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On an Extension of Hardy-Hilbert's Inequality

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Abstract. In this paper, by introducing three parameters A, B and λ, and estimating the weight coefficient, we give a new extension of Hardy-Hilbert's inequality with a best constant factor, involving the Beta function. As applications, we consider its equivalent inequality.

1. Introduction

If $a_n, b_n \geq 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, such that $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then the famous Hardy-Hilbert's inequality is given by

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\{\sum_{n=1}^{\infty} a_n^p\}^{1/p}\{\sum_{n=1}^{\infty} b_n^q\}^{1/q}, \tag{1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Its equivalent inequality is

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{m+n}\right)^p < \left[\frac{\pi}{\sin(\pi/p)}\right]^p\sum_{n=1}^{\infty} a_n^p, \tag{2}$$

where the constant factor $[\frac{\pi}{\sin(\pi/p)}]^p$ is still the best possible (see [1]).

Inequality (1) and (2) are important in analysis and its applications (see [2]). In recent years, (1) had been strengthened by [3], [4] as

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \{\sum_{n=1}^{\infty}[\frac{\pi}{\sin(\frac{\pi}{p})} - \frac{1-\gamma}{n^{1/p}}]a_n^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}[\frac{\pi}{\sin(\frac{\pi}{p})} - \frac{1-\gamma}{n^{1/q}}]b_n^q\}^{\frac{1}{q}}, \tag{3}$$

where $1-\gamma = 0.42278433^{+}$ ($\gamma = 0.57721566^{+}$ is Euler constant).

By introducing three parameters A, B and λ, Yang et al. [5] gave a generalization of (1) as

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(Am+Bn)^{\lambda}} < \frac{B(\varphi_\lambda(p),\varphi_\lambda(q))}{A^{\varphi_\lambda(p)}B^{\varphi_\lambda(q)}}\{\sum_{n=1}^{\infty} n^{1-\lambda}a_n^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty} n^{1-\lambda}b_n^q\}^{\frac{1}{q}}, \tag{4}$$

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where the constant factor $\frac{B(\varphi_\lambda(p),\varphi_\lambda(q))}{A^{\varphi_\lambda(p)}B^{\varphi_\lambda(q)}}$ ($\varphi_\lambda(r) = \frac{r+\lambda-2}{r}$, $\lambda > 2-r$, $r = p, q$) is still the best possible ($B(u,v)$ is the Beta function). For $A = B = 1$, inequality (4) reduces to

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n)^\lambda} < B(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q})\{\sum_{n=1}^{\infty}n^{1-\lambda}a_n^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}n^{1-\lambda}b_n^q\}^{\frac{1}{q}}. \tag{5}$$

Both (4) and (5) are generalizations of (1).

The main objective of this paper is to estimate the following weight coefficient:

$$\omega_\lambda(A,B,p,m) = \sum_{n=1}^{\infty}\frac{m^{(1-\lambda)(p-1)}(Am)^{(1-\lambda/q)(p-1)}}{(Am+Bn)^\lambda(Bn)^{1-\lambda/p}} \quad (A,B>0,\ 0<\lambda\le p,\ m\in N) \tag{6}$$

and then to obtain a new inequality related to the double series

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(Am+Bn)^\lambda}$$

with a best constant factor, but different from (4).

We need some lemmas and the following formula of the Beta function (see [6]):

$$B(u,v) = \int_0^\infty \frac{t^{-1+u}}{(1+t)^{u+v}}dt = B(v,u) \quad (u,v>0). \tag{7}$$

2. Some lemmas

Lemma 2.1. *If $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $0 < \lambda \le p$ and A, $B > 0$, $\omega_\lambda(A,B,p,m)$ is defined by* (6)*, then for any $m \in N$, we have*

$$\omega_\lambda(A,B,p,m) < \frac{1}{A^{(\lambda-1)(p-1)}B}B(\frac{\lambda}{p},\frac{\lambda}{q}). \tag{8}$$

Proof. Since $0 < \lambda \le p$ and $A, B > 0$, we have

$$\omega_\lambda(A,B,p,m) < m^{(1-\lambda)(p-1)}(Am)^{(1-\frac{\lambda}{q})(p-1)}\int_0^\infty \frac{1}{(Am+By)^\lambda(By)^{1-\lambda/p}}dy.$$

Putting $u = (By)/(Am)$ in the above inequality, we obtain

$$\omega_\lambda(A,B,p,m) < \frac{1}{A^{(\lambda-1)(p-1)}B}\int_0^\infty \frac{u^{-1+\lambda/p}}{(1+u)^\lambda}du.$$

