KYUNGPOOK Math. J. 46(2006), 169-180

On the Value Distribution of $ff^{(k)}$

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Abstract. This paper proves the following results: Let f be a transcendental entire function, and let $k(\geq 2)$ be a positive integer. If $T(r,f) \not\equiv N_{1)}(r,1/f)+S(r,f)$, then $ff^{(k)}$ assumes every finite nonzero value infinitely often. Also the case when f is a transcendental meromorphic function has been considered and some results are obtained.

1. Sketch of value distribution theory

We first briefly introduce the value distribution theory of meromorphic function found by R. Nevanlinna and its standard notations as well as some main classic results(see [4] or [14]).

Definition 1. For every real number $x \geq 0$, we define

$$\log^+ x = \begin{cases} \log x, & x \geq 1; \\ 0, & 0 \leq x < 1. \end{cases}$$

The basic properties of this truncated logarithm can be found in [4] or [14].

Definition 2. Let $f(z)(\not\equiv \infty)$ be a meromorphic function and a be a complex value, and let $r \in (0,+\infty)$. We call, as usual, the following functions:

$$\begin{aligned} m(r,f) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|d\theta, \\ N(r,f) &= \int_0^r \frac{n(t,f)-n(0,f)}{t}dt + n(0,f)\log r, \\ \overline{N}(r,f) &= \int_0^r \frac{\overline{n}(t,f)-\overline{n}(0,f)}{t}dt + \overline{n}(0,f)\log r, \end{aligned}$$

Received March 19, 2004, and, in revised form, April 11, 2006.
2000 Mathematics Subject Classification: 30D35, 30D45.
Key words and phrases: meromorphic function, entire function, Nevanlinna theory.
Project supported by the National Natural Science Foundation (Grant No. 10371065) of China and the Zhejiang Provincial Natural Science Foundation (Grant No. M103006) of China.

$$T(r,f) \quad = \quad m(r,f)+N(r,f)$$

the mean value function of $f(z)$; the counting function of poles of $f(z)$; the reduced counting function of poles of $f(z)$ and the Nevanlinna characteristic function of $f(z)$, respectively, where the term $n(t,f)$ denotes the number of poles of $f(z)$ inside of the disc $|z|\le t$, each taken into account according to its multiplicity; by $n(0,f)$ we denote the multiplicity of poles of $f(z)$ at the origin ($n(0,f)=\overline{n}(0,f)=0$ if $f(0)\ne\infty$; $\overline{n}(0,f)=1$ if $f(0)=\infty$); and by $\overline{n}(t,f)$ we denote the number of distinct poles of $f(z)$ in $|z|\le t$. Similarly, we can define $m(r,1/(f-a)), N(r,1/(f-a)), T(r,1/(f-a))\cdots$, for any finite complex value a.

Now we list two famous theorems founded by R. Nevanlinna(see [4]).

Theorem A (Nevanlinna's first main theorem)**.** *Let $f(z)$ be meromorphic in the domain $|z|<R$ $(\le+\infty)$, and let a be a finite value. If $f(z)\not\equiv a$, then for $r\in(0,R)$ we have*

$$T(r,\frac{1}{f-a})=T(r,f)+\log|c_\lambda|+\varepsilon(a,r),$$

where c_λ is the first nonzero coefficient of the Laurent expansion at the origin of the function $\frac{1}{f-a}$, and satisfying

$$|\varepsilon(a,r)|\le\log^+|a|+\log 2.$$

For simplicity, we sometimes write the above formula as follows:

$$T(r,\frac{1}{f-a})=T(r,f)+O(1).$$

Theorem B (Nevanlinna's second main theory)**.** *Let $f(z)$ be a nonconst meromorphic function, and let $a_1,a_2,\cdots,a_q$ be $q(\ge 3)$ pairwise distinct values, one of which can be infinite. Then*

$$(q-2)T(r,f)<\sum_{j=1}^{q}N\left(r,\frac{1}{f-a_j}\right)-N_1(r)+S(r,f),$$

where

$$\begin{aligned} N_1(r) &= 2N(r,f)-N(r,f')+N(r,\frac{1}{f'}),\\ S(r,f) &= m(r,\frac{f'}{f})+\sum_{j=1}^{q}m(r,\frac{f'}{f-a_j})+O(1). \end{aligned}$$

In this article, for every nonconstant meromorphic function $f(z)$, we always denote by $S(r,f)$ any quantity satisfying

$$S(r,f)=o(T(r,f))\quad(r\to+\infty,\quad r\notin E),$$

where E denotes a set of r of finite linear measure in the interval $(0, +\infty)$, but not necessarily the same at each occurrence.

