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Permuting Tri-Derivations in Prime and Semi-Prime Gamma Rings

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Abstract. We study permuting tri-derivations in Γ-rings and give an example.

1. Introduction

The notion of a Γ-ring, a concept more general than a ring, was defined by Nobusawa [3]. Barnes [1] weakened slightly the conditions in the definition of Γ-ring in the sense of Nobusawa. Barnes [1], Kyuno [2] and Öztürk et al. ([4]-[9]) studied the structure of Γ-rings and obtained various generalizations analogous to corresponding parts in ring theory. In [7], Öztürk proved some results related with permuting tri-derivation on prime and semi-prime rings. As a continuation of [7], we study permuting tri-derivations on Γ-rings and give an example.

2. Preliminaries

We first recall some basic concepts for the sake of completeness. Let M and Γ be additive abelian groups. M is called a Γ*-ring* if the following conditions are satisfied: for any $a, b, c \in M$ and $\alpha, \beta \in \Gamma$,

- $a\alpha b \in M$
- $(a+b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha+\beta)b = a\alpha b + a\beta b$, $a\alpha(b+c) = a\alpha b + a\alpha c$
- $(a\alpha b)\beta c = a\alpha(b\beta c)$.

Every ring is a Γ-ring and many notions on the ring theory are generalized to Γ-rings. Let M be a Γ-ring. A Γ*-subring* of M is an additive subgroup N such that

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$N\Gamma N \subset N$. A *right* (resp. *left*) *ideal* of M is an additive abelian group I such that $I\Gamma M \subset I$ (resp. $M\Gamma I \subset I$). If I is both a right and left ideal, then we say that I is an *ideal*. M is called a *prime* Γ*-ring* if $a\Gamma M\Gamma b = 0$ imply $a = 0$ or $b = 0$ $(a, b \in M)$. *Semi-prime* Γ*-ring* is defined similarly. A map $D(\cdot,\cdot) : M \times M \to M$ is said to be *symmetric bi-additive* if it is additive both argument and $D(x, y) = D(y, x)$ for all $x, y \in M$. Then the map $d : M \to M$ defined by $d(x) = D(x, x)$ is called the *trace* of D. A symmetric bi-additive map is called a *symmetric bi-derivation* if $D(x\alpha y, z) = D(x, z)\alpha y + x\alpha D(y, z)$ for all $x, y \in M$ and $\alpha \in \Gamma$.

Definition 2.1. Let M be a Γ-ring. For a subset I of M,

$$Ann_l I = \{a \in M \mid a\Gamma I = 0\}$$

is called the *left annihilator* of I. A *right annihilator* $Ann_r I$ can be defined similarly.

We shall need the following well-known and frequently used lemmas:

Lemma 2.2 [10, Lemma 3.4.5]**.** *Let M be a semi-prime Γ-ring and I a non-zero ideal of M. Then $Ann_r I = Ann_l I$.*

Let M be a semi-prime Γ-ring and I a non-zero ideal of M. Then we will denote $AnnI = Ann_r I = Ann_l I$.

Lemma 2.3 [10, Lemma 3.4.6]**.** *Let M be a semi-prime Γ-ring and I a non-zero ideal of M. Then*

(i) *$AnnI$ is an ideal of M.*

(ii) *$I \cap AnnI = 0$.*

Lemma 2.4 [8, Lemma 3]**.** *Let M be a 2-torsion free semi-prime Γ-ring, I a non-zero ideal of M and $a, b \in M$. Then the following are equivalent:*

(i) *$a\alpha x\beta b = 0$ for all $x \in I$ and $\alpha, \beta \in \Gamma$*

(ii) *$b\alpha x\beta a = 0$ for all $x \in I$ and $\alpha, \beta \in \Gamma$*

(iii) *$a\alpha x\beta b + b\alpha x\beta a = 0$ for all $x \in I$ and $\alpha, \beta \in \Gamma$.*

If one of the conditions is fulfilled and $Ann_l I = 0$ then $a\alpha b = 0 = b\alpha a$ for all $\alpha \in \Gamma$. Moreover if M is a prime Γ-ring then $a = 0$ or $b = 0$.

Lemma 2.5 [11, Lemma 3(ii)]**.** *Let M be a prime Γ-ring, I a non-zero ideal of M, and $a \in R$. If $a\Gamma d(I) = 0$ $(d(I)\Gamma a = 0)$, then $a = 0$ or $d = 0$, where d is a derivation of M.*

3. The results

Let M be a Γ-ring. A mapping $D(\cdot,\cdot,\cdot) : M \times M \times M \to M$ is said to be *tri-additive* if it satisfies:

- $D(x+w,y,z) = D(x,y,z) + D(w,y,z)$,
- $D(x,y+w,z) = D(x,y,z) + D(x,w,z)$,
- $D(x,y,z+w) = D(x,y,z) + D(x,y,w)$

for all $x,y,z,w \in M$. A tri-additive mapping $D(\cdot,\cdot,\cdot)$ is said to be *permuting tri-additive* if $D(x,y,z) = D(x,z,y) = D(y,x,z) = D(y,z,x) = D(z,x,y) = D(z,y,x)$ for all $x,y,z \in M$. A mapping $d : M \to M$ defined by $d(x) = D(x,x,x)$ is called the *trace* of $D(\cdot,\cdot,\cdot)$, where $D(\cdot,\cdot,\cdot)$ is a permuting tri-additive mapping. It is obvious that if $D(\cdot,\cdot,\cdot)$ is a permuting tri-additive mapping, then the trace of $D(\cdot,\cdot,\cdot)$ satisfies the relation

$$d(x+y) = d(x) + d(y) + 3D(x,x,y) + 3D(x,y,y) \tag{1}$$

for all $x,y \in M$. A permuting tri-additive mapping $D(\cdot,\cdot,\cdot)$ is called a *permuting tri-derivation* if $D(x\alpha w,y,z) = D(x,y,z)\alpha w + x\alpha D(w,y,z)$ for all $x,y,z,w \in M$ and $\alpha \in \Gamma$. Then the relations