Hence by (7), we have (8). The lemma is proved. □

Note. If $0 < \lambda \leq q$, by (8) and (7), for any $n \in N$, we also have

$$(9)\quad \omega_\lambda(B, A, q, n) = \sum_{n=1}^{\infty} \frac{m^{(1-\lambda)(q-1)}(Bn)^{(1-\lambda/p)(q-1)}}{(Bn + Am)^\lambda (Am)^{1-\lambda/q}} < \frac{1}{B^{(\lambda-1)(q-1)}A} B(\frac{\lambda}{p}, \ \frac{\lambda}{q}).$$

Lemma 2.2. *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \lambda \leq \min\{p, \ q\}$ and $0 < \varepsilon < \lambda(q-1)$, then we have*

$$(10)\qquad \begin{aligned} I \ &= \ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{(Am + Bn)^\lambda} m^{\lambda-1-\frac{\lambda+\varepsilon}{p}} n^{\lambda-1-\frac{\lambda+\varepsilon}{q}} \\ &> \ A^{-\frac{\lambda-\varepsilon}{q}} B^{\frac{\varepsilon}{q}-\frac{\lambda}{p}} \{\frac{1}{\varepsilon} B(\frac{\lambda}{p} - \frac{\varepsilon}{q}, \frac{\lambda+\varepsilon}{q}) - (\frac{\lambda}{p} - \frac{\varepsilon}{q})^{-2} (\frac{B}{A})^{\frac{\lambda}{p}-\frac{\varepsilon}{q}}\}. \end{aligned}$$

Proof. We have $\lambda - 1 - \frac{\lambda+\varepsilon}{r} < 0$ $(r = p, \ q)$ and $\frac{\lambda}{p} - \frac{\varepsilon}{q} > 0$. Hence we find

$$I > \int_1^\infty x^{\lambda-1-\frac{\lambda+\varepsilon}{p}} [\int_0^\infty \frac{1}{(Ax + By)^\lambda} y^{\lambda-1-\frac{\lambda+\varepsilon}{q}} dy] dx.$$

Setting $u = (By)/(Ax)$ in the above integral, we obtain

$$\begin{aligned} I \ &> \ A^{-\frac{\lambda-\varepsilon}{q}} B^{\frac{\varepsilon}{q}-\frac{\lambda}{p}} \int_1^\infty x^{-1-\varepsilon} [\int_{B/(Ax)}^\infty \frac{1}{(1+u)^\lambda} u^{-1+\frac{\lambda}{p}-\frac{\varepsilon}{q}} du] dx \\ &= \ A^{-\frac{\lambda-\varepsilon}{q}} B^{\frac{\varepsilon}{q}-\frac{\lambda}{p}} \{\int_1^\infty x^{-1-\varepsilon} [\int_0^\infty \frac{1}{(1+u)^\lambda} u^{-1+\frac{\lambda}{p}-\frac{\varepsilon}{q}} du] dx \\ &\quad - \int_1^\infty x^{-1-\varepsilon} [\int_0^{B/(Ax)} \frac{1}{(1+u)^\lambda} u^{-1+\frac{\lambda}{p}-\frac{\varepsilon}{q}} du] dx\} \\ &> \ A^{-\frac{\lambda-\varepsilon}{q}} B^{\frac{\varepsilon}{q}-\frac{\lambda}{p}} \{\frac{1}{\varepsilon} \int_0^\infty \frac{u^{-1+\frac{\lambda}{p}-\frac{\varepsilon}{q}}}{(1+u)^\lambda} du - \int_1^\infty x^{-1} [\int_0^{B/(Ax)} u^{-1+\frac{\lambda}{p}-\frac{\varepsilon}{q}} du] dx\} \\ &= \ A^{-\frac{\lambda-\varepsilon}{q}} B^{\frac{\varepsilon}{q}-\frac{\lambda}{p}} \{\frac{1}{\varepsilon} \int_0^\infty \frac{u^{-1+(\frac{\lambda}{p}-\frac{\varepsilon}{q})}}{(1+u)^\lambda} du - (\frac{\lambda}{p} - \frac{\varepsilon}{q})^{-2} (\frac{B}{A})^{\frac{\lambda}{p}-\frac{\varepsilon}{q}}\}. \end{aligned}$$

