KYUNGPOOK Math. J. 46(2006), 445-448

Path-connected Group Extensions

Laurie A. Edler
Department of Mathematics, Georgia College & State University, Milledgeville, GA 31061, U.S.A.
e-mail : laurie.edler@gcsu.edu

Victor P. Schneider
Department of Mathematics, University of Louisiana at Lafayette, Lafayette, Louisiana 70504, U.S.A.
e-mail : vps3252@louisiana.edu

Abstract. Let N be a normal subgroup of a path-connected topological group (G, t). In this paper, the authors consider the existence of path-connectedness in refined topologies in order to address the property of maximal path-connectedness in topological groups. In particular, refinements on t and refinements on the quotient topology on G/N are studied. The preservation of path-connectedness in extending topologies and translation topologies is also considered.

1. Introduction

When a path-connected topology is refined, a natural question arises about the continued existence of paths in the new topology. Tkachenko addressed a similar question in [3] by asking when connected group topologies have connected topological group refinements. He demonstrated that any separable connected abelian torsion-free topological group has a connected separable refinement which is also a group topology. In [2], compact subsets of topological groups are studied to determine their behavior when topologies are refined.

In this paper, we discuss two methods for refining a path-connected topological group with the goal of retaining path-connectedness. We use the term *refinement* of a topology to mean a strict refinement of the topology. Also, we use the notation $t \subset T$ to mean that t is strictly contained in T. The discussion in this paper will be limited to topological groups, and therefore all topologies mentioned will be group topologies. A path-connected topological group (G, t) is *maximally path-connected* if every group topology which refines t fails to be path-connected. Suppose that N is a normal subgroup of a topological group (G, t), and let $h : G \longrightarrow G/N$ be the canonical homomorphism. Then $h(t)$ is the quotient topology on G/N. If τ

Received May 16, 2005.
2000 Mathematics Subject Classification: 22A05, 54H1.
Key words and phrases: topological group, path-connected, normal subgroup, extending topologies.

is any topology on G/N, then $h^{-1}(\tau) = \{h^{-1}(U) : U \in \tau\}$ is called the *pull-back topology*, and $t \vee h^{-1}(\tau)$ is the *pull-back join topology* on G. If $h : (G,t) \to (G/N,\tau)$ is discontinuous, the pull-back join topology is a natural candidate for a refinement of the topology t. We make the following definition.

Definition 1. A path-connected topological group (G,t) has a *pull-back extension* with respect to a normal subgroup N if and only if there is a path-connected group topology τ finer than $h(t)$ such that $t \vee h^{-1}(\tau)$ is path connected.

We notice that a maximally path-connected topological group can have no pull-back extension. Additionally, if $N = \{e\}$, the maximal path connectedness of (G,t) is equivalent to having no pull back extension with respect to N.

Definition 2. A path-connected topological group (G,t) has a *same-quotient extension* with respect to a normal subgroup N if and only if there is a path-connected group topology T finer than t such that $h(t) = h(T)$.

It is clear that a maximally path-connected topological group will have no same-quotient extension. Similarly, if $N = G$, maximal path-connectedness is equivalent to having no same-quotient extension with respect to N. Theorem 4 will show that a path-connected topology is maximal if neither of these extensions exists.

2. Main results

Theorem 3. *A path-connected topological group (G,t) has a pull back extension with respect to N if and only if for every $x \in G$, there is a path p in (G,t) from the identity $e \in G$ to x such that $h \circ p$ is a path in $(G/N,\tau)$ for a path-connected topology τ that refines $h(t)$.*

Proof. Suppose (G,t) has a pull-back extension with respect to N. Then there is a topology τ such that $t \vee h^{-1}(\tau)$ is path-connected. Consider $h : (G,t) \longrightarrow (G/N,\tau)$. For $x \in G$, there exists a path p in $(G, t \vee h^{-1}(\tau))$ from the identity e to x. Then p is a path in (G,t). Since $h : (G, t \vee h^{-1}(\tau)) \longrightarrow (G/N,\tau)$ is continuous, we have a path $h \circ p$ in $(G/N,\tau)$ from the identity in G/N to $h(x)$.

Now suppose that τ is a path-connected refinement of $h(t)$ and that for every $x \in G$, there is a path p in (G,t) from the identity $e \in G$ to x such that $h \circ p$ is a path in $(G/N,\tau)$. Then consider the following commutative diagram:

$$\begin{array}{lcl} (G, t \vee h^{-1}(\tau)) & & \\ p \uparrow & & \downarrow h \\ I & \longrightarrow & (G/N,\tau) \end{array}$$

Let $U = h^{-1}(V)$ where $V \in \tau$. Then $p^{-1}(U) = p^{-1}(h^{-1}(V))$ is an open set since $h \circ p$ is a path in $(G/N,\tau)$. Thus p is a path from e to x in $(G, t \vee h^{-1}(\tau))$. Thus $(G, t \vee h^{-1}(\tau))$ is path-connected. $\square$

Theorem 4. *Let (G,t) be a path-connected topological group. Then the following statements are equivalent.*

(1) (G,t) *is maximally path-connected.*

(2) *For some choice of* N*,* (G,t) *has no same-quotient extension and has no pull-back extension.*

(3) *For every choice of* N*,* (G,t) *has no same-quotient extension and has no pull-back extension.*

Proof. Suppose (G,t) is maximally path-connected. As previously discussed, (G,t) has neither a same-quotient extension nor does (G,t) have a pull-back extension.