$$D(x,y\alpha w,z) = D(x,y,z)\alpha w + y\alpha D(w,y,z)$$

and

$$D(x,y,z\alpha w) = D(x,y,z)\alpha w + z\alpha D(w,y,z)$$

are fulfilled for all $x,y,z,w \in M$ and $\alpha \in \Gamma$. Let $D(\cdot,\cdot,\cdot)$ be a permuting tri-additive mapping of M where M is a Γ-ring. Since

$$D(0,x,y) = D(0+0,x,y) = D(0,x,y) + D(0,x,y),$$

we have $D(0,x,y) = 0$ for all $x,y \in M$. Thus

$$0 = D(0,y,z) = D(-x+x,y,z) = D(-x,y,z) + D(x,y,z),$$

and so $D(-x,y,z) = -D(x,y,z)$ for all $x,y,z \in M$. Therefore the mapping $d : M \to M$ defined by $d(x) = D(x,x,x)$ is an odd function.

Example 3.1. For a commutative ring R, let

$$M = \left\{ \begin{pmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| \; a,b,c \in R \right\} \text{ and } \Gamma = \left\{ \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| \; \alpha \in R \right\}.$$

It is obvious that M and Γ are both abelian groups under matrix addition. Now it is easy to show that M is a Γ-ring under matrix multiplication. A map $D(\cdot,\cdot,\cdot) : M \times M \times M \to M$ defined by

$$\left(\begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} a_3 & b_3 & c_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \mapsto \begin{pmatrix} 0 & 0 & a_1\alpha a_2\beta a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is a permuting tri-derivation.

Lemma 3.2. *Let M be a semi-prime Γ-ring of characteristic not 2, and 3, 5-torsion free, I a non-zero ideal of M. Let $D_1(\cdot,\cdot,\cdot)$ and $D_2(\cdot,\cdot,\cdot)$ be permuting tri-derivations of M with the traces d_1 and d_2 respectively. Then*

(i) *If $d_1(I)\Gamma I\Gamma d_2(I) = 0$ then $d_1(M)\Gamma I\Gamma d_2(M) = 0$.*

(ii) *If $Ann_l I = 0$ and $d_1(M)\Gamma I\Gamma d_2(M) = 0$ then $d_1(M)\Gamma M\Gamma d_2(M) = 0$.*

Proof. (i). Suppose for all $x, y, z \in I$ and $\alpha, \beta \in \Gamma$

$$d_1(x)\alpha z\beta d_2(x) = 0. \tag{2}$$

Linearizing (2) implies that

$$\begin{aligned} 0 &= d_1(x+y)\alpha z\beta d_2(x+y) \\ &= d_1(x)\alpha z\beta d_2(x) + d_1(x)\alpha z\beta d_2(y) + 3d_1(x)\alpha z\beta D_2(x,x,y) \\ &\quad + 3d_1(x)\alpha z\beta D_2(x,y,y) + d_1(y)\alpha z\beta d_2(x) + d_1(y)\alpha z\beta d_2(y) \\ &\quad + 3d_1(y)\alpha z\beta D_2(x,x,y) + 3d_1(y)\alpha z\beta D_2(x,y,y) \\ &\quad + 3D_1(x,x,y)\alpha z\beta d_2(x) + 3D_1(x,x,y)\alpha z\beta d_2(y) \\ &\quad + 9D_1(x,x,y)\alpha z\beta D_2(x,x,y) + 9D_1(x,x,y)\alpha z\beta D_2(x,y,y) \\ &\quad + 3D_1(x,y,y)\alpha z\beta d_2(x) + 3D_1(x,y,y)\alpha z\beta d_2(y) \\ &\quad + 9D_1(x,y,y)\alpha z\beta D_2(x,x,y) + 9D_1(x,y,y)\alpha z\beta D_2(x,y,y) \end{aligned} \tag{3}$$

and by using (2), we have for all $x, y, z \in I$ and $\alpha, \beta \in \Gamma$

$$\begin{aligned} & d_1(x)\alpha z\beta d_2(y) + 3d_1(x)\alpha z\beta D_2(x,x,y) + 3d_1(x)\alpha z\beta D_2(x,x,y) \\ &\quad + d_1(y)\alpha z\beta d_2(x) + 3d_1(y)\alpha z\beta D_2(x,x,y) + 3d_1(y)\alpha z\beta D_2(x,y,y) \\ &\quad + 3D_1(x,x,y)\alpha z\beta d_2(x) + 3D_1(x,x,y)\alpha z\beta d_2(y) \\ &\quad + 9D_1(x,x,y)\alpha z\beta D_2(x,x,y) + 9D_1(x,x,y)\alpha z\beta D_2(x,y,y) \\ &\quad + 3D_1(x,y,y)\alpha z\beta d_2(x) + 3D_1(x,y,y)\alpha z\beta d_2(y) \\ &\quad + 9D_1(x,y,y)\alpha z\beta D_2(x,x,y) + 9D_1(x,y,y)\alpha z\beta D_2(x,y,y) \\ &= 0. \end{aligned} \tag{4}$$