Now we can rewrite the Nevanlinna second main theory as the following more practical form:

Theorem C. *Let $f(z)$ be a nonconstant meromorphic function, let $q(\geq 3)$ be an integer, and let $a_1 \cdots, a_q$ be distinct values, one of which can be infinite. Then*

$$(q-2)T(r,f) < \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{f-a_j}\right) + S(r,f).$$

Moreover, let k be a positive integer, we denote by $N_{k)}(r, \frac{1}{f})$ the counting function of those zeros of f whose multiplicity are less than or equal to k and by $N_{(k+1}(r, \frac{1}{f})$ the counting function of those zeros of f whose multiplicity are greater than k. We denote by $\overline{N}_{k)}(r, \frac{1}{f})$ and $\overline{N}_{(k+1}(r, \frac{1}{f})$ the corresponding reduced counting functions respectively. Similarly define for $N_{k)}(r,f)$, $N_{(k+1}(r,f)$, $\overline{N}_{k)}(r,f)$ and $\overline{N}_{(k+1}(r,f)$ etc.

2. Main results

In 1959, W. K. Hayman proved the following well known theorem.

Theorem D (see [6])**.** *Let f be a transcendental meromorphic function, and let n be a positive integer. If $n \geq 3$, then $f^n f'$ assumes every finite nonzero value infinitely often.*

Also Hayman conjectured in [5] that the conclusion of Theorem D remained valid for $n = 1$ and $n = 2$. The case $n = 2$ was settled by E. Mues[7] in 1979. The last case $n = 1$ was solved by W. Bergweiler and A. Eremenko[1], H. Chen and M. Fang[2], independently. They proved the following theorem.

Theorem E (see [1]-[2])**.** *Let f be a transcendental meromorphic function, then ff' assumes every finite nonzero value infinitely often.*

A natural question is: Whether the analogous conclusion still holds if in Theorem D, f' is replaced by $f^{(k)}$ even by a more general differential monomial $\psi = (f')^{n_1} \cdots, (f^{(k)})^{n_k}$, where k is a positive integer, $n_1, \cdots, n_k$ are all nonnegative integers with $n_1 + \cdots + n_k \geq 1$. This question has been studied by a number of authors such as L. R. Sons, N. Steinmetz, C. C. Yang, L. Yang and Y. F. Wang, etc and a series of results have been obtained (see [8]-[13], etc).

Recently, X. C. Pang and L. Zalcman proved the following theorem.

Theorem F (see [8])**.** *Let f be a transcendental entire function all of whose zeros have multiplicity at least k, and let n be a positive integer. Then $f^n f^{(k)}$ assumes every finite nonzero value infinitely often.*

Remark 1. It seems that Pang-Zalcman's method used in the proof of Theorem F does not work on meromorphic functions; Moreover, in Theorem F the restriction on the multiplicity of zeros of the function is somewhat strong.

In view of the fact that, in the above Theorem F, the case $n \geq 2$ was resolved (for meromorphic function) by N. Steinmetz with no additional hypothesis (see [10], Theorem 1). So the purpose of this article is only to study the value distribution of the meromorphic (entire) function $ff^{(k)}$; In the case when f is entire, the restriction on the multiplicity of zeros of f has been relaxed greatly. We proved the following results.

Theorem 1. *Let f be a transcendental entire function, and let $k(\geq 2)$ be a positive integer. If $T(r,f) \neq N_{1)}(r, \frac{1}{f}) + S(r,f)$, then $ff^{(k)}$ assumes every finite nonzero value infinitely often.*

Corollary 1.1. *Let f be a transcendental entire function all of whose zeros have multiplicity at least 2, then for any positive integer $k(\geq 2)$, $ff^{(k)}$ assumes every finite nonzero value infinitely often.*

Remark 2. Obviously, Theorem 1 and Corollary 1.1 improve Theorem F by greatly relaxing the restriction on the multiplicity of zeros of f.