Now suppose that there exists a normal subgroup N, such that (G,t) has no same-quotient extension and has no pull-back extension with respect to N. Then any path-connected refinement T of t will induce a quotient topology $h(T)$ on G/N that is strictly finer than $h(t)$. By Theorem 3, there is an element $x \in G$ such that for every path $p : I \longrightarrow (G,t)$ from the identity $e \in G$ to x, the composition $h \circ p$ fails to be a path in $(G/N, h(T))$. Since (G,T) is path connected, there exists a path $p : I \longrightarrow (G,T)$ from e to x, and p is also a path in (G,t). But $h \circ p$ is the composition of continuous functions and is therefore continuous in $(G/N, h(T))$. This contradiction shows that (G,t) is maximally path-connected. □

Lemma 5. *If* T *is a same-quotient extension for* t*, then* $t_{|N} \subset T_{|N}$*.*

Proof. Since T is a same-quotient extension, we have that $h(t) = h(T)$ and $t_{|N} \subseteq T_{|N}$. If $t_{|N} = T_{|N}$, Theorem 3 of [1] would require $t = T$. □

This lemma is of particular interest when the subgroup N is discrete.

Theorem 6. *If* N *is a discrete subgroup of* (G,t)*, then* (G,t) *has no same-quotient extension with respect to* N*. Moreover, if* $(G/N, h(t))$ *is maximally path connected, then* (G,t) *is maximally path connected.*

Proof. Since N is a discrete subgroup, $t_{|N}$ has no refinement. Thus, by Lemma 5, (G,t) has no same-quotient extension.

If T is a path-connected refinement of t, then $h(t) = h(T)$, and so $T_{|N}$ refines $t_{|N}$. □

We now address the issue of the converse to Theorem 6. For this discussion, we use the notation in [1], and let $\mathfrak{T}$ be the collection of all group topologies on G. We say that T is an *extending topology* for $t_{|N}$ if and only if T is a group topology on G and $T_{|N} = t_{|N}$. Then let $\varepsilon_N = \{T \in \mathfrak{T} : t_{|N} = T_{|N}\}$ be the collection of extending topologies on N. The sets of the form $\{gU : g \in G, U \in t_{|N}\}$ form a basis for a topology T^*, the *translation topology* for $t_{|N}$.

Theorem 7. *Suppose that* N *is discrete and* (G,t) *is maximally path connected. Then* G/N *is not maximally path-connected if and only if there exists a proper path-connected normal subgroup of* G *which covers* G/N*.*

Proof. Suppose $(G/N, h(t))$ is not maximally path connected. Then there exists a path-connected topology τ on G/N such that $h(t) \subset \tau$. Since $h : (G,t) \longrightarrow$

$(G/N, \tau)$ is discontinuous, $t \subset t \vee h^{-1}(\tau)$. Since (G, t) is maximally path-connected, $(G, t \vee h^{-1}(\tau))$ is not path-connected. Let H be the path component of the identity in $(G, t \vee h^{-1}(\tau))$. Then H is a normal subgroup of G,and since $t \vee h^{-1}(\tau)$ is not path-connected, $H \subset G$. Now we show H covers G/N. Choose $g \in G/N$. Since $(G/N, \tau)$ is path-connected, there is a path q in $(G/N, \tau)$ from the identity $e \in G/N$ to g. But q is also a path in $(G/N, h(t))$. Since N is discrete, q has a lift to a path p in (G, t) from $e \in G$ to some $y \in G$ such that $h \circ p = q$. Thus $h(y) = g$, and $y \in H$. Thus $\{e\} \subset H \subset G$, and we have a proper path connected normal subgroup of G which covers G/N.

Now suppose that H is a proper path-connected normal subgroup of G which covers G/N. Then T^* is the translation topology of $t_{|H}$. By Theorem 2 of [1], t is contained in T^*. Thus, $h(t) \subseteq h(T^*)$. Suppose that $h(t) = h(T^*)$. We note that $t_{|N} = T^*_{|N}$ since N is discrete, and so $t, T^* \in \varepsilon_N$. Since t is path-connected and T^* is not path-connected, we have that $t \subset T^*$. However, by Theorem 3 of [1], $t = T^*$. This contradiction shows that $h(t) \subset h(T^*)$. The restriction of the projection map $h : (H, T^*_{|H}) \longrightarrow (G/N, h(T^*))$ is continuous and surjective. Thus, $h(T^*)$ is a path-connected refinement of $h(t)$. $\square$

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