Replacing x by $-x$ in (4) induces that

$$\begin{aligned} & -d_1(x)\alpha z\beta d_2(y) - 3d_1(x)\alpha z\beta D_2(x,x,y) + 3d_1(x)\alpha z\beta D_2(x,y,y) \\ &\quad - d_1(y)\alpha z\beta d_2(x) + 3d_1(y)\alpha z\beta D_2(x,x,y) - 3d_1(y)\alpha z\beta D_2(x,y,y) \\ &\quad - 3D_1(x,x,y)\alpha z\beta d_2(x) + 3D_1(x,x,y)\alpha z\beta d_2(y) \\ &\quad + 9D_1(x,x,y)\alpha z\beta D_2(x,x,y) - 9D_1(x,x,y)\alpha z\beta D_2(x,y,y) \\ &\quad + 3D_1(x,y,y)\alpha z\beta d_2(x) - 3D_1(x,y,y)\alpha z\beta d_2(y) \\ &\quad - 9D_1(x,y,y)\alpha z\beta D_2(x,x,y) + 9D_1(x,y,y)\alpha z\beta D_2(x,y,y) \\ &= 0 \end{aligned} \tag{5}$$

for all $x, y, z \in I$ and $\alpha, \beta \in \Gamma$. Since $CharM \neq 2$ and M is 3-torsion free, it follows from (4) and (5) that

$$d_1(x)\alpha z\beta D_2(x,y,y) + d_1(y)\alpha z\beta D_2(x,x,y) + D_1(x,x,y)\alpha z\beta d_2(y) + 3D_1(x,x,y)\alpha z\beta D_2(x,x,y) + D_1(x,y,y)\alpha z\beta d_2(x) + 3D_1(x,y,y)\alpha z\beta D_2(x,y,y) = 0 \tag{6}$$

for all $x, y, z \in I$ and $\alpha, \beta \in \Gamma$. Writing $2y$ for y in (6) and using the fact that $CharM \neq 2$, we get

$$d_1(y)\alpha z\beta D_2(x,x,y) + D_1(x,x,y)\alpha z\beta d_2(y) + 3D_1(x,y,y)\alpha z\beta D_2(x,y,y) = 0 \tag{7}$$

for all $x, y, z \in I$ and $\alpha, \beta \in \Gamma$. Writing $x + y$ for x in (7) and using (2) and the fact that M is 5-torsion free, we have

$$d_1(y)\alpha z\beta D_2(x,y,y) + D_1(x,y,y)\alpha z\beta d_2(y) = 0 \tag{8}$$

for all $x, y, z \in I$ and $\alpha, \beta \in \Gamma$. Replacing z by $z\beta d_2(y)\alpha' m\beta' D_1(x,y,y)\alpha z$ in (8), we get

$$D_1(x,y,y)\alpha z\beta d_2(y)\alpha' m\beta' D_1(x,y,y)\alpha z\beta d_2(y) = -d_1(y)\alpha z\beta d_2(y)\alpha' m\beta' D_1(x,y,y)\alpha z\beta D_2(x,y,y) \tag{9}$$

for all $x, y, z \in I, m \in M$ and $\alpha, \beta, \alpha', \beta' \in \Gamma$ and from (2), we get

$$D_1(x,y,y)\alpha z\beta d_2(y)\alpha' m\beta' D_1(x,y,y)\alpha z\beta d_2(y) = 0$$

for all $x, y, z \in I$, $m \in M$ and $\alpha, \beta, \alpha', \beta' \in \Gamma$. Since M is a semi-prime Γ-ring, we get

$$D_1(x,y,y)\alpha z\beta d_2(y) = 0 \tag{10}$$

for all $x, y, z \in I$ and $\alpha, \beta \in \Gamma$. Now writing $m\gamma z$ by z in (10), where $m \in M, \gamma \in \Gamma$, we get

$$D_1(x,y,y)\alpha m\gamma z\beta d_2(y) = 0 \tag{11}$$

for all $x, y, z \in I$, $m \in M$ and $\alpha, \beta, \gamma \in \Gamma$. Next replacing x by $x\gamma m$ in (10) and using (11), we have

$$x\gamma D_1(m,y,y)\alpha z\beta d_2(y) = 0 \tag{12}$$

for all $x, y, z \in I$, $m \in M$ and $\alpha, \beta, \gamma \in \Gamma$, which implies that

$$D_1(m,y,y)\alpha z\beta d_2(y) \in Ann_r I \text{ and also } D_1(m,y,y)\alpha z\beta d_2(y) \in I,$$

and so $D_1(m,y,y)\alpha z\beta d_2(y) \in (AnnI) \cap I = 0$ by Lemmas 2.2 and 2.3. Thus, for all $y, z \in I$, $m \in M$ and $\alpha, \beta \in \Gamma$

$$D_1(m,y,y)\alpha z\beta d_2(y) = 0 \tag{13}$$

Now replacing y by $x+y$ in (13), we get

$$\begin{aligned} &D_1(m,x,x)\alpha z\beta d_2(y) + 3D_1(m,x,x)\alpha z\beta D_2(x,x,y) \\ &\ + 3D_1(m,x,x)\alpha z\beta D_2(x,y,y) + D_1(m,y,y)\alpha z\beta d_2(x) \\ &\ + 3D_1(m,y,y)\alpha z\beta D_2(x,x,y) + 3D_1(m,y,y)\alpha z\beta D_2(x,y,y) \\ &\ + 2D_1(m,x,y)\alpha z\beta d_2(x) + 2D_1(m,x,y)\alpha z\beta d_2(y) \\ &\ + 6D_1(m,x,y)\alpha z\beta D_2(x,x,y) + 6D_1(m,x,y)\alpha z\beta D_2(x,y,y) \\ &= 0 \end{aligned} \tag{14}$$

for all $x, y, z \in I$, $m \in M$ and $\alpha, \beta \in \Gamma$. Writing $-x$ for x in (14) and using the fact that $CharM \neq 2$, we get

$$\begin{aligned} &D_1(m,x,x)\alpha z\beta d_2(y) + 3D_1(m,x,x)\alpha z\beta D_2(x,x,y) \\ &\ + 3D_1(m,y,y)\alpha z\beta D_2(x,x,y) + 2D_1(m,x,y)\alpha z\beta d_2(x) \\ &\ + 6D_1(m,x,y)\alpha z\beta D_2(x,y,y) = 0 \end{aligned} \tag{15}$$