Theorem 2. *Let f be a transcendental meromorphic function all of whose zeros have multiplicity at least t, then for any positive integer $k(\geq 2)$, $ff^{(k)}$ assumes every finite nonzero value infinitely often, where $t = k+1$ if $2 \leq k \leq 4$; $t = 5$ if $k = 5$ and $t = 6$ if $k \geq 6$ respectively.*

By Theorem 2 and Theorem E, we obtain the following corollary immediately.

Corollary 2.1. *Let f be a transcendental meromorphic function all of whose zeros have multiplicity at least k, then $ff^{(k)}$ assumes every finite nonzero value infinitely often, except for at most three positive integers k with $2 \leq k \leq 4$.*

3. Lemmas

Let f be a nonconstant meromorphic function in the complex plane C, and let k be a positive integer. We call $M[f] = f^{n_0}(f')^{n_1} \cdots (f^{(k)})^{n_k}$ a differential monomial in f, where $n_0, n_1, \cdots, n_k$ are nonnegative integers, and $\gamma_M := \sum_{j=0}^{k} n_j$ its degree. Furthermore, let $M_j[f]$ denote differential monomials in f of degree γ_{M_j} for $j = 1, \cdots, n$, and let $a_j(z)$ be meromorphic functions satisfying $T(r, a_j(z)) = S(r,f)$ for $j = 1, \cdots, n$, then $Q[f] = \sum_{j=1}^{n} a_j(z) M_j[f]$ is called a differential polynomial in f of degree $\gamma_Q := \max_{1 \leq j \leq n} \gamma_{M_j}$ with coefficients $a_j(z)$. If the coefficients $a_j(z)$ only satisfy $m(r, a_j(z)) = S(r,f)$, then we call the function $Q[f]$ a quasi-differential polynomial in f.

Lemma 1 (see [3]). *Let f be a nonconstant meromorphic function and $Q_1[f]$, $Q_2[f]$*

be quasi-differential polynomials in f with $Q_2[f] \not\equiv 0$. Let n be a positive integer and $f^nQ_1[f] = Q_2[f]$. If $\gamma_{Q_2} \le n$, then $m(r, Q_1[f]) = S(r, f)$, where *γ_{Q_2} is the degree of $Q_2[f]$.*

Lemma 2 (see [14], p.41)**.** *Let f be a transcendental meromorphic function and k be a positive integer, then $N(r, \frac{1}{f^{(k)}}) \le N(r, \frac{1}{f}) + k\overline{N}(r, f) + S(r, f)$.*

We now state the main lemma of this article which is interesting by itself.

Lemma 3. *Let f be a transcendental meromorphic function, and let k be a positive integer. Set*

$$F = ff^{(k)} - 1, \tag{1}$$

then we have

$$T(r, f) < 4\overline{N}(r, \frac{1}{F}) + 4\overline{N}_{k)}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{f}) + S(r, f).$$

Proof. From (1) we obtain

$$T(r, F) = O(T(r, f)), \tag{2}$$

$$fa = -\frac{F'}{F}, \tag{3}$$

where

$$a = \frac{f'}{f} f^{(k)} + f^{(k+1)} - f^{(k)} \frac{F'}{F}. \tag{4}$$

Since f is transcendental, it follows from (1) that $F \not\equiv$ *constant*. Thus by (3) we can deduce that $a \not\equiv 0$. By applying Lemma 1 to (3) and noting (2), we have

$$m(r, a) = S(r, f). \tag{5}$$

From (1) we can see that any pole of f must be a simple pole of $\frac{F'}{F}$. Therefore, it follows from (3) that any pole of f with multiplicity q (≥ 2) must be a zero of a with multiplicity $q - 1$. Thus we get

$$N_{(2}(r, f) \le N(r, \frac{1}{a}) + \overline{N}(r, \frac{1}{a}) \le 2N(r, \frac{1}{a}). \tag{6}$$

Suppose that z_0 is a zero of f with multiplicity q ($\ge k + 1$), then we find that z_0 is a zero of $F' = ff^{(k+1)} + f'f^{(k)}$ with multiplicity at least $2q - (k + 1)$. So from (3) we can deduce that z_0 will never be a pole of a. Hence, the poles of a come only from the zeros of F and those zeros of f with multiplicity less than or equal to k. Which together with (4) gives

$$N(r, a) \le \overline{N}_{k)}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{F}). \tag{7}$$