By (7), we have (10). The lemma is proved. □

3. Main results and applications

Theorem 3.1. *If $a_n, \ b_n \geq 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \lambda \leq \min\{p, q\}$, such that $0 < \sum_{n=1}^\infty n^{(1-\lambda)(p-1)} a_n^p < \infty$ and $0 < \sum_{n=1}^\infty n^{(1-\lambda)(q-1)} b_n^q < \infty$, then for*

$A,\ B>0$, we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(Am+Bn)^{\lambda}} < \frac{B(\frac{\lambda}{p},\frac{\lambda}{q})}{A^{\lambda/q}B^{\lambda/p}}\{\sum_{n=1}^{\infty}n^{(1-\lambda)(p-1)}a_n^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}n^{(1-\lambda)(q-1)}b_n^q\}^{\frac{1}{q}}, \tag{11}$$

where the constant factor $B(\frac{\lambda}{p},\frac{\lambda}{q})/(A^{\lambda/q}B^{\lambda/p})$ is the best possible. In particular, for $A=B=1$, we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < B(\frac{\lambda}{p},\frac{\lambda}{q})\{\sum_{n=1}^{\infty}n^{(1-\lambda)(p-1)}a_n^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}n^{(1-\lambda)(q-1)}b_n^q\}^{\frac{1}{q}}. \tag{12}$$

Proof. By Hölder's inequality and in view of (6) and (9), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(Am+Bn)^{\lambda}}\\
=\ &\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}[\frac{a_m}{(Am+Bn)^{\lambda/p}}\cdot\frac{(Am)^{(q-\lambda)/q^2}}{(Bn)^{(p-\lambda)/p^2}}][\frac{b_m}{(Am+Bn)^{\lambda/q}}\cdot\frac{(Bn)^{(p-\lambda)/p^2}}{(Am)^{(q-\lambda)/q^2}}]\\
\le\ &\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m^p}{(Am+Bn)^{\lambda}}\frac{(Am)^{(q-\lambda)p/q^2}}{(Bn)^{(p-\lambda)/p}}\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{b_m^q}{(Bn+Am)^{\lambda}}\frac{(Bn)^{(p-\lambda)q/p^2}}{(Am)^{(q-\lambda)/q}}\}^{\frac{1}{q}}\\
=\ &\{\sum_{m=1}^{\infty}\omega_{\lambda}(A,B,p,m)m^{(1-\lambda)(p-1)}a_m^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}\omega_{\lambda}(B,A,q,n)n^{(1-\lambda)(q-1)}b_n^q\}^{\frac{1}{q}}.
\end{aligned}$$

Hence by (8) and (9), we have (11).

For $0<\varepsilon<\lambda(q-1)$, setting $\widetilde{a_m}$ and $\widetilde{b_n}$ as

$$\widetilde{a_m}=m^{\lambda-1-\frac{\lambda+\varepsilon}{p}},\widetilde{b_n}=n^{\lambda-1-\frac{\lambda+\varepsilon}{q}}\ (m,n\in N),$$

then we have

$$\begin{aligned}
J &= \{\sum_{n=1}^{\infty}n^{(1-\lambda)(p-1)}\widetilde{a}_n^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}n^{(1-\lambda)(q-1)}\widetilde{b}_n^q\}^{\frac{1}{q}}\\
&= \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}=1+\sum_{n=2}^{\infty}\frac{1}{n^{1+\varepsilon}}<1+\int_1^{\infty}\frac{1}{t^{1+\varepsilon}}dt=1+\frac{1}{\varepsilon}.
\end{aligned} \tag{13}$$

If there exists $A,B>0$ and $0<\lambda\le\min\{p,\ q\}$, such that the constant factor $B(\frac{\lambda}{p},\frac{\lambda}{q})/(A^{\lambda/q}B^{\lambda/p})$ in (11) is not the best possible, then there exists a positive number $K < B(\frac{\lambda}{p}, \frac{\lambda}{q})/(A^{\lambda/q}B^{\lambda/p})$, such that (11) is valid if we replace $B(\frac{\lambda}{p}, \frac{\lambda}{q})/(A^{\lambda/q}B^{\lambda/p})$ by K. In particular, we have

$$\varepsilon I = \varepsilon \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\widetilde{a}_m \widetilde{b}_n}{(Am+Bn)^{\lambda}} < \varepsilon K J.$$