for all $x, y, z \in I$, $m \in M$ and $\alpha, \beta \in \Gamma$. Now replacing y by $x+y$ in (15) and using (13) the fact that M is 3-torsion free, we obtain

$$\begin{aligned} &6D_1(m,x,x)\alpha z\beta D_2(x,x,y) + 3D_1(m,x,x)\alpha z\beta D_2(x,y,y) \\ &\ + D_1(m,y,y)\alpha z\beta d_2(x) + 4D_1(m,x,y)\alpha z\beta d_2(x) \\ &\ + 6D_1(m,x,y)\alpha z\beta D_2(x,x,y) = 0 \end{aligned} \tag{16}$$

for all $x, y, z \in I$, $m \in M$ and $\alpha, \beta \in \Gamma$. Writing $-x$ for x in (16) and using the fact that $CharM \neq 2$, we get

$$3D_1(m,x,x)\alpha z\beta D_2(x,x,y) + 2D_1(m,x,y)\alpha z\beta d_2(x) = 0 \tag{17}$$

for all $x, y, z \in I$, $m \in M$ and $\alpha, \beta \in \Gamma$. Writing $z\beta d_2(x)\alpha' m'\beta' D_1(m,x,y)\alpha z$ for z in (17) and using (13), we get

$$2D_1(m,x,y)\alpha z\beta d_2(x)\alpha' m'\beta' D_1(m,x,y)\alpha z\beta d_2(x) = 0 \tag{18}$$

for all $x, y, z \in I$, $m, m' \in M$ and $\alpha, \beta, \alpha', \beta' \in \Gamma$. Since $CharM \neq 2$ and M is semi-prime Γ-ring, (18) implies that

$$D_1(m,x,y)\alpha z\beta d_2(x) = 0 \tag{19}$$

for all $x, y, z \in I$, $m \in M$ and $\alpha, \beta \in \Gamma$. Now writing $m\gamma z$ by z in (19), we get

$$D_1(m,x,y)\alpha m\gamma z\beta d_2(x) = 0 \tag{20}$$

for all $x, y, z \in I$, $m \in M$ and $\alpha, \beta, \gamma \in \Gamma$. Next replacing y by $y\gamma m$ in (19) and using (20), we have

$$y\gamma D_1(m,m,x)\alpha z\beta d_2(x) = 0 \tag{21}$$

for all $x, y, z \in I$, $m \in M$ and $\alpha, \beta, \gamma \in \Gamma$. It follows that

$$D_1(m,m,x)\alpha z\beta d_2(x) \in Ann_r I \text{ and } D_1(m,m,x)\alpha z\beta d_2(x) \in I.$$

So we get $D_1(m,m,x)\alpha z\beta d_2(x) \in (AnnI) \cap I = 0$ by Lemmas 2.2 and 2.3. Thus, for all $x, z \in I$, $m \in M$ and $\alpha, \beta \in \Gamma$

$$D_1(m,m,x)\alpha z\beta d_2(x) = 0. \tag{22}$$

Now replacing x by $x + y$ in (22), we get

$$\begin{aligned} &D_1(m,m,x)\alpha z\beta d_2(y) + 3D_1(m,m,x)\alpha z\beta D_2(x,x,y) \\ &\quad + 3D_1(m,m,x)\alpha z\beta D_2(x,y,y) + D_1(m,m,y)\alpha z\beta d_2(x) \\ &\quad + 3D_1(m,m,y)\alpha z\beta D_2(x,x,y) + 3D_1(m,m,y)\alpha z\beta D_2(x,y,y) \\ &= 0 \end{aligned} \tag{23}$$

for all $x, y, z \in I$, $m \in M$ and $\alpha, \beta \in \Gamma$. Writing $-x$ for x in (23) and using the fact that M is 3-torsion free, we get

$$D_1(m,m,x)\alpha z\beta D_2(x,y,y) + D_1(m,m,y)\alpha z\beta D_2(x,x,y) = 0 \tag{24}$$

for all $x, y, z \in I$, $m \in M$ and $\alpha, \beta \in \Gamma$. Writing $x + y$ for x in (24) and using (22), we get

$$D_1(m,m,x)\alpha z\beta d_2(y) + 3D_1(m,m,y)\alpha z\beta D_2(x,y,y) = 0 \tag{25}$$

for all $x, y, z \in I$, $m \in M$ and $\alpha, \beta \in \Gamma$. Writing $z\beta d_2(y)\alpha' m'\beta' D_1(m,m,x)\alpha z$ for z in (25) and using (22), we get

$$D_1(m,m,x)\alpha z\beta d_2(y)\alpha' m'\beta' D_1(m,m,x)\alpha z\beta d_2(y) = 0 \tag{26}$$

for all $x, y, z \in I$, $m, m' \in M$ and $\alpha, \beta, \alpha', \beta' \in \Gamma$. Since M is semi-prime Γ-ring, (26) implies that

$$D_1(m,m,x)\alpha z\beta d_2(y) = 0 \tag{27}$$

for all $x, y, z \in I$, $m \in M$ and $\alpha, \beta \in \Gamma$. Now writing $m\gamma z$ by z in (27), we get

$$D_1(m,m,x)\alpha m\gamma z\beta d_2(y) = 0 \tag{28}$$

for all $x, y, z \in I$, $m \in M$ and $\alpha, \beta, \gamma \in \Gamma$. Next replacing x by $x\gamma m$ in (27) and using (28), we have

$$x\gamma d_1(m)\alpha z\beta d_2(y) = 0 \tag{29}$$

for all $x, y, z \in I$, $m \in M$ and $\alpha, \beta, \gamma \in \Gamma$. It follows that

$$d_1(m)\alpha z\beta d_2(y) \in Ann_r I \text{ and } d_1(m)\alpha z\beta d_2(y) \in I$$

so that $d_1(m)\alpha z\beta d_2(y) \in (AnnI) \cap I = 0$ by Lemmas 2.2 and 2.3. Thus, for all $y, z \in I$, $m \in M$ and $\alpha, \beta \in \Gamma$