From (5) and (7), we have

$$T(r,a) \le \overline{N}_{k)}(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{F}) + S(r,f). \tag{8}$$

By using (6), (8) and the first fundamental theorem, we get

$$N_{(2}(r,f) \le 2\overline{N}_{k)}(r,\frac{1}{f}) + 2\overline{N}(r,\frac{1}{F}) + S(r,f). \tag{9}$$

From (2), (3), (8) and the first fundamental theorem, it follows that

$$\begin{aligned} m(r,f) &\le m(r,\frac{1}{a}) + S(r,f) = T(r,a) - N(r,\frac{1}{a}) + S(r,f) \\ &\le \overline{N}_{k)}(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{F}) - N(r,\frac{1}{a}) + S(r,f). \end{aligned} \tag{10}$$

If there exist only finitely many simple poles of f, then from (9) and (10) we can deduce that the conclusion of Lemma 3 holds. We now suppose that there exist infinitely many simple poles of f. Let z_0 be a simple pole of f, from (3) we can find that $a(z_0) \ne 0, \ \infty$. Suppose that f and a have the following expansions in a neighborhood of z_0 respectively:

$$f(z) = \frac{c_1}{z-z_0} + c_0 + O(z-z_0), \tag{11}$$

$$a(z) = a(z_0) + a'(z_0)(z-z_0) + O((z-z_0)^2), \tag{12}$$

where the constant $c_1 \ne 0$ and $a(z_0) \ne 0$. By taking the derivatives on both sides of (11), we have

$$f^{(j)}(z) = \frac{(-1)^j j! c_1}{(z-z_0)^{j+1}} + O(1) \text{ for } j = 1, 2, \cdots . \tag{13}$$

Moreover, from (3) and (4), we obtain

$$fa = f'f^{(k)} + ff^{(k+1)} + f^2 f^{(k)} a. \tag{14}$$

By substituting (11), (12) and (13) into (14) and comparing the coefficients of both sides of equality (14), we get

$$c_1 = \frac{k+2}{a(z_0)}, \quad c_0 = -\frac{(k+2)^2 a'(z_0)}{(k+3)a^2(z_0)}. \tag{15}$$

From (11), (12) and (13) we get

$$\frac{f'}{f} = -\frac{1}{z-z_0} + \frac{c_0}{c_1} + O(z-z_0), \quad \frac{F'}{F} = -\frac{k+2}{z-z_0} + \frac{c_0}{c_1} + O(z-z_0). \tag{16}$$

Define

$$h(z) := \frac{F'}{F} - (k+2)\frac{f'}{f} - \frac{(k+1)(k+2)}{k+3}\frac{a'(z)}{a(z)}. \tag{17}$$

Then from (15)-(17), it can be seen that $h(z_0) = 0$. That is to say, the simple poles of f must be the zeros of $h(z)$. Moreover, from (17), (2) and (4), we get

$$m(r, h(z)) = S(r, f). \tag{18}$$

In the sequel we shall prove that $h(z) \not\equiv 0$. Suppose to the contrary that $h(z) \equiv 0$, then from (17) we obtain $\frac{F'}{F} \equiv (k+2)\frac{f'}{f} + \frac{(k+1)(k+2)}{k+3}\frac{a'(z)}{a(z)}$. By integrating, we have

$$F^{k+3} \equiv cf^{(k+2)(k+3)} \cdot a^{(k+1)(k+2)}, \tag{19}$$

where c is a nonzero constant.

From (1) we can find that any zero of f is neither a zero nor a pole of F. Suppose that there exists a zero point z_0 of f with multiplicity q, then we can see from (19) that z_0 must be a pole of the function a with multiplicity $\frac{k+3}{k+1}q \ \ (> q)$, and so by (3) we know that z_0 must be a pole of $\frac{F'}{F}$, which is a contradiction. Thus we have $f(z) \neq 0$. On the other hand, suppose that there exists a pole point z_1 of $a(z)$ defined by (4), then we can see from (19) that z_1 must be a pole of F because f has no zero, which when combined with (1) shows that z_1 is also a pole of f. This contradicts (3) again. Therefore, we have $a(z) \neq \infty$, which together with (5) gives that

