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Uniqueness of Meromorphic Functions and a Question of Gross

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Abstract. In this paper, we deal with the uniqueness of meromorphic functions concerning one question of Gross (see [5, Question 6]), and obtain some results that are improvements of that of former authors. Moreover, the example shows that the result is sharp.

1. Introduction and main results

In this paper, the term "meromorphic" will always mean meromorphic in the complex plane C. We assume that the reader is familiar with the basic results and notations of Nevanlinna's value distribution theory (see [6]), such as $T(r,f)$, $N(r,f)$ and $m(r,f)$. Meanwhile, we need the following notations. Let $f(z)$ be a meromorphic function. We denote by $n_{1)}(r,f)$ the number of simple poles of f in $|z|\le r$, $N_{1)}(r,f)$ is defined in terms of $n_{1)}(r,f)$ in the usual way (see [19]). We further define

$$\delta_{1)}(\infty,\ f)=1-\limsup_{r\to\infty}\frac{N_{1)}(r,f)}{T(r,f)}.$$

By the definition of $N_{1)}(r,f)$, we have

$$N_{1)}(r,f)\le \overline{N}(r,f)\le \frac{1}{2}\,N_{1)}(r,f)+\frac{1}{2}\,N(r,f)\le \frac{1}{2}\,N_{1)}(r,f)+\frac{1}{2}\,T(r,f).$$

From this we obtain

$$(1)\qquad \frac{1}{2}\,\delta_{1)}(\infty,f)\le \frac{1}{2}\,\delta_{1)}(\infty,f)+\frac{1}{2}\,\delta(\infty,f)\le \Theta(\infty,f)\le \delta_{1)}(\infty,f).$$

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Let S be a subset of distinct elements in $\hat{C}$. Define

$$E(S,f) = \bigcup_{a\in S}\{z|f(z)-a=0,\ \text{counting multiplicity}\},$$
$$\overline{E}(S,f) = \bigcup_{a\in S}\{z|f(z)-a=0,\ \text{ignoring multiplicity}\}.$$

Let f and g be two nonconstant meromorphic functions. If $E(S,f)=E(S,g)$, we say f and g share the set S CM (counting multiplicity). If $\overline{E}(S,f)=\overline{E}(S,g)$, we say f and g share the set S IM (ignoring multiplicity). Especially, let $S=\{a\}$, where $a\in\hat{C}$, we say f and g share the value a CM if $E(S,f)=E(S,g)$, and say f and g share the value a IM if $\overline{E}(S,f)=\overline{E}(S,g)$ (see [19]).

In [5] F. Gross proved that there exist three finite sets S_j $(j=1,2,3)$ such that any two nonconstant entire functions f and g satisfying $E(S_j,f)=E(S_j,g)$ for $j=1,2,3$ must be identical, and asked the following question (see [5, Question 6]):

Question A. Can one find two (even one set) finite sets S_j $(j=1,2)$ such that any two entire functions f and g satisfying $E(S_j,f)=E(S_j,g)$ $(j=1,2)$ must be identical?

H. Yi seems to have been the first to draw the affirmative answer to the above question A completely (see [14]). Since then, many results have been obtained for this and related topics (see [2], [3], [8]-[13], [16] and [17]).

In 1995, H. Yi proved the following theorems.

Theorem A ([18]). *Let $S_1=\{w|w^n+aw^{n-1}+b=0\}$, where $n\,(\geq 7)$ is an integer, a and b are two nonzero constants such that the algebraic equation $w^n+aw^{n-1}+b=0$ has no multiple roots. If f and g are two entire functions satisfying $E(S_1,f)=E(S_1,g)$, then $f\equiv g$.*

Theorem B ([16]). *Let S_1 be defined as Theorem A and $S_2=\{\infty\}$. If f and g are two distinct meromorphic functions satisfying $E(S_j,f)=E(S_j,g)$ for $j=1,2$ and $n\geq 9$, then*

$$f(z)=-\frac{ah(h^{n-1}-1)}{h^n-1},\qquad g(z)=-\frac{a(h^{n-1}-1)}{h^n-1},$$

where h is a nonconstant meromorphic function.

Afterwards, I. Lahiri proved the result as follows.

Theorem C ([7]). *Let S_1 be defined as Theorem A and $S_2=\{\infty\}$. Assume that f and g are two meromorphic functions satisfying $E(S_j,f)=E(S_j,g)$ for $j=1,2$. If f has no simple poles and $n\geq 8$, then $f\equiv g$.*

Recently, M. Fang and I. Lahiri improved Theorem C.