$$d_1(m)\alpha z\beta d_2(y) = 0. \tag{30}$$

Writing $x + y$ for y in (30) and using the fact that M is 3-torsion free, we get

$$d_1(m)\alpha z\beta D_2(x, x, y) + d_1(m)\alpha z\beta D_2(x, y, y) = 0 \tag{31}$$

for all $x, y, z \in I$, $m \in M$ and $\alpha, \beta \in \Gamma$. Replacing x for $-x$ in (31) and using the fact that $CharM \neq 2$, we get

$$d_1(m)\alpha z\beta D_2(x, x, y) = 0 \tag{32}$$

for all $x, y, z \in I$, $m \in M$ and $\alpha, \beta \in \Gamma$. Now writing $z\gamma n$ by z in (32), we get

$$d_1(m)\alpha z\gamma n\beta D_2(x, x, y) = 0 \tag{33}$$

for all $x, y, z \in I$, $m, n \in M$ and $\alpha, \beta, \gamma \in \Gamma$. Next replacing y by $n\gamma y$ in (32) and using (33), we have

$$d_1(m)\alpha z\beta D_2(n, x, x)\gamma y = 0 \tag{34}$$

for all $x, y, z \in I$, $m, n \in M$ and $\alpha, \beta, \gamma \in \Gamma$. It follows that

$$d_1(m)\alpha z\beta D_2(n, x, x) \in Ann_l I \text{ and } d_1(m)\alpha z\beta D_2(n, x, x) \in I$$

so that $d_1(m)\alpha z\beta D_2(n, x, x) \in (AnnI) \cap I = 0$ by Lemmas 2.2 and 2.3. Thus, for all $x, z \in I$, $m, n \in M$ and $\alpha, \beta \in \Gamma$,

$$d_1(m)\alpha z\beta D_2(n, x, x) = 0. \tag{35}$$

Writing $x + y$ for x in (35) and using the fact that $CharM \neq 2$, we get

$$d_1(m)\alpha z\beta D_2(n, x, y) = 0 \tag{36}$$

for all $x, y, z \in I$, $m, n \in M$ and $\alpha, \beta \in \Gamma$. Now writing $z\gamma n$ by z in (36), we get

$$d_1(m)\alpha z\gamma n\beta D_2(n, x, y) = 0 \tag{37}$$

for all $x, y, z \in I$, $m, n \in M$ and $\alpha, \beta, \gamma \in \Gamma$. Next replacing x by $n\gamma x$ in (36) and using (37), we have

$$d_1(m)\alpha z\beta D_2(n, n, y)\gamma x = 0 \tag{38}$$

for all $x, y, z \in I$, $m, n \in M$ and $\alpha, \beta, \gamma \in \Gamma$. It follows that

$$d_1(m)\alpha z\beta D_2(n,n,y) \in Ann_l I \text{ and } d_1(m)\alpha z\beta D_2(n,n,y) \in I$$

so that $d_1(m)\alpha z\beta D_2(n,n,y) \in (AnnI) \cap I = 0$ by Lemmas 2.2 and 2.3. Thus, for all $y, z \in I$, $m, n \in M$ and $\alpha, \beta \in \Gamma$

$$d_1(m)\alpha z\beta D_2(n,n,y) = 0. \tag{39}$$

Replacing z by $z\gamma n$ in (39), we get

$$d_1(m)\alpha z\gamma n\beta D_2(n,n,y) = 0 \tag{40}$$

for all $y, z \in I$, $m, n \in M$ and $\alpha, \beta, \gamma \in \Gamma$. Next replacing y by $n\gamma y$ in (39) and using (40), we get

$$d_1(m)\alpha z\beta d_2(n)\gamma y = 0 \tag{41}$$

for all $y, z \in I$, $m, n \in M$ and $\alpha, \beta, \gamma \in \Gamma$. It follows that

$$d_1(m)\alpha z\beta d_2(n) \in Ann_l I \text{ and } d_1(m)\alpha z\beta d_2(n) \in I$$

so that $d_1(m)\alpha z\beta d_2(n) \in (AnnI) \cap I = 0$ by Lemmas 2.2 and 2.3. Thus, for all $z \in I$, $m, n \in M$ and $\alpha, \beta \in \Gamma$

$$d_1(m)\alpha z\beta d_2(n) = 0.$$

(ii). Suppose that $Ann_l I = 0$ and for all $z \in I$, $m, n \in M$ and $\alpha, \beta \in \Gamma$,

$$d_1(m)\alpha z\beta d_2(n) = 0. \tag{42}$$

Replacing z by $m'\beta d_2(n)\gamma z\beta' n'\gamma' d_1(m)\alpha m'$ in (42), we get

$$d_1(m)\alpha m'\beta d_2(n)\gamma z\beta' n'\gamma' d_1(m)\alpha m'\beta d_2(n) = 0$$

for all $z \in I$, $m, n, m', n' \in M$ and $\alpha, \beta, \gamma, \beta', \gamma' \in \Gamma$. Since M is a semi-prime Γ-ring, we have

$$d_1(m)\alpha m'\beta d_2(n)\gamma z = 0 \tag{43}$$

for all $z \in I$, $m, n, m' \in M$ and $\alpha, \beta, \gamma \in \Gamma$, and so $d_1(m)\alpha m'\beta d_2(n) \in Ann_l I = 0$. Thus we conclude that

$$d_1(m)\alpha m'\beta d_2(n) = 0$$

for all $m, n, m' \in M$ and $\alpha, \beta \in \Gamma$. This completes the proof. □

Lemma 3.3. *Let M be a 2, 3-torsion free Γ-ring and I a non-zero one-sided ideal of M. Let $D(\cdot,\cdot,\cdot)$ be a permuting tri-derivation with the trace d. Consider the following conditions:*

(i) $d(x) = 0$ *for all* $x \in I$

(ii) $D(x, y, z) = 0$ *for all* $x, y, z \in I$

(iii) $D(m, x, y) = 0$ *for all* $x, y \in I$ *and* $m \in M$

(iv) $D(m, n, x) = 0$ *for all* $x \in I$ *and* $m, n \in M$

(v) $D(m, n, r) = 0$ *for all* $m, n, r \in M$.