$$T(r, a) = m(r, a) + N(r, a) = S(r, f). \tag{20}$$

Since f is transcendental, we can rewrite (4) as: $\frac{a}{f^{(k)}} = \frac{f'}{f} + \frac{f^{(k+1)}}{f^{(k)}} - \frac{F'}{F}$, which when combined with (20) and (2) gives

$$S(r, f) = m(r, \frac{f'}{f} + \frac{f^{(k+1)}}{f^{(k)}} - \frac{F'}{F}) = m(r, \frac{a}{f^{(k)}}) = m(r, \frac{1}{f^{(k)}}) + S(r, f),$$

that is, $m(r, \frac{1}{f^{(k)}}) = S(r, f)$, from which, Lemma 2, the first fundamental theorem and the fact that $f \neq 0$ it follows that

$$\begin{aligned} & T(r, f^{(k)}) \\ = \ & T(r, \frac{1}{f^{(k)}}) + O(1) = N(r, \frac{1}{f^{(k)}}) + S(r, f) \leq N(r, \frac{1}{f}) + k\overline{N}(r, f) + S(r, f) \\ = \ & N(r, f^{(k)}) - N(r, f) + S(r, f) \leq T(r, f^{(k)}) - N(r, f) + S(r, f). \end{aligned}$$

From the above we have $N(r, f) = S(r, f)$. Moreover, we can see from (1) that any zero of F will never be a zero of $f^{(k)}$, from which, (4) and the fact that $f \neq 0$ we can deduce that any zero of F must be a pole of a, which and the fact that $a(z) \neq \infty$ show that $F(z) \neq 0$. Hence, from (10), and noting the facts that $f \neq 0$, $F \neq 0$ and $N(r,f) = S(r,f)$, we get $T(r,f) = S(r,f)$, which contradicts the assumption that f is transcendental. So we must have $h(z) \not\equiv 0$.

We know that any simple pole of f is a zero of h, and from (3) we can see that any multiple pole of f must be a zero of a. Moreover, we note the facts that any pole of F must be a pole of f and that any zero of F must be a pole of a. It follows from the above analysis that

$$N(r,h) \leq \overline{N}(r,a) + \overline{N}(r,\frac{1}{a}) + \overline{N}(r,\frac{1}{f}). \tag{21}$$

By (18), (21) and the first fundamental theorem, we get

$$N_{1)}(r,f) \leq N(r,\frac{1}{h}) \leq N(r,h)+S(r,f) \leq \overline{N}(r,a)+\overline{N}(r,\frac{1}{a})+\overline{N}(r,\frac{1}{f})+S(r,f). \tag{22}$$

From (6), (8), (10) and (22) we obtain

$$\begin{aligned} T(r,f) &= m(r,f) + N_{1)}(r,f) + N_{(2}(r,f) \\ &\leq \overline{N}_{k)}(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{F}) - N(r,\frac{1}{a}) + \overline{N}(r,a) + \overline{N}(r,\frac{1}{a}) \\ &\quad +\overline{N}(r,\frac{1}{f}) + 2N(r,\frac{1}{a}) + S(r,f) \\ &\leq \overline{N}_{k)}(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{F}) + 3T(r,a) + \overline{N}(r,\frac{1}{f}) + S(r,f) \\ &\leq 4\overline{N}_{k)}(r,\frac{1}{f}) + 4\overline{N}(r,\frac{1}{F}) + \overline{N}(r,\frac{1}{f}) + S(r,f). \end{aligned}$$

This proves Lemma 3. □

4. Proof of Theorem 2

We shall divide our argument into three cases.

Case (i). $2 \leq k \leq 4$

In this case, we know from the assumption of Theorem 2 that all zeros of f have multiplicity at least $t = k+1$. So by Lemma 3, we have

$$\begin{aligned} T(r,f) &\leq 4\overline{N}(r,\frac{1}{F}) + 5\overline{N}_{k)}(r,\frac{1}{f}) + \overline{N}_{(k+1}(r,\frac{1}{f}) + S(r,f) \\ &= 4\overline{N}(r,\frac{1}{F}) + \overline{N}_{(k+1}(r,\frac{1}{f}) + S(r,f) \\ &\leq 4\overline{N}(r,\frac{1}{F}) + \frac{1}{k+1}T(r,f) + S(r,f) \leq 4\overline{N}(r,\frac{1}{F}) + \frac{1}{3}T(r,f) + S(r,f). \end{aligned}$$

From the above we can deduce that F has infinitely many zeros.

