of R for each i. Then there exits a subset Q_i of R such that $R = R_i = \cup\{q_i + M_i : q_i \in Q_i\}$ and if $q_i, q_i' \in Q_i$ then $q_i = q_i'$ if and only if $(q_i + M_i) \cap (q_i' + M_i) \neq \emptyset$ for each i. Define $Q = \left\{ \sum_{\text{finite}} q_i x^i : q_i \in Q_i, \text{ for each } i \right\}$. Let $\sum_{i=0}^{n} r_i x^i \in R[x]$, where $r_i \in R$. For each i, $r_i = q_i + a_i$, for some $a_i \in M_i$. Thus there exists a polynomial $h(x) \in M$ such that $a_i x^i$ is a term of $h(x)$. Since M is monic, $a_i x^i \in M$. Now $\sum_{i=0}^{n} r_i x^i = \sum_{i=0}^{n} q_i x^i + \sum_{i=0}^{n} a_i x^i$, where $\sum_{i=0}^{n} q_i x^i \in Q$ and $\sum_{i=0}^{n} a_i x^i \in M$. Now $\sum_{i=0}^{n} r_i x^i \in \cup\{q(x) + M : q(x) \in Q\}$. So $R[x] = \cup\{q(x) + M : q(x) \in Q\}$. Let $\sum_{\text{finite}} q_i x^i, \sum_{\text{finite}} q_i' x^i \in Q$ where $q_i, q_i' \in Q_i$. Suppose $\left(\sum q_i x^i + M\right) \cap \left(\sum q_i' x^i + M\right) \neq \emptyset$. Let $\sum q_i x^i + \sum a_i x^i = \sum q_i' x_i + \sum b_i x^i$ for some $\sum a_i x^i, \sum b_i x^i \in M$. Then $q_i + a_i = q_i' + b_i$ for some $a_i, b_i \in M_i$. Hence $(q_i + M_i) \cap (q_i' + M_i) \neq \emptyset$. So $q_i = q_i'$. Now $\sum_{\text{finite}} q_i x^i = \sum_{\text{finite}} q_i' x^i$. Hence M is a partitioning ideal of $R[x]$. □

Theorem 3.3. *Let M be a monic ideal of $R[x]$ where R is a strongly euclidean semiring. Then M is a partitioning ideal of $R[x]$ if and only if it is a subtractive ideal.*

Proof. Let M be a partitioning ideal of $R[x]$. By Lemma 1.1, M is subtractive. Conversely, let M be a subtractive ideal of $R[x]$. By Lemma 3.1, M_i is subtractive for each i. By Theorem 1.4, M_i is a partitioning ideal of R. Hence M is a partitioning ideal of $R[x]$, by Lemma 3.2. □

Acknowledgement. The authors are thankful to the referee for his helpful suggestions.

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