Then (i) *and* (ii) *are equivalent. Moreover if* M *is a prime* Γ*-ring or* $Ann_r I = 0$ (or $Ann_l I = 0$)*, the above conditions are equivalent.*

Proof. Let I be a right ideal of M and let $m, n, r \in M$, $x, y, z \in I$ and $\alpha, \beta, \gamma \in \Gamma$. Since M is 3-torsion free, it follows from (1) that

$$D(x, x, y) + D(x, y, y) = 0. \tag{44}$$

Writing $y + z$ for y in (44) and using the fact that M is 2-torsion free, we know that (i) and (ii) are equivalent. Replacing z by $z\alpha m$ in (ii) implies that

$$0 = D(x, y, z\alpha m) = D(x, y, z)\alpha m + z\alpha D(m, x, y) = z\alpha D(m, x, y).$$

If M is a prime Γ-ring then by Lemma 2.5, the above condition shows that (ii) and (iii) are equivalent. If $Ann_r I = 0$, then the above condition shows that (ii) and (iii) are equivalent. Replacing y by $y\beta n$ in (iii), we have

$$0 = D(m, x, y\beta n) = D(m, x, y)\beta n + y\beta D(m, n, x) = y\beta D(m, n, x).$$

If M is a prime Γ-ring then by Lemma 2.5, the above condition shows that (iii) and (iv) are equivalent. If $Ann_r I = 0$, then the above condition shows that (iii) and (iv) are equivalent. Replacing x by $x\gamma r$ in (iv), we have

$$0 = D(m, n, x\gamma r) = D(m, n, x)\gamma n + x\gamma D(m, n, r) = x\gamma D(m, n, r).$$

If M is a prime Γ-ring then by Lemma 2.5, the above condition shows that (iv) and (v) are equivalent. If $Ann_r I = 0$, then the above condition shows that (iv) and (v) are equivalent. Similarly we can prove the result for a left ideal I. □

Theorem 3.4. *Let* M *be a* 2, 3*-torsion free prime* Γ*-ring,* I *a non-zero ideal of* M*. Let* $D_1(\cdot, \cdot, \cdot)$ *and* $D_2(\cdot, \cdot, \cdot)$ *be permuting tri-derivations of* M *with traces* d_1 *and* d_2 *respectively. If* $D_1(d_2(x), x, x) = 0$ *for all* $x \in I$*, then* $D_1 = 0$ *or* $D_2 = 0$.

Proof. Assume that $D_1(d_2(x), x, x) = 0$ for all $x \in I$. For any $x, y \in I$ we have

$$D_1(d_2(x + y), x + y, x + y) + D_1(d_2(-x + y), x + y, x + y) = 0.$$

Since M is 2-torsion free, it follows that

$$\begin{aligned} 2D_1(d_2(x), x, y) + D_1(d_2(y), x, x) + 3D_1(D_2(x, x, y), x, x) \\ + 3D_1(D_2(x, x, y), y, y) + 6D_1(D_2(x, y, y), x, y) = 0 \end{aligned} \tag{45}$$

for all $x, y \in I$. Writing $x+y$ for y in (45) and using the fact that M is 3-torsion free, we get

$$D_1(d_2(x), y, y) + 4D_1(d_2(x), x, y) + 6D_1(D_2(x, x, y), x, x) + 6D_1(D_2(x, x, y), x, y) + 3D_1(D_2(x, y, y), x, x) = 0 \tag{46}$$

for all $x, y \in I$. Writing $-x$ for x in (46) and using the fact that M is 2-torsion free, we get

$$4D_1(d_2(x), x, y) + 6D_1(D_2(x, x, y), x, x) = 0 \tag{47}$$

for all $x, y \in I$. Replacing y for $x\alpha y$ in (47) and using the hypothesis and the fact that M is 2, 3-torsion free, we get

$$d_2(x)\alpha D_1(x, x, y) + d_1(x)\alpha D_2(x, x, y) = 0 \tag{48}$$

for all $x, y \in I$ and $\alpha \in \Gamma$. Writing $y\beta z$ for y in (48) implies that

$$d_2(x)\alpha y\beta D_1(x, x, z) + d_1(x)\alpha y\beta D_2(x, x, z) = 0 \tag{49}$$

for all $x, y, z \in I$ and $\alpha, \beta \in \Gamma$. Writing x for z in (49) and using Lemma 2.4, we have

$$d_1(x)\alpha y\beta d_2(x) = 0 \tag{50}$$

for all $x, y \in I$ and $\alpha, \beta \in \Gamma$. In this case, suppose that d_1 and d_2 are both different from zero. Then there exist $x_1, x_2 \in I$ such that $d_1(x_1) \neq 0$ and $d_2(x_2) \neq 0$. In particular, $d_1(x_1)\alpha y\beta d_2(x_1) = 0$ for all $y \in I$ and $\alpha, \beta \in \Gamma$. Since $d_1(x_1) \neq 0$ and M is prime Γ-ring we have $d_2(x_1) = 0$. Similarly, we get $d_1(x_2) = 0$. Then the relation (49) reduces to the equation $d_1(x_1)\alpha y\beta D_2(x_1, x_1, z) = 0$ for all $y, z \in I$ and $\alpha, \beta \in \Gamma$. Using this relation and Lemma 2.5 we obtain that $D_2(x_1, x_1, z) = 0$ for all $z \in I$ because of $d_1(x_1) \neq 0$ (the mapping $z \to D_2(x_1, x_1, z)$ is a derivation). Thus, we have $D_2(x_1, x_1, z) = 0$. In the same way, we get $D_1(x_1, x_1, z) = 0$. Substituting $x_1 + x_2$ for z, we obtain

$$\begin{aligned} d_1(z) &= d_1(x_1 + x_2) \\ &= d_1(x_1) + d_1(x_2) + 3D_1(x_1, x_1, x_2) + 3D_1(x_1, x_2, x_2) \\ &= d_1(x_1) \neq 0 \end{aligned}$$

and

$$\begin{aligned} d_2(z) &= d_2(x_1 + x_2) \\ &= d_2(x_1) + d_2(x_2) + 3D_2(x_1, x_1, x_2) + 3D_2(x_1, x_2, x_2) \\ &= d_2(x_2) \neq 0. \end{aligned}$$