Case (ii). $k \geq 6$

In this case, we get from the assumption of Theorem 2 that all zeros of f have multiplicity at least $t = 6$. Moreover, it follows from (3) that any multiple zero of f must be a zero of F', thus from (1), (2), (6), (8), (22) and Lemma 2 we get

$$\begin{aligned}
&5\overline{N}_{(6}(r, \frac{1}{f}) \\
\leq\ & N(r, \frac{1}{F'}) \leq N(r, \frac{1}{F}) + N_{1)}(r, f) + \overline{N}_{(2}(r, f) + S(r, f) \\
\leq\ & N(r, \frac{1}{F}) + \overline{N}(r, a) + \overline{N}(r, \frac{1}{a}) + \overline{N}(r, \frac{1}{f}) + \frac{1}{2}\{\overline{N}(r, \frac{1}{a}) + N(r, \frac{1}{a})\} + S(r, f) \\
\leq\ & N(r, \frac{1}{F}) + 3T(r, a) + \overline{N}(r, \frac{1}{f}) + S(r, f) \\
\leq\ & 4N(r, \frac{1}{F}) + 4\overline{N}_{k)}(r, \frac{1}{f}) + \overline{N}_{(k+1}(r, \frac{1}{f}) + S(r, f) \\
\leq\ & 4N(r, \frac{1}{F}) + 4\overline{N}_{(6}(r, \frac{1}{f}) + S(r, f).
\end{aligned}$$

That is $\overline{N}_{(6}(r, \frac{1}{f}) \leq 4N(r, \frac{1}{F}) + S(r, f)$. If $\overline{N}_{(6}(r, \frac{1}{f}) \neq S(r, f)$, then we can deduce that F has infinitely many zeros. Otherwise, we will have $\overline{N}(r, \frac{1}{f}) = \overline{N}_{(6}(r, \frac{1}{f}) = S(r, f)$, and so we can also deduce from Lemma 3 that there exist infinitely many zeros of F.

Case (iii). $k = 5$

In this case, it follows from the assumption of Theorem 2 that all zeros of f have multiplicity at least $t = 5$. We now denote by $N_k(r, \frac{1}{f})$ the counting function of those zeros of f with multiplicity k; by $\overline{N}_k(r, \frac{1}{f})$ the corresponding reduced counting function. Since all zeros of f have multiplicity at least k, thus we have $\overline{N}_{k)}(r, \frac{1}{f}) = \overline{N}_k(r, \frac{1}{f})$ and $N_{k)}(r, \frac{1}{f}) = N_k(r, \frac{1}{f})$, which gives

$$\begin{aligned}
\overline{N}_{k)}(r, \frac{1}{f}) &= \overline{N}_k(r, \frac{1}{f}) = \frac{1}{k}N_k(r, \frac{1}{f}) = \frac{1}{k}[N(r, \frac{1}{f}) - N_{(k+1}(r, \frac{1}{f})] \\
&\leq \frac{1}{k}[T(r, f) - N_{(k+1}(r, \frac{1}{f})] + O(1).
\end{aligned} \tag{23}$$

We now shall divide our following argument into three subcases again.

Subcase (iii-1). $\overline{N}_{(k+1}(r, \frac{1}{f}) \neq S(r, f)$

By (23), we get $\overline{N}_{k)}(r, \frac{1}{f}) \leq \frac{1}{k}T(r, f) - \frac{k+1}{k}\overline{N}_{(k+1}(r, \frac{1}{f}) + O(1)$, that is

$$\overline{N}(r, \frac{1}{f}) \leq \frac{1}{k}T(r, f) - \frac{1}{k}\overline{N}_{(k+1}(r, \frac{1}{f}) + O(1). \tag{24}$$