Theorem D ([1]). *Let S_1 be defined as Theorem A and $S_2 = \{\infty\}$. Assume that f and g are two meromorphic functions satisfying $E(S_j, f) = E(S_j, g)$ for $j = 1, 2$. If f has no simple poles and $n \geq 7$, then $f \equiv g$.*

In this paper, we prove the following theorems, which improve the above results.

Theorem 1. *The condition changed from $n \geq 9$ to $n \geq 8$, Theorem B still holds.*

Theorem 2. *Let S_1 be defined as Theorem A and $S_2 = \{\infty\}$. Assume that f and g are two meromorphic functions satisfying $E(S_j, f) = E(S_j, g)$ for $j = 1, 2$. If $n \geq 8$ and $\Theta(\infty, f) > \frac{2}{n-1}$, then $f \equiv g$.*

Remark 1. The following example shows that Theorem 2 is sharp.

Example 1. Let

$$f(z) = -\frac{ah(h^{n-1} - 1)}{(h^n - 1)}, \qquad g(z) = -\frac{a(h^{n-1} - 1)}{(h^n - 1)},$$

where $h = \frac{u^2 e^z - u}{e^z - 1}$ and $u = \exp \frac{2\pi i}{n}$. It is easy to see that f and g satisfy $E(S_j, f) = E(S_j, g)$ for $j = 1, 2$, and $\Theta(\infty, f) = \frac{2}{n-1}$. However, $f \not\equiv g$. This shows that the assumption "$\Theta(\infty, f) > \frac{2}{n-1}$" in Theorem 2 is best possible.

Theorem 3. *Let S_1 be defined as Theorem A and $S_2 = \{\infty\}$. Assume that f and g are meromorphic functions satisfying $E(S_j, f) = E(S_j, g)$ for $j = 1, 2$. If $n \geq 7$ and $\Theta(\infty,\ f) > 1/2$, then $f \equiv g$.*

Theorem 4. *Let S_1 be defined as Theorem A and $S_2 = \{\infty\}$. Assume that f and g are meromorphic functions satisfying $E(S_j, f) = E(S_j, g)$ for $j = 1, 2$. If $n \geq 7$ and $\delta_{1)}(\infty, f) > 9/14$, then $f \equiv g$.*

Remark 2. Obviously, if f has no simple poles, then $\delta_{1)}(\infty, f) = 1$, so Theorem 4 improve Theorem C and Theorem D. In case that f is an entire function, we have $\Theta(\infty, f) = \delta_{1)}(\infty, f) = 1$, so both of Theorem 3 and Theorem 4 improve Theorem A.

2. Some Lemmas

In order to prove the theorems, we need the following results.

Lemma 1 ([19]). *Let f be a nonconstant meromorphic function and let*

$$R(f) = \sum_{k=0}^{n} a_k f^k / \sum_{j=0}^{m} b_j f^j$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$, *where* $a_n \neq 0$ *and* $b_m \neq 0$. *Then*

$$T(r,\ R(f)) = dT(r,\ f) + S(r,\ f),$$

where $d = max\{n,\ m\}$.

Below, let f and g be two nonconstant meromorphic functions, and $S_1 = \{w | w^{n-1}(w+a)+b=0\}$, where $n\,(\geq 7)$ is an integer, a and b are two nonzero constants such that the algebraic equation $w^{n-1}(w+a)+b=0$ has no multiple roots. We denote by

$$F = -\frac{f^{n-1}(f+a)}{b}, \qquad G = -\frac{g^{n-1}(g+a)}{b}. \tag{2}$$

Obviously, if $E(S_1, f) = E(S_1, g)$ then F and G share 1 CM.

Lemma 2. *Suppose that and* $\Theta(\infty, f) > \frac{2}{n-1}$ *and* $F \equiv G$, *where* F *and* G *are defined as (2), then* $f \equiv g$.

Proof. Suppose that $f \not\equiv g$. Since $F \equiv G$, we have

$$f^n(f+a) = g^n(g+a), \tag{3}$$

We assume that $\frac{f}{g} = h$, where h is a meromorphic function. By $f \not\equiv g$, we obtain that $h \not\equiv 1$. From (3) we deduce

$$f(z) = -\frac{ah(h^{n-1}-1)}{(h^n-1)}, \qquad g(z) = -\frac{a(h^{n-1}-1)}{(h^n-1)}. \tag{4}$$

Now we distinguish the following two cases.

Case 1. If h is a constant, then it follows from (4) that f is also constant. This is a contradiction.