Therefore we have $d_1(z) \neq 0$ and $d_2(z) \neq 0$, a contradiction. Hence, we get $d_1(x) = 0$ for all $x \in I$ or $d_2(x) = 0$ for all $x \in I$. Thus $D_1 = 0$ or $D_2 = 0$. □

Corollary 3.5. *Let M be a semi-prime Γ-ring of characteristic not 2 and 3, 5-torsion free, I a non-zero ideal of M. Let $D(\cdot,\cdot,\cdot)$ be a permuting tri-derivation of M and d be the trace of $D(\cdot,\cdot,\cdot)$ such that $d(I) \subset I$. If $Ann_l I = 0$ and $D(d(x),x,x) = 0$ for all $x \in I$, then $D = 0$.*

Proof. Take $D_1 = D_2 = D$ in Theorem 3.4. By (50) we get $d(x)\alpha y \beta d(x) = 0$ for all $x \in I$ and $\alpha, \beta \in \Gamma$. Since M is a semi-prime Γ-ring, it follows from Lemma 3.2 that $d(m) = 0$ for all $m \in M$ so from Lemma 3.3 that $D = 0$. □

Theorem 3.6. *Let M be a prime Γ-ring of characteristic not 2 and 3, 5-torsion free, I a non-zero ideal of M. Let $D_1(\cdot,\cdot,\cdot)$ and $D_2(\cdot,\cdot,\cdot)$ be permuting tri-derivations of M and let d_1 and d_2 be traces of $D_1(\cdot,\cdot,\cdot)$ and $D_2(\cdot,\cdot,\cdot)$, respectively, such that $d_2(I) \subset I$. If $Ann_l I = 0$ and $D_1(d_2(x), d_2(x), x) = 0$ for all $x \in I$, then $D_1 = 0$ or $D_2 = 0$.*

Proof. For any $x, y \in I$, we have

$$D_1(d_2(x+y), d_2(x+y), x+y) + D_1(d_2(-x+y), d_2(-x+y), -x+y) = 0.$$

Since $CharM \neq 2$, it follows that

$$\begin{aligned} &2D_1(d_2(y), d_2(x), x) + 6D_1(D_2(x,x,y), d_2(x), x) \\ &+ 6D_1(D_2(x,y,y), d_2(y), x) + 18D_1(D_2(x,x,y), D_2(x,y,y), x) \\ &+ D_1(d_2(x), d_2(x), y) + 6D_1(D_2(x,y,y), d_2(x), y) \\ &+ 6D_1(D_2(x,x,y), d_2(y), y) + 9D_1(D_2(x,y,y), D_2(x,y,y), y) \\ &+ 9D_1(D_2(x,x,y), D_2(x,x,y), y) = 0 \end{aligned} \tag{51}$$

for all $x, y \in I$. Writing $2x$ for x in (51) and using the fact that $CharM \neq 2$ and M is 3-torsion free, we get

$$\begin{aligned} &2D_1(d_2(y), d_2(x), x) + 30D_1(D_2(x,x,y), d_2(x), x) \\ &+ 18D_1(D_2(x,x,y), D_2(x,y,y), x) + 5D_1(d_2(x), d_2(x), y) \\ &+ 6D_1(D_2(x,y,y), d_2(x), y) + 9D_1(D_2(x,x,y), D_2(x,x,y), y) = 0 \end{aligned} \tag{52}$$

for all $x, y \in I$. Writing $2x$ for x in (52) and using the fact that $CharM \neq 2$ and M is 3, 5-torsion free, we get

$$6D_1(D_2(x,x,y), d_2(x), x) + D_1(d_2(x), d_2(x), y) = 0 \tag{53}$$

for all $x, y \in I$. Replacing y for $y\beta x$ in (53) implies that

$$D_2(x,x,y)\beta D_1(d_2(x), x, x) + D_1(d_2(x), x, y)\beta d_2(x) = 0 \tag{54}$$

for all $x, y \in I$ and $\beta \in \Gamma$. Replacing y for $x\alpha y$ in (54) induces

$$d_2(x)\alpha y \beta D_1(d_2(x), x, x) + D_1(d_2(x), x, x)\alpha y \beta d_2(x) = 0 \tag{55}$$

for all $x, y \in I$ and $\alpha, \beta \in \Gamma$. We now show that $D_1(d_2(x), x, x) = 0$ for all $x \in I$. Assume that there exists $x_1 \in I$ such that $D_1(d_2(x_1), x_1, x_1) \neq 0$. Replacing x by x_1 in (55), then $d_2(x_1) = 0$ by Lemma 2.4. Therefore $D_1(d_2(x_1), x_1, x_1) = D_1(0, x_1, x_1) = 0$, a contradiction. It follows from Theorem 3.4 that $D_1 = 0$ or $D_2 = 0$. □

Corollary 3.7. *Let M be a semi-prime Γ-ring of characteristic not 2 and 3, 5-torsion free, I a non-zero ideal of M. Let $D(\cdot,\cdot,\cdot)$ be a permuting tri-derivation of M, d the trace of $D(\cdot,\cdot,\cdot)$ such that $d(I) \subset I$. If $Ann_l I = 0$ and $D(d(x), d(x), x) = 0$ for all $x \in I$, then $D = 0$.*