Noting that $k=5$, it follows from Lemma 3 and (24) that

$$\begin{aligned} T(r,f) &\le 4\overline{N}(r,\frac{1}{F})+4\overline{N}_{k)}(r,\frac{1}{f})+\frac{1}{k}T(r,f)-\frac{1}{k}\overline{N}_{(k+1}(r,\frac{1}{f})+S(r,f)\\ &\le 4\overline{N}(r,\frac{1}{F})+\frac{4}{k}N_{k}(r,\frac{1}{f})+\frac{1}{k}T(r,f)-\frac{1}{k}\overline{N}_{(k+1}(r,\frac{1}{f})+S(r,f)\\ &\le 4\overline{N}(r,\frac{1}{F})+\frac{5}{k}T(r,f)-\frac{1}{k}\overline{N}_{(k+1}(r,\frac{1}{f})+S(r,f)\\ &= 4\overline{N}(r,\frac{1}{F})+T(r,f)-\frac{1}{k}\overline{N}_{(k+1}(r,\frac{1}{f})+S(r,f), \end{aligned}$$

from the above we get $\frac{1}{k}\overline{N}_{(k+1}(r,\frac{1}{f})\le 4\overline{N}(r,\frac{1}{F})+S(r,f)$. This implies that F has infinitely many zeros.

Subcase (iii-2). $\overline{N}_{(k+1}(r,\frac{1}{f})=S(r,f)$, but $N_{(k+1}(r,\frac{1}{f})\ne S(r,f)$

From Lemma 3 and (23), we obtain

$$T(r,f)\le 4\overline{N}(r,\frac{1}{F})+5\overline{N}_{k)}(r,\frac{1}{f})+S(r,f)\le 4\overline{N}(r,\frac{1}{F})+\frac{5}{k}T(r,f)-\frac{5}{k}N_{(k+1}(r,\frac{1}{f})+S(r,f).$$

Considering the fact that $k=5$, by the above we get $\frac{5}{k}N_{(k+1}(r,\frac{1}{f})\le 4\overline{N}(r,\frac{1}{F})+S(r,f)$, which shows that F has infinitely many zeros.

Subcase (iii-3). $N_{(k+1}(r,\frac{1}{f})=S(r,f)$

Noting that all zeros of f have multiplicity at least k, so in this subcase we have

$$N(r,\frac{1}{f})=N_{k}(r,\frac{1}{f})+S(r,f). \tag{25}$$

If $N_k(r,\frac{1}{f})=S(r,f)$, then from Lemma 3 and (25) we can deduce that the conclusion of Theorem 2 holds. We now suppose that $N_k(r,\frac{1}{f})\ne S(r,f)$ and that z_0 is a zero of f with multiplicity k, then we have $f^{(k)}(z_0)\ne 0,\infty$. Moreover, we can see from (1) that $F(z_0)=-1$. Hence, by (4) we can deduce that z_0 must be a simple pole of $a(z)$. Suppose that there exists a point z_1 such that $F(z_1)=0$, then from (1) it follows that $f(z_1)\ne 0,\ \infty$ and that $f^{(k)}(z_1)\ne 0,\ \infty$. Which when combined with (4) we get $a(z_1)=\infty$. From the above analysis we can see that any zero of both f and F will be a pole of $a(z)$ on condition that $N_{(k+1}(r,\frac{1}{f})=S(r,f)$. Moreover, we can see from (3) that any multiple pole of f must be a zero of a, and note that each simple pole of f is a zero of $h(z)$. So from (17) it follows that all poles of h arise only from the zeros or poles of a. Which implies that

$$N(r,h)\le \overline{N}(r,a)+\overline{N}(r,\frac{1}{a}). \tag{26}$$

From (6), (8), (10)and (26), we get

$$\begin{aligned} T(r,f) &= m(r,f)+N_{1)}(r,f)+N_{(2}(r,f) \\ &\le T(r,a)-N(r,\frac{1}{a})+N(r,\frac{1}{h})+2N(r,\frac{1}{a})+S(r,f) \\ &\le T(r,a)+N(r,\frac{1}{a})+N(r,h)+S(r,f) \\ &\le T(r,a)+N(r,\frac{1}{a})+\overline{N}(r,a)+\overline{N}(r,\frac{1}{a})+S(r,f) \\ &\le 4T(r,a)+S(r,f)\le 4\overline{N}(r,\frac{1}{F})+4\overline{N}_k(r,\frac{1}{f})+S(r,f) \\ &\le 4\overline{N}(r,\frac{1}{F})+\frac{4}{k}N_k(r,\frac{1}{f})+S(r,f)\le 4\overline{N}(r,\frac{1}{F})+\frac{4}{k}T(r,f)+S(r,f). \end{aligned}$$