Hence by (10) and (13), we find

$$A^{-\frac{\lambda-\varepsilon}{q}} B^{\frac{\varepsilon}{q}-\frac{\lambda}{p}} \{B(\frac{\lambda}{p}-\frac{\varepsilon}{q}, \frac{\lambda+\varepsilon}{q}) - \varepsilon(\frac{\lambda}{p}-\frac{\varepsilon}{q})^{-2}(\frac{B}{A})^{\frac{\lambda}{p}-\frac{\varepsilon}{q}}\} < K(1+\varepsilon).$$

Setting $\varepsilon \to 0^+$ in the above inequality, we conclude that $B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)/(A^{\lambda/q}B^{\lambda/p}) \leq K$. This contradicts the fact that $K < B(\frac{\lambda}{p}, \frac{\lambda}{q})/(A^{\lambda/q}B^{\lambda/p})$. Thus the constant factor $B(\frac{\lambda}{p}, \frac{\lambda}{q})/(A^{\lambda/q}B^{\lambda/p})$ in (11) is the best possible. The theorem is proved. □

Theorem 3.2. *If $a \geq 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \lambda \leq \min\{p, q\}$, such that $0 < \sum_{n=1}^{\infty} n^{(1-\lambda)(p-1)} a_n^p < \infty$, then for $A, B > 0$, we have*

$$\sum_{n=1}^{\infty} n^{\lambda-1}\left[\sum_{m=1}^{\infty} \frac{a_m}{(Am+Bn)^{\lambda}}\right]^p < \frac{[B(\frac{\lambda}{p}, \frac{\lambda}{q})]^p}{A^{\lambda(p-1)}B^{\lambda}} \sum_{n=1}^{\infty} n^{(1-\lambda)(p-1)} a_n^p, \tag{14}$$

where the constant factor $[B(\frac{\lambda}{p}, \frac{\lambda}{q})]^p/(A^{\lambda(p-1)}B^{\lambda})$ is the best possible; Inequality (14) is equivalent to (11). In particular, for $A = B = 1$, we have

$$\sum_{n=1}^{\infty} n^{\lambda-1}\left[\sum_{m=1}^{\infty} \frac{a_m}{(m+n)^{\lambda}}\right]^p < [B(\frac{\lambda}{p}, \frac{\lambda}{q})]^p \sum_{n=1}^{\infty} n^{(1-\lambda)(p-1)} a_n^p. \tag{15}$$

Proof. Since $0 < \sum_{n=1}^{\infty} n^{(1-\lambda)(p-1)} a_n^p < \infty$, then there exists $k_0 \in N$, such that for any $k > k_0$, that makes $0 < \sum_{n=1}^{k} n^{(1-\lambda)(p-1)} a_n^p < \infty$. We set $b_n(k) = n^{\lambda-1}\left[\sum_{m=1}^{k} \frac{a_m}{(Am+Bn)^{\lambda}}\right]^{p-1}$, and use (11) to obtain

$$\begin{aligned} 0 &< \sum_{n=1}^{k} n^{(1-\lambda)(q-1)} b_n^q(k) = \sum_{n=1}^{k} n^{\lambda-1}\left[\sum_{m=1}^{k} \frac{a_m}{(m+n)^{\lambda}}\right]^p \\ &= \sum_{n=1}^{k}\sum_{m=1}^{k} \frac{a_m b_n(k)}{(Am+Bn)^{\lambda}} \\ &< \frac{B(\frac{\lambda}{p}, \frac{\lambda}{q})}{A^{\lambda/q}B^{\lambda/p}}\{\sum_{n=1}^{k} n^{(1-\lambda)(p-1)} a_n^p\}^{\frac{1}{p}}\{\sum_{n=1}^{k} n^{(1-\lambda)(q-1)} b_n^q\}^{\frac{1}{q}}. \end{aligned} \tag{16}$$

Hence we find

$$\{\sum_{n=1}^{k} n^{(1-\lambda)(q-1)} b_n^q(k)\}^{\frac{1}{p}} < \frac{B(\frac{\lambda}{p}, \frac{\lambda}{q})}{A^{\lambda/q}B^{\lambda/p}}\{\sum_{n=1}^{k} n^{(1-\lambda)(p-1)} a_n^p\}^{\frac{1}{p}}. \tag{17}$$

It follows that $0 < \sum_{n=1}^{\infty} n^{(1-\lambda)(q-1)} b_n^q(\infty) < \infty$. Hence, (16) keeps strict inequality as $k \to \infty$ by (11); so dose (17) . Thus inequality (14) holds.