Case 2. Suppose that h is nonconstant, by Lemma 1 and (4), we have

$$T(r, f) = (n-1)T(r, h) + S(r, f). \tag{5}$$

In addition, suppose that $c_j \in C \setminus \{1\} (j = 1, 2, \ldots, n-1)$ are the distinct roots of algebra equation $w^n - 1 = 0$. Using the second main theorem, from (4) we have

$$\overline{N}(r, f) = \sum_{j=1}^{n-1} \overline{N}(r, \frac{1}{h - c_j}) \geq (n-3)T(r, h) + S(r, h). \tag{6}$$

It follows from (5) and (6) that $\Theta(\infty, f) \leq \frac{2}{n-1}$. This contradicts $\Theta(\infty,\ f) > \frac{2}{n-1}$.

This completes the proof of Lemma 2. □

Lemma 3. *Let* S_j $(j = 1, 2)$ *be defined as in Theorem 3, and let* F *and* G *be defined as* (2). *If* $E(S_j, f) = E(S_j, g)$ *for* $j = 1, 2$ *and* $F \not\equiv G$, *then*

$$\overline{N}(r, f) = \overline{N}(r, g) \leq \frac{2}{n-1}(T(r, f) + T(r, g)) + S(r, f) + S(r, g), \tag{7}$$

and

$$(8)\quad \overline{N}(r,f)=\overline{N}(r,g)\le \frac{2}{13}(T(r,f)+T(r,g))+\frac{7}{13}N_{1)}(r,f)+S(r,f)+S(r,g).$$

Proof. Set

$$(9)\qquad H:=(\frac{F'}{F-1}-\frac{G'}{G-1})-(\frac{F'}{F}-\frac{G'}{G}),$$

then

$$H=\frac{F'}{F(F-1)}-\frac{G'}{G(G-1)}.$$

It follows that

$$(10)\qquad N(r,H)\le \overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{g})+\overline{N}(r,\frac{1}{f+a})+\overline{N}(r,\frac{1}{g+a}).$$

Therefore, by a logarithmic derivative theorem and (10), we get that

$$(11)\qquad T(r,H)\le 2T(r,f)+2T(r,g)+S(r,f)+S(r,g).$$

We discuss the following two cases.

Case 1. Suppose that $H\equiv 0$. By integration, we have from (9)

$$(12)\qquad \frac{F-1}{F}=B\,\frac{G-1}{G},$$

where B is nonzero constant. Since $F\not\equiv G$, we have $B\ne 1$. Again by (12), we deduce that ∞ is a Picard exceptional value of f. Therefore, (7) and (8) hold.

Case 2. Suppose that $H\not\equiv 0$. Assume that z_1 is a pole of f with multiplicity p, then an elementary calculation gives that z_1 is the zero of H with multiplicity at least $np-1$. From this and (11), we obtain

$$(13)\quad \begin{aligned}(2n-1)\,\overline{N}(r,f)-n\,N_{1)}(r,f) &\le N(r,\frac{1}{H})\\ &\le 2T(r,f)+2T(r,g)+S(r,f)+S(r,g).\end{aligned}$$

Noting that $(2n-1)\,\overline{N}(r,f)-n\,N_{1)}(r,f)\ge(n-1)\,\overline{N}(r,f)$, we obtain from (13) that (7) holds. Again by (13), we have

$$(2n-1)\,\overline{N}(r,f)\le 2T(r,f)+2T(r,g)+n\,N_{1)}(r,f)+S(r,f)+S(r,g).$$

It follows from $n\ge 7$ that (8) holds.

This completes the proof of Lemma 3. □

Finally, we need the following important lemma due to Yi (see [15]). We first introduce some notations.

Let $F(z)$ be a meromorphic function, we denote by $n_2(r,F)$ the number of poles of F in $|z|\le r$, where a simple pole is counted once and a multiple pole is counted two times, $N_2(r,F)$ is defined as the counting function of $n_2(r,F)$. Moreover, we denote by E any set with finite linear measure.

Lemma 4. *Let F and G be two nonconstant meromorphic functions such that F and G share $1, \infty$ CM. If*

$$N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G})+2\,\overline{N}(r,F)<\lambda\,T(r)+S(r),$$

where $\lambda<1$, $T(r)=\max\{T(r,F),T(r,G)\}$ and $S(r)=o\{T(r)\}\,(r\to\infty, r\notin E)$, then $F\equiv G$ or $F\,G\equiv 1$.

3. Proof of main results

Proof of Theorem 1. We define F and G as (2), then F and G share 1 CM.