Since $k=5$, it follows from the above that F must have infinitely many zeros. This completes the proof of Theorem 2. □

5. Proof of Theorem 1

Since f is an entire function and $a(z)$ has only simple poles, it follows from (3) that any multiple zero of f must be a zero of F'. So it follows from Lemma 2 and (1) that

$$\text{(27)}\qquad \begin{aligned} \overline{N}_{(2}(r,\frac{1}{f}) &\le N(r,\frac{1}{F'})\le N(r,\frac{1}{F})+\overline{N}(r,F)+S(r,f) \\ &= N(r,\frac{1}{F})+\overline{N}(r,f)+S(r,f)=N(r,\frac{1}{F})+S(r,f). \end{aligned}$$

By (10) and (27) we have

$$\text{(28)}\qquad \begin{aligned} T(r,f) &= m(r,f)\le \overline{N}_{k)}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{F})-N(r,\frac{1}{a})+S(r,f) \\ &\le N_{1)}(r,\frac{1}{f})+2N(r,\frac{1}{F})-N(r,\frac{1}{a})+S(r,f). \end{aligned}$$

If F has only finitely many zeros, then from (28) we get $T(r,f)=N_{1)}(r,\frac{1}{f})+S(r,f)$, which contradicts the condition of Theorem 1. The proof of Theorem 1 is complete.

Final Remark. According to Bloch's principle (see [15], p.222), we guess that there exists a corresponding normal criterion to Corollary 1.1. So we have the following

Conjecture. *Let $\mathcal{F}$ be a family of holomorphic functions on the unit disc Δ such that all zeros of functions in $\mathcal{F}$ have multiplicity greater than or equal to 2. Suppose that there exist a positive integer $k(\geq 2)$ and a finite nonzero value a such that $ff^{(k)}\neq a$ for every $f\in\mathcal{F}$. Then $\mathcal{F}$ is a normal family.*

Acknowledgement. The author thanks the referee for his or her valuable suggestions about this paper.

References

[1] W. Bergweiler and A. Eremenko, *On the singularities of the inverse to a meromorphic function of finite order*, Rev. Mat. Iberoamericana, **11**(1995), 355-373.

[2] H. H. Chen and M. L. Fang, *On the value distribution of* $f^n f'$, Science in China, **38A**(1995), 789-798.

[3] W. Doeringer, *Exceptional Values of Differetial Polynomials*, Pacific J. of Math., **98**(1982), 55-62.

[4] W. K. Hayman, Meromorphic Functions, Clarendon Press, Oxford, 1964.

[5] W. K. Hayman, Research Problems in Function Theory, Athlone Press, London, 1967.

[6] W. K. Hayman, *Picard value of meromorphic functions and their derivatives*, Ann. Math., **70**(1959), 9-42.

[7] E. Mues, *Über ein problem von Hayman*, Math. Z., **164**(1979), 239-259.

[8] X. C. Pang and L. Zalcmam, *On theorems of Hayman and Clunie*, New Zealand J. Math., **28**(1999), 71-75.

[9] L. R. Sons, *Deficiencies of monomials*, Math. Z., **111**(1969), 53-68.

[10] N. Steinmetz, *Über die Nullstellen von Differentialpolynomen*, Math. Z., **176**(1981), 255-264.

[11] Jian-Ping Wang, *On the conjecture for zeros of the entire function* $f^{(k)}f - a$, Chin. Advances in Math., **31**(2002), 41-46.

[12] C. C. Yang and P. C. Hu, *On the value distribution of* $ff^{(k)}$, Kodai Math. J., **19**(1996), 157-167.

[13] C. C. Yang, L. Yang and Y. F. Wang, *On zeros of* $(f^{(k)})^n f - a$, Chin. Sci. Bull., **38**(1993), 2215-2218.

[14] H. X. Yi and C. C. Yang, Uniqueness theory of meromorphic functions, Pure and Applied Math. Monography, Science Press, Beijing, 1995.

[15] L. Zalcman, *Normal families: New perspectives*, Bull. Amer. Math. Soc., **35**(1998), 215-230.