We prove that (11) implies (14). We need show that (14) implies (11). By Hölde's inequality, we have

$$
\begin{aligned}
(18)\qquad & \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{(Am+Bn)^{\lambda}} \\
= & \sum_{n=1}^{\infty} \left[n^{(\lambda-1)/p} \sum_{m=1}^{\infty} \frac{a_m}{(Am+Bn)^{\lambda}} \right] \left[n^{(1-\lambda)/p} b_n \right] \\
\le & \{\sum_{n=1}^{\infty} n^{\lambda-1} \left[\sum_{m=1}^{\infty} \frac{a_m}{(Am+Bn)^{\lambda}} \right]^p \}^{\frac{1}{p}} \{\sum_{n=1}^{\infty} n^{(1-\lambda)(q-1)} b_n^q \}^{\frac{1}{q}}.
\end{aligned}
$$

Hence by (14), we have (11). It follows that inequality (14) is equivalent to (11).

If the constant factor in (14) is not the best possible, we may get a contradiction that the constant factor in (11) is not the best possible by (18). The theorem is proved. □

If $\lambda = p \le q$, we find that

$$\frac{B(\frac{p}{p}, \frac{p}{q})}{A^{p/q} B^{p/p}} = \frac{\Gamma(1)\Gamma(p-1)}{\Gamma(p) A^{p-1} B} = \frac{1}{(p-1)A^{p-1}B},$$

and in view of (11) and (14), we have

Corollary 3.3. *If $a_n,\ b_n \ge 0,\ 1 < p \le q,\ \frac{1}{p} + \frac{1}{q} = 1$, such that $0 < \sum_{n=1}^{\infty} n^{-(p-1)^2} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} n^{-1} b_n^q < \infty$, then for $A, B > 0$, we have the following two equivalent inequalities*:

$$(19)\qquad \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{(Am+Bn)^p} < \frac{1}{(p-1)A^{p-1}B} \{\sum_{n=1}^{\infty} n^{-(p-1)^2} a_n^p\}^{\frac{1}{p}} \{\sum_{n=1}^{\infty} n^{-1} b_n^q\}^{\frac{1}{q}};$$

$$(20)\qquad \sum_{n=1}^{\infty} n^{\lambda-1} \left[\sum_{m=1}^{\infty} \frac{a_m}{(Am+Bn)^p} \right]^p < \left[\frac{1}{(p-1)A^{p-1}B} \right]^p \sum_{n=1}^{\infty} n^{-(p-1)^2} a_n^p,$$

where both of the constant factors in (19) *and* (20) *are the best possible. In particular, for $A = B = 1$, we have*

$$(21)\qquad \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)^p} < \frac{1}{(p-1)} \{\sum_{n=1}^{\infty} n^{-(p-1)^2} a_n^p\}^{\frac{1}{p}} \{\sum_{n=1}^{\infty} n^{-1} b_n^q\}^{\frac{1}{q}};$$

$$(22)\qquad \sum_{n=1}^{\infty} n^{\lambda-1} \left[\sum_{m=1}^{\infty} \frac{a_m}{(m+n)^p} \right]^p < \frac{1}{(p-1)^p} \sum_{n=1}^{\infty} n^{-(p-1)^2} a_n^p.$$

Remark 3.4. (i). For $\lambda = 1$ and $A, B > 0$, both (4) and (11) reduce to

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{Am+Bn} < \frac{\pi}{A^{1/q}B^{1/p}\sin(\frac{\pi}{p})}\{\sum_{n=1}^{\infty} a_n^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty} b_n^q\}^{\frac{1}{q}}, \tag{23}$$

and for $p = q = 2$ and $0 < \lambda \leq 2$, both (4) and (11) reduce to

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(Am+Bn)^{\lambda}} < \frac{B(\frac{\lambda}{2},\frac{\lambda}{2})}{(AB)^{\lambda/2}}\{\sum_{n=1}^{\infty} n^{1-\lambda}a_n^2\sum_{n=1}^{\infty} n^{1-\lambda}b_n^2\}^{\frac{1}{2}}, \tag{24}$$

which is just (3.17) in [5] and similar to (3.5) in [7] for $C = A + B$. Inequalities (4) and (11) are two distinct extensions of (1) with distinct best constant factors; so are (5) and (12).

(ii). Inequality (4) is a new extension of (2). Inequality (22) is only dependent on $p > 1$, which is not an extension of (2).

(iii). Since all the extended inequalities are with the best constant factors, we give some new results.

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