Suppose that $F\not\equiv G$. Lemma 3 implies that

$$\overline{N}(r,\ f)\le\frac{2}{n-1}(T(r,f)+T(r,g))+S(r). \tag{14}$$

Therefore, we have

$$\begin{aligned}(15)\qquad & N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G})+2\,\overline{N}(r,F)\\ \le\ & 2N(r,\frac{1}{f})+2N(r,\frac{1}{g})+N(r,\frac{1}{f+a})+N(r,\frac{1}{g+a})+2\,\overline{N}(r,f)+S(r)\\ \le\ & 3T(r,f)+3T(r,g)+2\,\overline{N}(r,F)+S(r).\end{aligned}$$

Set $T_1(r):=\max\{T(r,f),\,T(r,g)\}$, then we obtain from (2) that

$$T(r)=nT_1(r)+O(1), \tag{16}$$

where $T(r)=\max\{T(r,F),\,T(r,G)\}$. From (14), (15) and (16) we deduce that

$$N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G})+2\,\overline{N}(r,F)\le\frac{6+\frac{8}{n-1}}{n}\,T(r)+S(r). \tag{17}$$

Since $n\ge 8$, we have $6+\frac{8}{n-1}<n$. Using Lemma 4, we have $F\,G\equiv 1$. From (2) we obtain

$$f^n\,(f+a)\,g^n\,(g+a)\equiv b^2. \tag{18}$$

Since $E(S_2,f)=E(S_2,g)$, from (18), we obtain that 0, $-a$ and ∞ are all Picard exceptional values of f. This is a contradiction. And hence, we obtain that $F\equiv G$, i.e.,

$$f^{n-1}(f+a)\equiv g^{n-1}(g+a). \tag{19}$$

Let $h = \frac{f}{g}$, then $h \not\equiv 1$. Therefore, by (19) we get that Theorem 1 holds. □

Proof of Theorem 2. By Theorem 1, we have $f \equiv g$ or

$$f(z) = -\frac{ah(h^{n-1}-1)}{h^n - 1}, \qquad g(z) = -\frac{a(h^{n-1}-1)}{h^n - 1}, \tag{20}$$

where h is a nonconstant meromorphic function. If $f \not\equiv g$, from (20) we could deduce that $\Theta(\infty, f) \le \frac{2}{n-1}$, which is a contradiction. Therefore, $f \equiv g$. □

Proof of Theorem 3. If $n \ge 8$, by Theorem 2 and $\Theta(\infty, f) > \frac{1}{2} > \frac{2}{n-1}$, we get that $f \equiv g$. Next we assume that $n = 7$. Proceeding as in the proof of Theorem 1, we have (15) and (16). From (15) and (16), we deduce that

$$N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + 2\,\overline{N}(r, F) \le \frac{8 - 2\,\Theta(\infty, f)}{7}\, T(r) + S(r). \tag{21}$$

Noting that $8-2\Theta(\infty, f) < 7$ when $\Theta(\infty, f) > 1/2$, and using Lemma 4 and Lemma 2, we also obtain the conclusion of Theorem 3. □

Proof of Theorem 4. Suppose that $n \ge 8$. From (1), we get $\Theta(\infty, f) \ge \frac{1}{2}\delta_{1)}(\infty, f) > \frac{2}{n-1}$. Therefore, by Theorem 2, we have $f \equiv g$. Next we assume that $n = 7$. Proceeding as in the proof of Theorem 1, we also have (15) and (16). From (8) and (16), we deduce that

$$N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + 2\,\overline{N}(r, F) \le \frac{86}{13}\, T_1(r) + \frac{14}{13}\, N_{1)}(r, f) + S(r). \tag{22}$$

From (19) and (22) we obtain

$$N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + 2\,\overline{N}(r, F) \le \frac{\frac{100}{13} - \frac{14}{13}\,\delta_{1)}(\infty, f)}{7}\, T(r) + S(r).$$

Noting that $\delta_{1)}(\infty, f) > 9/14$ and using Lemma 4 and a similar method to the above proof, we obtain the conclusion of Theorem 4. □

4. Applications

As an application of Theorem 4, we obtain the following result.

Theorem 5. *Let S_1, S_2 be defined as in Theorem 4. For a positive integer k, if f and g are meromorphic functions satisfying $E(S_j, f^{(k)}) = E(S_j, g^{(k)})$ for $j = 1, 2$ and $n \ge 7$, then $f^{(k)} \equiv g^{(k)}$.*

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