Proof. Replacing $D_1(\cdot,\cdot,\cdot)$ and $D_2(\cdot,\cdot,\cdot)$ by $D(\cdot,\cdot,\cdot)$ in (54) implies that

$$D(x, x, y)\beta D(d(x), x, x) + D(d(x), x, y)\beta d(x) = 0 \tag{56}$$

for all $x, y \in I$ and $\alpha, \beta \in \Gamma$. Replacing y for $y\alpha z$ in (56), then

$$D(x, x, y)\alpha z\beta D(d(x), x, x) + D(d(x), x, y)\alpha z\beta d(x) = 0 \tag{57}$$

for all $x, y, z \in I$ and $\alpha, \beta \in \Gamma$. Replacing y by $d(x)$ in (57) induces

$$D(d(x), x, x)\alpha z\beta D(d(x), x, x) = 0$$

for all $x, y, z \in I$ and $\alpha, \beta \in \Gamma$. Thus, since M is a semi-prime Γ-ring, we have $D = 0$ by Corollary 3.5. □

Theorem 3.8. *Let M be a prime Γ-ring of characteristic not 2, 3 and 5, 7-torsion free, I a non-zero ideal of M. Let $D_1(\cdot,\cdot,\cdot)$ and $D_2(\cdot,\cdot,\cdot)$ be permuting tri-derivations of M, and d_1 and d_2 traces of $D_1(\cdot,\cdot,\cdot)$ and $D_2(\cdot,\cdot,\cdot)$, respectively, such that $d_2(I) \subset I$. If $d_1(d_2(x)) = f(x)$ for all $x \in I$, then $D_1 = 0$ or $D_2 = 0$, where a permuting tri-additive mapping $F(\cdot,\cdot,\cdot) : M \times M \times M \to M$ and f is the trace of $F(\cdot,\cdot,\cdot)$.*

Proof. For any $x, y \in I$, we have

$$d_1(d_2(x + y)) + d_1(d_2(-x + y)) = f(x + y) + f(-x + y).$$

Using the hypothesis and $Char M \neq 2, 3$, we have

$$\begin{aligned} &D_1(d_2(x), d_2(x), d_2(y)) + 27D_1(D_2(x, x, y), D_2(x, y, y), D_2(x, y, y)) \\ &+ 9d_1(D_2(x, x, y)) + 3D_1(d_2(x), d_2(x), D_2(x, x, y)) \\ &+ 6D_1(d_2(x), d_2(y), D_2(x, y, y)) + 3D_1(d_2(y), d_2(y), D_2(x, x, y)) \\ &+ 18D_1(d_2(x), D_2(x, x, y), D_2(x, y, y)) \\ &+ 9D_1(d_2(y), D_2(x, y, y), D_2(x, y, y)) \\ &+ 9D_1(d_2(y), D_2(x, x, y), D_2(x, x, y)) = F(x, x, y) \end{aligned} \tag{58}$$

for all $x, y \in I$. Writing $2x$ for x in (58) and using the fact that $CharM \neq 2, 3$, we get

$$5D_1(d_2(x), d_2(x), d_2(y)) + 27D_1(D_2(x, x, y), D_2(x, y, y), D_2(x, y, y)) + 45d_1(D_2(x, x, y)) + 63D_1(d_2(x), d_2(x), D_2(x, x, y)) + 6D_1(d_2(x), d_2(y), D_2(x, y, y)) + 18D_1(d_2(x), D_2(x, x, y), D_2(x, y, y)) + 9D_1(d_2(y), D_2(x, x, y), D_2(x, x, y)) = 0 \tag{59}$$

for all $x, y \in I$. Writing $2x$ for x in (59) and using the fact that $CharM \neq 2, 3$, we get

$$5D_1(d_2(x), d_2(x), d_2(y)) + 315D_1(d_2(x), d_2(x), D_2(x, x, y)) + 45d_1(D_2(x, x, y)) + 18D_1(d_2(x), D_2(x, x, y), D_2(x, y, y)) = 0 \tag{60}$$

for all $x, y \in I$. Writing $2x$ for x in (60) and using the fact that $CharM \neq 2, 3$ and M is 5, 7-torsion free, we get

$$D_1(d_2(x), d_2(x), D_2(x, x, y)) = 0 \tag{61}$$

for all $x, y \in I$. Replacing y for $y\beta z$ in (61) implies

$$D_2(x, x, y)\beta D_1(d_2(x), d_2(x), z) + D_1(d_2(x), d_2(x), y)\beta D_2(x, x, z) = 0 \tag{62}$$

for all $x, y, z \in I$ and $\beta \in \Gamma$. Replacing y for $x\alpha y$ in (62), then

$$d_2(x)\alpha y\beta D_1(d_2(x), d_2(x), z) + D_1(d_2(x), d_2(x), x)\alpha y\beta D_2(x, x, z) = 0 \tag{63}$$

for all $x, y, z \in I$ and $\alpha, \beta \in \Gamma$. Replacing z for x in (63) and using Lemma 2.4, we get

$$D_1(d_2(x), d_2(x), x)\alpha y\beta d_2(x) = 0 \tag{64}$$

for all $x, y \in I$ and $\alpha, \beta \in \Gamma$. Suppose that $D_1(d_2(x_1), d_2(x_1), x_1) \neq 0$ for some $x_1 \in I$. Replacing x by x_1 in (64), then $d_2(x_1) = 0$ since M is a prime Γ-ring. Therefore $D_1(d_2(x_1), d_2(x_1), x_1) = D_1(0, 0, x_1) = 0$, a contradiction. Hence $D_1(d_2(x), d_2(x), x) = 0$ for all $x \in I$, and so $D_1 = 0$ or $D_2 = 0$ by Theorem 3.6. □

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